Received 04/01/11

Hermite-Hadamard Inequality in the Geometry of Banach Spaces

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This thesis is presented in fulfilment of
the requirements for the degree of Doctor of Philosophy

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Victoria University

February 2010

Abstract

The theory of inequalities has made significant contributions in many areas of mathematics. The purpose of this dissertation is to employ inequalities in studying the geometry of a Banach space.

Motivated by the Hermite-Hadamard inequality, a new family of norms is defined, which is called the p-HH-norm. The p-HH-norms are equivalent to the p-norms (the vector-valued analogue of the ℓ^p-spaces) in $\mathbf{X}^2$. Evidently, the p-HH-norms preserve the completeness (as well as the reflexivity) of the underlying space in $\mathbf{X}^2$. The p-HH-norm of two positive real numbers is the well-known generalized logarithmic mean. The sensitivity of the p-HH-norms to the geometry of the underlying space is markedly different than the p-norms. The reason for this is that the p-HH-norms depend on the relative positions of the original vectors, not just the size of the vectors. The smoothness and convexity of the p-HH-norms in $\mathbf{X}^2$ are inherited from $\mathbf{X}$, when $p \neq 1$. The 1-HH-norm preserves the smoothness, in contrast to the 1-norm.

Despite the equivalence, the p-HH-norms are different to the p-norms. Some Ostrowski type inequalities are established to give quantitative comparison between the p-norm and the p-HH-norm. In the same spirit, some inequalities of Grüss type are employed to give comparison amongst the p-HH-norms.

By utilizing the 2-HH-norm, some new notions of orthogonality in normed spaces are introduced. These orthogonalities are shown to be closely connected to the classical ones, namely Pythagorean, Isosceles and Carlsson's orthogonalities. Some characterizations of inner product spaces follow by the homogeneity, as well as the additivity, of these new orthogonalities.

The p-HH-norms are then extended to the nth Cartesian power $\mathbf{X}^n$ of a normed space $\mathbf{X}$. When $\mathbf{X} = \mathbb{R}$, the p-HH-norms resemble the unweighted hypergeometric mean. As in the case of $n = 2$, the metrical and geometrical properties of the p-HH-norms in $\mathbf{X}^n$ are closely connected to those of $\mathbf{X}$, in the same manner. The extension of the p-HH-norm from $\mathbf{X}^n$ to a suitable space of sequences of elements in $\mathbf{X}$ reveals their fundamental differences with the p-norms. When $\mathbf{X} = \mathbb{R}$ the norm provides an extension of the hypergeometric mean to infinite sequences. The resulting sequence spaces all lie between $\ell^1(\mathbf{X})$ and $\ell^\infty(\mathbf{X})$. These spaces need not be lattices, are not necessarily complete spaces, and need not even be closed under a permutation of the terms of the sequence.

The research outcomes of this thesis make significant contributions in Banach space theory, the theory of means and the theory of inequalities. These contributions including the characterization of inner product spaces via orthogonality; the extension of means of positive numbers to a vector space setting; and the developments of some important inequalities, namely the Hermite-Hadamard inequality, Ostrowski inequality and Grüss inequality in linear spaces.

Declaration

"I, Eder Kikianty, declare that the PhD thesis entitled *Hermite-Hadamard Inequality in the Geometry of Banach Spaces* is no more than 100,000 words in length including quotes and exclusive of tables, figures, appendices, bibliography, references and footnotes. This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work".

Signature Date

Acknowledgements

There are many people that have assisted and supported me during my PhD candidacy; and I would like to express my gratitude to them.

First of all, my deepest gratitude goes to my principal supervisor, Professor Sever S. Dragomir for his guidance, the countless hours of supervision and the encouraging advice throughout my candidature. I thank him for showing me the beauty of mathematical inequalities, teaching me the art of writing research papers, for his support on many research-related matters and his advice on my PhD research, as well as future research career. I would also like to thank my associate supervisor, Professor Pietro Cerone, for his advice and feedback on my research, for helping me in correcting my papers, for supporting me in attending conferences and for countless hours of encouraging discussions.

This dissertation would not be completed without the help of Professor Gord Sinnamon from the University of Western Ontario, Canada. I thank him for his contributions in Chapter 7 of this dissertation, for introducing me to the study of function spaces, for giving up many hours to help me in completing this research and for fruitful discussions and conversations on mathematics-related matters.

I acknowledge the financial support from Victoria University by awarding me the Faculty of Health, Engineering and Science Research Scholarship. I would also like to thank the Secomb Travel and Fund, the Australian Mathematical Science Institute (AMSI) and the Australian Mathematical Society (AustMS) for the support that had been given to me to attend the graduate school, summer school and conferences.

My gratitude to Professor Terry Mills and Dr. David Yost, for inviting me to share my work at La Trobe University (Bendigo Campus) and the University of Ballarat, respectively; and for their useful comments on my research. I would also like to express a deep appreciation to my former supervisor, Professor Hendra Gunawan from Institut Teknologi Bandung (ITB), for his constant support and encouraging advice. I would also like to thank my former lecturers and colleagues in the Analysis and Geometry Research group of ITB, for their constant support and advice.

Special thanks goes to my best friend Ivanky Saputra, for endless hours of mathematics-related conversations, for his moral support, for fruitful discussions on LaTeX and for his companionship throughout the years I have spent in Melbourne. My warmest thanks to my dear friends, officemates and colleagues that have shared the wonderful years at Victoria University: David Fitrio, Michael Grubinger, Anand Mohan, Jaideep Chandran, Mark Mojic, Hayrettin Arisoy, Nikhil Joglekar, Khai Hoang, Gabe Sorrentino and George Hanna; for their support and useful comments on my research and discussions on many research-related matters and/or any random matters during our lunch/coffee-breaks. I would also like to thank fellow students of AustMS: Ray Vozzo, Kenneth Chan, Simon James, Melissa Tacy, Allison Plant, Ric Green, Rongmin Lu, Ryan Mickler, Shreya Bhattarai and Brian Corr; for the joyous moments that we have shared in conferences and winter/summer schools. I am truly grateful to my best friends, Carolyn Sugiarto and Retta Nugrahani for all the laughter and tears that we have shared, which have helped me to endure the hardships throughout these years. I would also like to thank my dear friends Agus Susanto, Cindy Ekaputri, Suryanto Loa, Regina Anggraini, Kartika Dewi, Bianca Sondakh, Vincent Sanjaya, Daniel Adhiguna and David Irawan for their prayers and constant moral support. A special thanks to Stojan and Pato, for accompanying me in long office hours and many sleepless nights.

Most of all, to my family, Papa, Mama, my sister Juni and her husband Sony; my brother Adi, his wife Lina and my lovely niece Clarissa; my dearest cousin Wira Wijaya; and last but not least, to my dearest sister and best friend, Wenny; I thank you for your endless support, for always being there for me and for always believing in me. Above all, I thank my God and Saviour, Lord Jesus Christ, for He is my shield and my strength.

Papers published/submitted during the author's candidature

This dissertation contains the results from a number of papers that have been published and submitted in refereed publications.

- Chapter 3 of this dissertation contains the results of the following research paper: "Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space" with S.S. Dragomir, *Mathematical Inequalities and Application*, Vol. 13, 2010, no. 1, 1–32.
- Chapter 4 of this dissertation contains the results of the following research papers:
  - "Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and applications" with S.S. Dragomir and P. Cerone, *Computer Mathematics and Applications*, Vol. 56, 2008, 2235–2246.
  - "Ostrowski type inequality for absolutely continuous functions on segments in linear space" with S.S. Dragomir and P. Cerone, *Bulletin of Korean Mathematical Society*, Vol. 45, no. 4, 2008, 763–780.
- Chapter 5 of this dissertation contains the results of the following research paper: "Inequalities of Grüss type involving the p-HH-norms in the Cartesian product space" with S.S. Dragomir and P. Cerone, *Journal of Mathematical Inequalities*, Vol. 3, no. 4, 2009, 543–557.
- Chapter 6 of this dissertation contains the results of the following research papers:
  - "Orthogonality connected with integral means and characterizations of inner product spaces" with S.S. Dragomir, *submitted*.
  - "On Carlsson type orthogonality" with S.S. Dragomir, *submitted*.
- Chapter 7 of this dissertation contains the results of the following research paper: "The p–HH norms on Cartesian powers and sequence spaces" with Gord Sinnamon,

Journal of Mathematical Analysis and Applications, Vol. 359 Issue 2, 2009, 765–779.

Presentations given during the author's candidature

During my candidature, I have given a number of presentations in Conferences, Workshops and Seminars in conjunction with the material in this dissertation.

- **School of Computer Science and Mathematics Seminar Series**, "Orthogonality in normed and n-normed spaces", Victoria University, Melbourne, 22 September 2006.
- **ICE-EM Australian Graduate School in Mathematics** Student talks, "Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and applications", University of Queensland, Brisbane, 2-20 July 2007.
- **Invited talk in the Seminar Program of the Department of Mathematics and Statistics, La Trobe University**, "Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space", La Trobe University, Bendigo, 2 May 2008.
- **Mathematical Inequalities and Application 2008**, a conference in honour of Prof. Josip Pečarić on the occasion of his 60th birthday, "Hermite-Hadamard's inequality and the p-HH-norm", Trogir-Split, Croatia, 8-14 June 2008.
- **Faculty of Health, Engineering and Science Postgraduate Research Seminar**, "Hermite-Hadamard's inequality and the p-HH-norm", Victoria University, Melbourne, 31 October 2008.
- **7th joint Australia-New Zealand Mathematics Convention 2008**, "The p-HH-norm on the Cartesian product of n copies of a normed space", University of Canterbury, Christchurch, New Zealand, 7-14 December 2008.
- **Invited talk in the Seminar Program of Analysis and Geometry Research Group, Institut Teknologi Bandung** "The p-HH-norms on Cartesian powers and sequence spaces", Institut Teknologi Bandung, Indonesia, 2 February 2009.

- **Invited talk in the Seminar Program of School of Information Technology and Mathematical Sciences, University of Ballarat**, "Orthogonality in normed spaces and characterisations of inner product spaces," University of Ballarat, Ballarat, 18 June 2009.
- **1$^{\text{st}}$ Pacific Rim Mathematical Association (PRIMA) Congress**, "On new notions of orthogonality via integral means in normed spaces", University of New South Wales, Sydney, Australia, 6–10 July 2009.
- **AMSI-ANU Workshop in Spectral Theory and Harmonic Analysis**, "On norms connected with the hypergeometric mean", Poster presentation, Australian National University, Canberra, 13–17 July 2009.
- **53$^{\text{rd}}$ Annual Meeting of the Australian Mathematical Society**, "Hypergeometric means in normed spaces", University of South Australia, Adelaide, 27 September – 1 October 2009.
- **Invited talk in the Seminar Program of the Department of Mathematics and Statistics, La Trobe University**, "Integral means and orthogonality in normed spaces", La Trobe University, Bendigo, 16 October 2009.
- The **Research Group of Mathematical Inequalities and Applications (RGMIA) Seminar Series**, Victoria University, Melbourne:
  - "Sharp inequalities of Ostrowski type for convex functions defined on linear spaces and applications", 23 April 2007.
  - "Notions of orthogonality in normed spaces", 22 May 2008.
  - "Hermite-Hadamard inequality and orthogonality in normed spaces", 4 May 2009.

Contents

Notation

$\boldsymbol{\ell^p}$**:** The space of all p-summable sequences $(1 \leq p < \infty)$

$\boldsymbol{\ell^\infty}$**:** The space of all bounded sequences

$\boldsymbol{L^p}$**:** The space of all (classes of) p-integrable functions $(1 \leq p < \infty)$

$\boldsymbol{L^\infty}$**:** The space of all (classes of) bounded measurable functions

$\boldsymbol{\ell^p(\mathrm{X})}$**:** The space of all $\mathbf{X}$-valued p-summable sequences $(1 \leq p < \infty)$

$\boldsymbol{L^p(\Omega, \mathrm{X})}$**:** The space of all (classes of) $\mathbf{X}$-valued p-integrable functions on a finite measure space Ω $(1 \leq p < \infty)$

$\mathbb{R}$**:** The real numbers

$\mathbb{R}^+$**:** The positive real numbers

$\mathbf{X}^*$**:** The dual space of $\mathbf{X}$ (the space of bounded linear functionals on $\mathbf{X}$)

$\langle\cdot,\cdot\rangle_{s(i)}$**:** The superior (inferior) semi-inner product

$\boldsymbol{S_{\mathrm{X}}}$**:** The unit circle in the normed space $\mathbf{X}$ with respect to the given norm

$\boldsymbol{(\nabla_{\pm} f(x))(y)}$**:** The right-(left-)Gâteaux derivative of f at x in y direction

$\boldsymbol{(\nabla f(x))(y)}$**:** The Gâteaux derivative of f at x in y direction

$\boldsymbol{D(f)}$**:** The domain of f

$\boldsymbol{A(x,y)}$**:** The arithmetic mean of two positive (real) numbers x and y, that is, $\frac{x+y}{2}$

$\boldsymbol{G(x,y)}$**:** The geometric mean of two positive numbers x and y, that is, $\sqrt{xy}$

$\boldsymbol{M_p(x,y)}$**:** The pth $(p \neq 0)$ power mean of two positive numbers x and y, that is, $(\frac{x^p+y^p}{2})^{1/p}$

$\boldsymbol{L(x,y)}$**:** The logarithmic mean of two positive numbers x and y, that is, $\frac{x-y}{\log x - \log y}$

$\boldsymbol{I(x,y)}$**:** The identric mean of two positive numbers x and y, that is, $\frac{1}{e}(\frac{y^y}{x^x})^{1/(y-x)}$

$\mathfrak{L}^{[p]}(\boldsymbol{x},\boldsymbol{y})$**:** The pth ($p \neq -1, 0$) order generalized logarithmic mean of two positive numbers x and y, that is, $(\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)})^{1/p}$

$\mathfrak{M}_{p}^{[n]}(\mathbf{x},\mathbf{w})$**:** The pth order weighted power mean of a positive n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ with respect to the weight $\mathbf{w} = (w_1, \ldots, w_n)$

$\mathfrak{C}(\boldsymbol{p},\boldsymbol{c};\mathbf{x},\mathbf{w})$**:** The pth order hypergeometric mean of $\mathbf{x} = (x_1, \ldots, x_n)$ with respect to the weight $\mathbf{w} = (w_1, \ldots, w_n)$, where c is a positive real number

$\mathfrak{M}_{[p]}^{[a,b]}(\boldsymbol{f})$**:** The pth power mean of f on $[a, b]$

"We are servants rather than masters in mathematics"

— Charles Hermite (1822-1901)

Chapter 1

Introduction

This chapter provides a concise introduction to the Hermite-Hadamard inequality. It also presents the main motivation for writing this dissertation in the field of Banach space geometry, and provides an overview of the outline and content of this dissertation.

1.1 Background

"Analysis abounds with inequalities" as pointed out by Aigner and Ziegler [1]. Numerous important inequalities have been employed as powerful tools not only in analysis, but also in other areas of mathematics, such as the theory of means, approximation theory, numerical analysis and so on. For example, the famous arithmetic-geometric mean inequality was elegantly used by Erdös and Grünwald [50] to estimate integrals by rectangles and tangential triangles (cf. Aigner and Ziegler [1, p. 111–114]). The importance of inequalities is mainly highlighted by their role in analysis; but the use of inequalities can sometimes be quite unexpected, for example in graph theory [1]. In their book [1, p. 114–115], Aigner and Ziegler discussed a simple form of Turan's theorem on the number of edges of graph without triangles, whose proof is done by applying the Cauchy-Schwarz inequality.

The theory of inequalities is now an important 'branch' of mathematics. In his essay, Fink traced the development of 'inequalities' as a discipline of mathematics [52]. Fink sketched the history of inequalities from ancient times, where inequalities were known as geometrical facts, to the awakening of inequality analysis in the early 18th century. One of the most notable books written in inequality theory is the famous classic "*Inequalities*" by Hardy, Littlewood and Polya [57], which was published in 1934. Some other famous books in this area are Beckenbach and Bellmann's "*Inequalities*" [10] which was published in 1961 and Mitrinović's "*Analytic Inequalities*" [88] in 1970. Numerous papers on inequalities appeared after the publication of these books. Several journals are devoted to inequalities, most notably "*Journal of Inequalities and Applications*" with the first volume in 1997, "*Mathematical Inequalities and Applications*" with the first volume in 1998, "*Journal of Inequalities in Pure and Applied Mathematics*" with the first volume in 2000 and "*Journal of Mathematical Inequalities*" which has been launched in 2007.

One of the most important inequalities, that has attracted many inequality experts in the last few decades, is the famous Hermite-Hadamard inequality. Although it was firstly known in the literature as a result by Jacques Hadamard (1865-1963), this result was actually due to Charles Hermite (1882-1901), as pointed out by Mitrinović and Lacković in 1985 [89]. Due to this fact, most experts refer to it as Hermite-Hadamard (or sometimes, Hadamard-Hermite) inequality.

The Hermite-Hadamard inequality plays a great role in the theory of convex functions. It provides a necessary and sufficient condition for a function to be convex in an open interval of real numbers [58]. In fact, the term 'convex' also stems from a result obtained by Hermite in 1881 and published in 1883 as a short note in *Mathesis*, a journal of elementary mathematics [42]. The Hermite-Hadamard inequality also interpolates Jensen's inequality, which is also an important inequality in the study of convex functions [42].

In their monograph [42], Dragomir and Pearce stated that the Hermite-Hadamard inequality is the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has attracted and continues to attract much interest in elementary mathematics. The Hermite-Hadamard inequality has made great contributions in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory and numerical analysis [42]. It has been developed for different classes of convexity, such as quasi-convex functions, Godunova-Levin class of functions, log-convex, r-convex functions, p-functions, etc. [42].

The Hermite-Hadamard inequality has also been developed for convex functions in linear spaces, particularly in linear spaces equipped with norms. It is well-known that every norm is a convex function on the associated linear space. The convexity of the norm enables us to apply the Hermite-Hadamard inequality. The aim of this dissertation is to study normed spaces by engaging it to the Hermite-Hadamard inequality.

The following section gives a background material of the Hermite-Hadamard inequality. We provide a historical consideration of this inequality, its extensions and its connection to the theory of means. Due to the large number of literature, some related theories are omitted.

1.2 The Hermite-Hadamard inequality

1.2.1 Historical consideration

For a convex function f, the double inequality:

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}, \tag{1.1}$$

was known in the literature as the Hadamard inequality. However, this inequality was actually suggested by Hermite. On 22 November 1881, Hermite sent a letter to the journal "*Mathesis*". An extract from that letter was then published in "*Mathesis*" **3** in 1883, page 82 (cf. Mitrinović and Lacković [89]). One of the inequalities which was mentioned by Hermite in this note is inequality (1.1). This note was nowhere mentioned in the mathematical literature and the important inequalities (of Hermite) were not widely known as Hermite's results [100].

E.F. Beckenbach, a leading expert on the history and theory of complex functions, wrote that inequality (1.1) was proven by Hadamard in 1893 and apparently was not aware of Hermite's result [9, 42]. Fejér in 1906, while studying trigonometric polynomials, obtained inequalities which generalize (1.1) but again Hermite's work was not acknowledged [100]. In 1905 (1906) Jensen defined convex functions using the first and last terms of inequality (1.1), that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

for all $a, b \in D(f)$ [100]. Inequality (1.2) is referred to as the *Jensen inequality.* It is important to note that inequality (1.1) provides a refinement to the Jensen inequality.

In 1974, D.S. Mitrinović found Hermite's note in "*Mathesis*" [89]. Due to these historical facts, inequality (1.1) is now referred to as the *Hermite-Hadamard inequality* [100].

Pečarić, Proschan and Tong [100, p. 140–141] noted that the first inequality is stronger than the second inequality in (1.1). Formally stated, the following inequality is valid for a convex function f

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.3}$$

Inequality (1.3) can be written as

$$\frac{2}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right],$$

which is

$$\begin{aligned} &\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx + \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx \\ \leq\ &\frac{1}{2}\left[f(a)+f\left(\frac{a+b}{2}\right)\right] + \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+f(b)\right]. \end{aligned}$$

This immediately follows by applying the second inequality in (1.1) twice (on the interval [a,(a+b)/2] and [(a+b)/2,b]) [100, p. 141].

1.2.2 Characterization of convexity

The importance of the Hermite-Hadamard inequality is mainly highlithed by its role in the theory of convex functions. In the famous work of Hardy, Littlewood and Pólya [58, p. 98], it is stated that a necessary and sufficient condition that a continuous function f should be convex on the interval (a, b) is that

$$f(x) \leq \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt \tag{1.4}$$

for $a \leq x - h < x < x + h \leq b$. This result is equivalent to (1.1) when f is continuous on $[a, b]$ (cf. Dragomir and Pearce [42, p. 3]). Pečarić, Proschan and Tong [100, p. 139] remarked that it remains unclear by who and when the transition from inequality (1.1) to the convexity criterion (1.4) was made.

This convexity criterion has been generalized by considering Steklov iterated operators [100, p. 139–140]. The *Steklov operator* S_h, associated to a positive number h, is defined by

$$S_h(f,x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt$$

for a continuous function f and $x \in I_1(h) := \{t : t-h, t+h \in I\}$. It is an operator mapping on $C(I)$ (that is, the set of all continuous functions on an interval I) into $C(I_1)$. For a finite interval $I = [a,b]$, the maximum value of h can be $(b-a)/2$. In this case, I_1 contains a single point; and S_h becomes a functional. Under these conditions, the Hermite-Hadamard inequality (1.1) now has the form

$$f(x) \leq S_h(f,x)$$

for $x \in I_1(h)$ and is equivalent to the convexity of the function (cf. Pečarić, Proschan and Tong [100, p. 139–140]).

The iterated Steklov operators (with step $h > 0$) S_h^n ($n \in \mathbb{N}$) are defined by

$$S_h^0(f,x) = f(x), \quad S_h^n(f,h) = \frac{1}{2h}\int_{x-h}^{x+h} S_h^{n-1}(f,x)dt,$$

where $n \in \mathbb{N}$, $x \in I_n(h) = \{t : t-nh, t+nh \in I\}$. For convenience, we write S_h instead of S_h^1; and (1.4) becomes $S_h^0(f,x) \leq S_h(f,x)$ (cf. Pečarić, Proschan and Tong [100, p. 139–140]). Kocić in his doctoral thesis [77] stated the following generalizations of the convexity criterion (of Hardy, Littlewood and Pólya):

1. A function $f \in C(I)$ is convex if and only if for every $h > 0$ and $x \in I_n(h)$ the inequality
$$f(x) \leq S_h^n(f,x)$$
holds for every fixed $n \in \mathbb{N}$.

2. A function $f \in C(I)$ is convex if and only if for every $h > 0$ and $x \in I_n(h)$ the inequality
$$S_h^{n-1}(f,x) \leq S_h^n(f,x)$$
holds for every fixed n.

The first result can also be found in Horová [59]. It is important to point out that (1.4) can be obtained by letting $h \to 0$ in the second result (cf. Pečarić, Proschan and Tong [100, p. 140]).

The second inequality in (1.1) can also be used as a convexity criterion (cf. Pečarić, Proschan and Tong [100, p. 141]). Roberts and Varberg [107, p. 15] stated that for any continuous function on $[a, b]]$, f is convex if and only if

$$\frac{1}{t-s}\int_s^t f(x)dx \leq \frac{1}{2}[f(s)+f(t)]$$

for all $a < s < t < b$.

1.2.3 Connection to special means

The theory of inequalities is closely related to the theory of means. In this subsection, we discuss the connection between the two theories, in particular, the connection between the Hermite-Hadamard inequality and some special means.

The Hermite-Hadamard inequality provides a refinement to the famous *Jensen inequality.* As stated in Subsection 1.2.1, a continuous function f is said to be convex if and only if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad \text{for all } x, y \in \mathbb{R}.$$

When the function is concave, this inequality is reversed. In particular, the logarithmic function is concave. Therefore

$$\log\left(\frac{x+y}{2}\right) \geq \log(\sqrt{xy}), \quad \text{for } x, y > 0,$$

which is the logarithmic of the well-known *arithmetic-geometric mean* inequality

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

We denote the *arithmetic mean* $\frac{x+y}{2}$ by $A(x, y)$ and the *geometric mean* $\sqrt{xy}$ by $G(x, y)$, for convenience.

These classical means have been extended to a more general form, which is called the *power mean.* The (unweighted) power mean (cf. Lin [80, p. 879–880] and Pittenger [102, p. 19–20]) of two positive numbers x and y and of order p is defined by

$$M_p = M_p(x,y) = \begin{cases} \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}}, & p \neq 0; \\ G(x,y), & p = 0, \end{cases} \tag{1.5}$$

where $G(x, y)$ is the geometric mean of x and y. It is noted that when $p = 1$, we obtain the arithmetic mean.

The power mean satisfies the following properties [113, p. 88]:

1. it is internal, that is, $\min\{x, y\} \leq M_p(x, y) \leq \max\{x, y\}$;
2. $M_p(x, y)$ is continuous in p;
3. $M_p(x, y) \leq M_q(x, y)$ if $p \leq q$;

for any positive numbers x and y. We refer the reader to the works by Bullen [14], Lin [80], Neuman [93], Pittenger [102] and Stolarsky [114] for further properties of power means.

Another type of mean, which is widely used in engineering, such as in heat transfer and fluid mechanics [80, p. 879], is the *logarithmic mean*. The logarithmic mean of two positive numbers x and y is defined by

$$L(x, y) = \begin{cases} \frac{x-y}{\log(x)-\log(y)}, & x \neq y; \\ x, & x = y, \end{cases}$$

(cf. Carlson [17, p. 615]). The logarithmic mean L is symmetric, homogeneous in x and y, and continuous at $x = y$ [17, p. 615].

The logarithmic mean is greater than the geometric mean and is less than the arithmetic mean, that is,

$$\sqrt{xy} \leq L(x, y) \leq \frac{x + y}{2}, \tag{1.6}$$

with strict inequalities if $x \neq y$ (cf. Carlson [17, p. 615]). Burk [15, p. 527] remarked that this inequality could be obtained by applying the Hermite-Hadamard inequality for the convex function $\int_{\ln x}^{\ln y} e^t dt$, where $x, y > 0$.

Stolarsky discussed the matter of understanding why $L(x, y)$ is a mean [113, p. 88]. For this purpose, Stolarsky considered the mean value theorem for differentiable functions f

$$\frac{f(x) - f(y)}{x - y} = f'(u), \quad x \neq y,$$

where u is strictly between x and y; and derived that if $f(x) = \log x$, then $u = L(x, y)$. This gives a motivation to 'create new means' by varying the function f. One of the functions that was considered by Stolarsky is

$$f(x) = x^{p+1}$$

where $p \in \mathbb{R}, p \neq -1, 0$. This gives us the *generalized logarithmic mean*, which is a special case of the *Stolarsky mean*. Formally stated, if p is a (extended) real number, the generalized logarithmic mean of order p of two positive numbers x and y is defined by

$$\mathfrak{L}^{[p]}(x,y) = \begin{cases} \left[\frac{1}{p+1}\left(\frac{y^{p+1}-x^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } p \neq -1, 0, \pm\infty; \\ \frac{y-x}{\log y - \log x}, & \text{if } p = -1; \\ \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}, & \text{if } p = 0; \\ \max\{x,y\}, & \text{if } p = +\infty; \\ \min\{x,y\}, & \text{if } p = -\infty, \end{cases} \tag{1.7}$$

and $\mathfrak{L}^{[p]}(x,x) = x$ (cf. Bullen [14, p. 385]). This mean is homogeneous and symmetric [14, p. 385]. In particular, there is no loss in generality by assuming $0 < x < y$.

The generalized logarithmic mean is closely related to the classical means. We summarized the relations as follows (cf. Bullen [14, p. 385]):

1. If $p = -1$, then $\mathfrak{L}^{[-1]}(x,y) = \frac{y-x}{\log y - \log x} = L(x,y)$, that is, the logarithmic mean of x and y.
2. If $p = 0$, then $\mathfrak{L}^{[0]}(x,y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} = I(x,y)$, that is, the *identric mean* of x and y.
3. If $p = 1$, then $\mathfrak{L}^{[1]}(x,y) = \frac{x+y}{2} = A(x,y)$, the arithmetic mean of x and y.
4. If $p = -\frac{1}{2}$, then $\mathfrak{L}^{[-\frac{1}{2}]}(x,y) = \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 = M_{\frac{1}{2}}(x,y)$, the power mean of x and y with exponent $\frac{1}{2}$.
5. If $p = -2$, then $\mathfrak{L}^{[-2]}(x,y) = \sqrt{xy} = G(x,y)$, the geometric mean of x and y.
6. If $p = 2$, then $\mathfrak{L}^{[2]}(x,y) = \sqrt{\frac{1}{3}(x^2 + xy + y^2)} = Q(x,y,G(x,y))$, the *quadratic mean* of x, y and $G(x,y)$.

The generalized logarithmic mean $\mathfrak{L}^{[p]}$ is a strictly increasing function of p on the extended real numbers [14, p. 387]. Thus, we have

$$G(x,y) < L(x,y) < I(x,y) < A(x,y), \text{ where } x \neq y. \tag{1.8}$$

In particular, the generalized logarithmic means are strictly internal [14, p. 387]. For further properties of the generalized logarithmic mean and its relationship with other means, we refer the reader to the works by Bullen [14], Burk [15], Carlson [16, 17], Lin [80], Neuman [93], Pittenger [102] and Stolarsky [113, 114].

The generalized logarithmic mean has been extended for positive n-tuples, which is known as the hypergeometric mean. Carlson [16, 32-33] (cf. Bullen [14, p. 366-367]) considered the following *hypergeometric R-function*

$$R(a,\mathbf{b},\mathbf{x}) = \int_{E_n} \left(\sum_{i=1}^{n} u_i x_i \right)^{-a} P(\mathbf{b},u')du',$$

where $\mathbf{b}$ and $\mathbf{x}$ are n-tuples of positive numbers, $a \in \mathbb{R}$, $u' = (u_1, u_2, \ldots, u_{n-1})$ and $du' = du_1 \ldots du_{n-1}$. The domain of integration is the simplex E_n, that is, the set of points satisfying $u_i > 0$ $(i = 1, \ldots, n-1)$ and $\sum_{i=1}^{n-1} u_i < 1$. We define $u_n = 1 - \sum_{i=1}^{n-1} u_i$, which are equivalent to the condition $u_n > 0$. The positive weight function

$$P(\mathbf{b},u') = \frac{\Gamma(b_1 + \cdots + b_n)}{\Gamma(b_1) \ldots \Gamma(b_n)} \prod_{i=1}^{n} u_i^{b_i - 1},$$

satisfies $\int_{E_n} P(\mathbf{b},u')du' = 1$.

A *hypergeometric mean* of $\mathbf{x} = (x_1, \ldots, x_n)$ with weight $\mathbf{w} = (w_1, \ldots, w_n)$, is then constructed as follows

$$\mathfrak{C}(p,c;\mathbf{x},\mathbf{w}) = [R(-p,c\mathbf{w},\mathbf{x})]^{\frac{1}{p}}$$

where $p \neq 0$, $c = \sum_{i=1}^{n} b_i$ and $w_i = \frac{b_i}{c}$. If $p = 0$ or $c = 0$, the mean value is defined by the limiting value of $\mathfrak{C}(p,c;\mathbf{x},\mathbf{w})$ as $p \to 0$ or $c \to 0$. Note that

$$\lim_{c \to 0} \mathfrak{C}(p,c;\mathbf{x},\mathbf{w}) = \mathfrak{M}_p^{[n]}(\mathbf{x},\mathbf{w}),$$

where $\mathfrak{M}_p^{[n]}$ is the *weighted power mean* of order t (cf. Carlson [16, p. 33]):

$$\mathfrak{M}_p^{[n]}(\mathbf{x},\mathbf{w}) = \begin{cases} \left(\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}}, & p \neq 0; \\ \left(\displaystyle\prod_{i=1}^{n} x_i^{w_i}\right)^{\frac{1}{W_n}}, & p = 0, \end{cases}$$

where $W_n = w_1 + \cdots + w_n$.

The logarithmic mean is a special case of the hypergeometric mean as shown in the following:

$$L(x,y) = R(1;1,1;x,y)^{-1} = \frac{x-y}{\log x - \log y}.$$

The generalized logarithmic mean is also a special case of the hypergeometric mean, by the following:

$$R(-p;1,1;x,y)^{1/p} = \left(\frac{x^{p+1}-y^{p+1}}{(p+1)(x-y)}\right)^{1/p}$$

for $p \neq -1$. It is noted that when $p \geq 1$ then

$$\left(\frac{x+y}{2}\right)^p \leq \frac{x^{p+1}-y^{p+1}}{(p+1)(x-y)} \leq \frac{x^p+y^p}{2}, \tag{1.9}$$

or, equivalently,

$$A(x,y) \leq \mathfrak{L}^{[p]}(x,y) \leq M_p(x,y).$$

The Hermite-Hadamard inequality (1.1) can be employed to prove (1.9), by considering the convex mapping $t \mapsto t^p$ on the interval $[x,y] \subset \mathbb{R}^+$ for the values of $p \geq 1$.

In this dissertation, we employ the Hermite-Hadamard inequality to study Banach spaces. The connection of this inequality to the theory of means is considered in Chapter 3. In particular, we consider the extension of special means from means of positive real numbers to means of vectors in normed spaces.

1.2.4 Some generalizations of the Hermite-Hadamard inequality

Many mathematicians have committed their thoughts and efforts to generalize and extend the Hermite-Hadamard inequality for different classes of functions. In this subsection, we recall some of the generalizations. For further results on the generalizations of the Hermite-Hadamard inequality, we refer to the monograph by Dragomir and Pearce [42].

In 1976, Vasić and Lacković [117], and Lupaş [83] (cf. Pečarić, Proschan and Tong [100]) obtained a generalization by considering a more general form of the upper and lower bounds of (1.1). Formally stated, for any continuous convex function f on the interval $[a_1, b_1]$, the inequalities

$$f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{2y}\int_{A-y}^{A+y} f(t)dt \leq \frac{pf(a)+qf(b)}{p+q}$$

hold for $A = (pa+qb)/(p+q)$ and $y > 0$, if and only if

$$y \leq \frac{b-a}{p+q}\min\{p, q\}$$

where p and q are positive numbers and $a_1 \leq a < b \leq b_1$.

In 1986, Pečarić and Beesack [98] generalized the result of by Vasić and Lacković. Before stating the result, we need to assume the following:

1. Let f be a continuous convex function on an interval $I \supset [m, M]$, where $-\infty < m < M < \infty$;
2. Suppose that $g : E \to \mathbb{R}$ satisfies $m \leq g(t) \leq M$ for all $t \in E$, $g \in L$, and $f(g) \in L$;
3. Let $A : L \to \mathbb{R}$ be an isotonic linear functional with $A(1) = 1$ and let $p = p_g$, $q = q_g$ be nonnegative real numbers (with $p + q > 0$) for which
$$A(g) = \frac{pm+qM}{p+q}.$$

Then, the following inequalities hold:

$$f\left(\frac{pm+qM}{p+q}\right) \leq A(f(g)) \leq \frac{pf(m)+qf(M)}{p+q}.$$

Wang and Wang [118] in 1982 considered the following generalization. For any convex function f on $[a,b]$, the following inequalities are valid:

$$\begin{aligned} f\left(\frac{\sum_{i=0}^{n} p_i x_i}{\sum_{i=0}^{n} p_i}\right) &\leq \prod_{j=1}^{n}(\beta_j-\alpha_j)^{-1}\int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_n}^{\beta_n} f\big(x_0(1-t_1) \\ &\quad + \sum_{j=1}^{n-1} x_j(1-t_{j+1})t_1\dots t_j + x_n t_1 t_2 \dots t_n\big)\prod_{i=1}^{n} dt_i \\ &\leq \frac{\sum_{i=0}^{n} p_i f(x_i)}{\sum_{i=0}^{n} p_i}, \end{aligned}$$

where $x_i \in [a,b]$, $p_i > 0$ for $i = 0, \dots, n$,

$$\frac{\alpha_i+\beta_i}{2} = \frac{\sum_{k=1}^{n} p_k}{\sum_{k=i-1}^{n} p_k} \quad \text{for } i = 1, \dots, n$$

and

$$0 \leq \alpha_i < \beta_i \leq 1 \quad \text{for } i = 1, \dots, n.$$

Another generalization of the Hermite-Hadamard inequality is pointed out by Neuman [92] in 1986 (cf. Pečarić, Proschan, Tong [100]). Let $t_0, \dots, t_n \geq 0$ and

$$m_r = m_r(t_0, \dots, t_n) = \frac{1}{\binom{n+k}{k}} \sum_{\substack{i_0,\dots,i_k \in \{0,\dots,n\} \\ i_0+\cdots+i_k=n}} t_0^{i_0} \dots t_k^{i_k}.$$

Suppose that $x(t) = \sum_{r=u}^{v} a_r t^r$, for $0 \leq u \leq v$ and $a_r \in \mathbb{R}$, is an algebraic polynomial of degree not exceeding v, $a = \min\{x(t) : c \leq t \leq d\}$ and $b = \max\{x(t) : c \leq t \leq d\}$. Let f be a convex function on (a,b). Then,

$$f\left(\sum_{r=u}^{v} a_r m_r\right) \leq \int_c^d M_n(t) f\left(\sum_{r=u}^{v} a_r t^r\right) dt,$$

where M_n is a B-spline of order n. We refer to Definition 4.12 of Schumaker [110] for the definition of B-spline.

The following results are generalizations of the Hermite-Hadamard inequality for multivariate convex functions.

1. For any convex function f on $\mathbb{R}^n$, we have

$$f(m_{\ell^1}, \ldots, m_{\ell^n}) \leq \int_{\mathbb{R}^n} f(x_1^{\ell^1}, \ldots, x_n^{\ell^n}) M(\mathbf{x}|\mathbf{x}_0, \ldots, \mathbf{x}_n) d\mathbf{x},$$

   where $\ell_i = 1, 2, \ldots,$ $i = 1, 2, \ldots, n$, and $\mathrm{vol}([\mathbf{x}_0, \ldots, \mathbf{x}_n]) > 0$, $\mathbf{x}_i \in \mathbb{R}^n$, $i = 0, \ldots, n$ (cf. Neuman and Pečarić [94]).

2. For any f be a convex function on $\mathbb{R}^n$, we have

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n} \mathbf{x}_i\right) \leq \int_{\mathbb{R}^n} f(\mathbf{x}) M(\mathbf{x}|\mathbf{x}_0, \ldots, \mathbf{x}_n) d\mathbf{x} \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\mathbf{x}_i),$$

   and equality holds if and only if $f \in \prod_1(\mathbb{R}^n)$, where $\mathrm{vol}([\mathbf{x}_0, \ldots, \mathbf{x}_n]) > 0$, $\mathbf{x}_i \in \mathbb{R}^n$, $i = 0, \ldots, n$, and $\prod_1(\mathbb{R}^n)$ is the set of all polynomials with degree of, at most 1 (cf. Pečarić, Proschan and Tong [100]).

As a special case, we obtain the following theorem (cf. Dragomir and Pearce [42]):

Theorem 1.2.1 (Neuman and Pečarić [94]). *Let $\sigma = [\mathbf{x}_0, \ldots, \mathbf{x}_n]$, where $n \geq 1$ and $\mathrm{vol}_n(\sigma) > 0$. If $f : \sigma \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n} \mathbf{x}_i\right) \leq \frac{1}{\mathrm{vol}_n(\sigma)}\int_{\sigma} f(\mathbf{x}) d\mathbf{x} \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\mathbf{x}_i),$$

and equalities hold if and only if $f \in \prod_1(\mathbb{R}^n)$.

Pečarić and Dragomir [99] in 1991 consider the extension of the Hermite-Hadamard inequality for isotonic linear functionals. By isotonic linear functional, we refer to the functionals $A : L \to \mathbb{R}$ which satisfy the following properties:

1. $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;

2. $f \in L$, $f \geq 0$ on E implies that $A(f) \geq 0$;

where L is a linear class of real valued functions g on a non-empty set E having the properties:

1. $f, g \in L$, $(af + bg) \in L$ for all $a, b \in \mathbb{R}$;

2. $\mathbf{1} \in L$, that is, if $f(t) = 1$ $(t \in E)$, then $f \in L$.

These isotonic linear functionals are commonly called the positive functionals.

Pečarić and Dragomir [99] considered the following extension of the Hermite-Hadamard inequality. Firstly, for any convex subset C of a linear space L, let $g_{x,y}$ be a real-valued mapping on $[0,1]$, associated to $x, y \in C$, defined by

$$g_{x,y}(t) := f(tx + (1-t)y),$$

which is a convex function on $[0,1]$. For any (real) convex function f on C, a linear class L on E, $h : E \to \mathbb{R}$, $0 \leq h(t) \leq 1$, $h \in L$, such that $g_{x,y} \circ h \in L$ for $x, y \in C$, and an isotonic functional A, with $A(\mathbf{1}) = 1$, we have

$$\begin{aligned} f(A(h)x + (1 - A(h))y) &\leq A[f(hx + (\mathbf{1} - h)y] \\ &\leq A(h)f(x) + (1 - A(h))f(y). \end{aligned}$$

It also remarked by Pečarić and Dragomir [99] that if $h : E \to [0,1]$ is such that $A(h) = \frac{1}{2}$, we have

$$f\left(\frac{x+y}{2}\right) \leq A[f(hx + (\mathbf{1} - h)y] \leq \frac{f(x) + f(y)}{2}.$$

As a consequence, we have the following inequality for any two vectors x and y in a normed space $(\mathbf{X}, \|\cdot\|)$, and $1 \leq p < \infty$

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2} \tag{1.10}$$

(note that $\int_0^1 \|(1-t)x + ty\|^p dt = \int_0^1 \|tx + (1-t)y\|^p dt$). Inequality (1.10) is the main focus of this dissertation. The integral mean $\int_0^1 \|(1-t)x + ty\|^p dt$ is utilized in the study of a new type of norm on the Cartesian square $\mathbf{X}^2$ of a normed space $(\mathbf{X}, \|\cdot\|)$.

1.3 Motivation

The Hermite-Hadamard inequality has been extended by considering the isotonic linear functional. As one of its applications, a Hermite-Hadamard type inequality in normed spaces was established by Pečarić and Dragomir [99, p. 106]. This result, however, follows by the fact that every norm (and the pth power of a norm) is a convex function on the associated normed space.

The study of normed spaces is one of the main focuses of functional analysis, in particular, when the norm induces a complete metric. On the suggestion of Fréchet, a complete normed space is referred to as Banach space [101, p. 2], as a tribute to Stefan Banach (1892-1945). The concept of Banach space appeared for the very first time in Banach's doctoral thesis "*Sur les opérations dans les ensembles abstraits, et leur application aux équations intégrales*" (On operations on abstract sets and their application to integral equations) [60]. A distinctive feature of Banach's concept was that the space in question was required to satisfy the crucial extra condition of completeness [60].

Banach submitted his doctoral thesis in 1920. Long before the concept of Banach space was introduced, all classical Banach spaces had been discovered. In 1903, Hadamard [56] considered the collection of all continuous real functions on a closed interval $[a, b]$, which is the simplest and most important Banach space, widely known as $C[a, b]$ (cf. Pietsch [101, p. 2]). The space of all square-summable sequences ℓ^2 was used by Hilbert in 1906, particularly its closed unit ball, as a domain of linear, bilinear and quadratic forms [101, p. 9]. Riesz referred to this space as '*l'espace hilbertien*' [101, p. 10]. In 1907, Fischer [53] and Riesz [104] invented the Banach space $L^2[a, b]$, which was more elegant than $C[a, b]$, since the norm in L^2 is induced by inner product space, giving a Hilbert space structure (cf. Pietsch [101, p. 3]). The completeness of L^2 is contained in the famous Fischer-Riesz theorem (cf. Pietsch [101, p. 10]). Subsequently, Riesz [105] extended this definition to exponents $1 < p < \infty$ [101, p. 3]. Pietsch noted that the concept of a norm was not yet in use, though Riesz was able to prove the Minkowski inequality [101, p. 3]. Interestingly, the simpler theory of ℓ^p formed by all p-summable scalar sequences, was treated only in 1913 [101, p. 4] (cf. Riesz [106]). Although, the concept of a complete normed space was ripe for discovery by 1913, it was Banach who then took a dominant role in the process of laying the foundation of this theory [101, p. 24].

Today, Banach space theory is one of the most powerful tools not only in functional analysis, but also in other areas of analysis, namely, harmonic analysis, functions of a complex variable, approximation theory, etc. [119]. Several books and monographs are devoted to Banach space theory, including those of Johnson and Lindenstrauss [64, 65], Lindenstrauss and Tzafriri [81], Pietsch [101] and Wojtaszczyk [119].

The purpose of this dissertation is to utilize the Hermite-Hadamard inequality in studying Banach spaces. Our main focus is the Hermite-Hadamard type inequality defined on the segment generated by two vectors, namely x and y, in a normed space

$(\mathbf{X}, \|\cdot\|)$:

$$\left\|\frac{x+y}{2}\right\| \leq \left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}} \leq \frac{1}{2^{\frac{1}{p}}} \left(\|x\|^p + \|y\|^p\right)^{\frac{1}{p}}, \tag{1.11}$$

for any $1 \leq p < \infty$ [99]. Note that $(\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ is a norm on the Cartesian square $\mathbf{X}^2 = \mathbf{X} \times \mathbf{X}$, for the pair (x, y) in $\mathbf{X}^2$. This norm is known as the p-norm, which is a vector-valued analogue of the ℓ^p-norms. Motivated by the above inequality, we investigate that the mapping

$$(x, y) \mapsto \left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}} \in \mathbb{R}$$

is a norm on $\mathbf{X}^2$. It is a complete norm, provided that the underlying space is a Banach space. Another fact which interests us to study this norm is that when the underlying normed space is the field of real numbers, the above mapping is the generalized logarithmic mean. Hence, this norm gives an extension from means of positive real numbers to means of vectors in normed spaces.

In contrast to the p-norm, this 'Hermite-Hadamard type' norm depends on the relative positions of the original vectors, not just the size of the vectors. As a consequence, the sensitivity of these norms to the geometry of the underlying space is markedly different than the p-norms. Our goal is to study the properties of these norms, their applications and extension to the Cartesian power spaces and sequence spaces.

1.4 Outline of the thesis

This dissertation is devoted to the study of Banach spaces by engaging it to the theory of inequalities. Our work begins with the study of the Cartesian square $\mathbf{X}^2$ of a normed space $(\mathbf{X}, \|\cdot\|)$, equipped with a *Hermite-Hadamard type norm* (cf. Section 1.1)

$$\|(x, y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,$$

for all $(x, y) \in \mathbf{X}^2$. It is well-known that the Cartesian square $\mathbf{X}^2$ is also a normed space, when equipped with any of the following norms

$$\|(x,y)\|_p = \begin{cases} (\|x\|^p + \|y\|^p)^{1/p}, & 1 \leq p < \infty; \\ \max\{\|x\|, \|y\|\}, & p = \infty, \end{cases}$$

that is, the so-called *p-norms*. This is a special case of the vector-valued analogue of the classical ℓ^p spaces, which is usually denoted by $\ell^p(\mathbf{X})$, where $\mathbf{X}$ is the underlying vector space. Some results concerning this space (cf. Leonard [78]) take the following general form: the space $\ell^p(\mathbf{X})$ (hence, $(\mathbf{X}^2, \|\cdot\|_p)$) has certain property if and only if $\mathbf{X}$ does. The properties that have been investigated mostly deal with metrical and geometrical properties, such as completeness and reflexivity; and also, smoothness and convexity of the unit ball.

Chapter 2 is written as a reference point for the later chapters. Some fundamental theories regarding Banach spaces are provided. Concerning the p-norms, a section is devoted to discuss the Banach sequence space $\ell^p(\mathbf{X})$ as a particular example of Banach space. The vector-valued analogue of the Lebesgue function spaces L^p, i.e. the Bochner spaces $L^p(\Omega, \mathbf{X})$ of functions defined on a normed space $\mathbf{X}$ and a finite measure space Ω, are also discussed as another example of a Banach space. The results concerning both $\ell^p(\mathbf{X})$ and $L^p(\Omega, \mathbf{X})$ serve as tools for the subsequent chapters.

The new norms (on the Cartesian square), which are called the p-HH-norms, are introduced and discussed in Chapter 3. These norms are equivalent to the p-norms. As stated in Section 1.1, when the underlying normed space is the field of real numbers, the above mapping is the generalized logarithmic mean. This norm then becomes an extension of the generalized logarithmic mean, which is not just restricted for positive real numbers, but generally in the setting of normed linear spaces. These norms are shown to be Banach norms, provided that the underlying space is a complete normed space. They also preserve reflexivity and smoothness of the underlying space in the Cartesian square space. The p-HH-norms, for $1 < p < \infty$, preserve the strict convexity and uniform convexity. However, the 1-HH-norm is neither strictly nor uniformly convex. It is important to note that the 1-HH-norm preserves smoothness, in contrast to the 1-norm.

Although they are equivalent, the p-HH-norms are essentially different to the p-norms. Some quantitative comparisons between the p-norm and the p-HH-norm, for a fixed $1 \leq p < \infty$ are given in Chapter 4. These comparisons are established via an Ostrowski type inequality. Two types of Ostrowski inequality are introduced for different classes of functions on linear spaces, namely absolutely continuous functions and convex functions. The results follow by the convexity (which implies the absolute continuity) property of the norm and the p-th power of the norm. The chapter is concluded by a comparison analysis between these results. Although the results obtained for absolutely continuous functions are more general than those for convex functions, they are proven to be coarser, in some particular cases. It is conjectured that this statement holds for any case.

More norm inequalities are discussed in Chapter 5 to give quantitative comparison amongst the p-HH-norms for different values of p. In order to establish such comparison, a particular type of Čebyšev difference is utilized. Some results are obtained as consequences of several classical results regarding upper bounds for a Čebyšev difference by Čebyšev, Grüss, Ostrowski and Lupaş. Some new bounds are introduced in a general setting; and they are proven to be sharp. Despite the sharpness, these bounds are complicated to compute. This chapter is concluded by suggesting simpler, but coarser, upper bounds. The sharpness of these bounds are yet to be addressed.

Chapter 6 is devoted to the study of a particular geometrical property, namely the orthogonality. It is well-known that in an inner product space, two vectors are orthogonal if and only if their inner product is zero. In a normed space, the notions of orthogonality are treated in a different manner. Some equivalent propositions to the usual orthogonality (that is, orthogonality in inner product space) have been adapted to define orthogonality in a normed space. In this chapter, the 2-HH-norm is utilized in introducing some new notions of orthogonality in normed spaces. The first part of Chapter 6 covers some classical notions of orthogonality in normed spaces, which also serves as reference for the later sections. The new notions of orthogonality are shown to have a close connection to the classical ones, namely the Pythagorean, Isosceles and Carlsson's orthogonalities. The main achievements in this chapter are some characterizations of inner product spaces via these orthogonalities. It is shown that the homogeneity, as well as the additivity, of these orthogonalities is a necessary and sufficient condition for the space to be an inner product space.

In Chapter 7, the definition of the p-HH-norms is extended to the nth Cartesian power of a normed space, for $n > 2$. These norms are related to the hypergeometric means but are not restricted to the positive real numbers. As in the case of $n = 2$, the reflexivity, convexity and smoothness of the norms are shown to be closely related to the corresponding property of the underlying space, in the same manner. Using a limit of isometric embeddings, the norms are extended to spaces of bounded sequences that include all summable sequences. Examples are given to show that the new sequence spaces have very different properties than the usual spaces of p-summable sequences.

Finally, we summarize the work of this dissertation in Chapter 8. We also recall the main achievements of this dissertation, explain some open problems that are yet to be addressed and the future research to be undertaken.

Chapter 2

Banach space theory

The main purpose of this chapter is to provide some fundamental theories of Banach spaces. In particular, some results regarding Banach sequence spaces and Bochner function spaces are provided as tools for later chapters.

2.1 Banach spaces

A (real) *normed space* $\mathbf{X}$ is a vector space, equipped with a real-valued mapping $\|\cdot\|$ defined on $\mathbf{X}$ which is called a norm, and satisfies the following properties:

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ *(positive definiteness)*;
2. $\|\alpha x\| = |\alpha|\|x\|$ *(positive homogeneity)*;
3. $\|x + y\| \leq \|x\| + \|y\|$ (*triangle inequality*);

for all $x, y \in \mathbf{X}$ and $\alpha \in \mathbb{R}$. Any normed space $(\mathbf{X}, \|\cdot\|)$ can be equipped with a metric which is induced by the norm of $\mathbf{X}$, that is, a metric d on $\mathbf{X}$ defined by

$$d(x, y) = \|x - y\|, \quad x, y \in \mathbf{X}.$$

A metric d on a metric space $\mathbf{X}$ is said to be *complete* if every *Cauchy sequence* (of points in $\mathbf{X}$) *converges* in $\mathbf{X}$. Intuitively, every convergent sequence in a complete metric space has its limit within the space. A Banach norm, or complete norm, is a norm that induces a complete metric. A normed space is a *Banach space*, or complete normed space, if its norm is a Banach norm.

The absolute value $|\cdot|$ is a Banach norm on $\mathbb{R}$. The space $\mathbb{R}^n$ of all n-tuples of real numbers is a Banach space with the Euclidean norm

$$\|(x_1,\ldots,x_n)\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}.$$

We consider more examples of Banach spaces in the following.

Example 2.1.1. Suppose that p is a real number satisfying $1 \le p < \infty$. The space ℓ^p is the set of all infinite sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{j=1}^{\infty} |x_j|^p < \infty$. These spaces are Banach spaces with the norm

$$\|x\|_{\ell^p} = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}}.$$

For $p = \infty$, we consider the Banach space ℓ^∞ as the space of all bounded sequences, with the norm

$$\|x\|_{\ell^\infty} = \max\{|x_1|, |x_2|, \ldots\}.$$

Example 2.1.2 (Megginson [85])**.** Suppose $1 \le p \le \infty$ and $n \in \mathbb{N}$. For any $(x_1,\ldots,x_n) \in \mathbb{R}^n$, the mapping $\|\cdot\|_{\ell^p} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|(x_1,\ldots,x_n)\|_{\ell^p} = \begin{cases} (|x_1|^p + \cdots + |x_n|^p)^{1/p}, & 1 \le p < \infty; \\ \max\{|x_1|,\ldots,|x_n|\}, & p = \infty \end{cases}$$

is a norm. The space $\mathbb{R}^n$ is a Banach space with the norm $\|\cdot\|_{\ell^p}$; and it is denoted by ℓ^p_n.

From the last two examples, it is important to note that we may equip a vector space with more than one norm. When two norms in a vector space induce the same topology, they are said to be *equivalent*. It is also important to note if $\|\cdot\|$ and $|||\cdot|||$ are two norms on a vector space $\mathbf{X}$, then they are equivalent if and only if there exist positive constants c_1 and c_2 such that

$$c_1|||x||| \le \|x\| \le c_2|||x|||,$$

for all $x \in \mathbf{X}$. We remark that the norms $\|\cdot\|_{\ell^p}$ are all equivalent on $\mathbb{R}^n$; in particular they are equivalent to the Euclidean norm.

The Euclidean norm is very important due to it is induced by an inner product. A (real) *inner product space* $\mathbf{X}$ is a vector space (over the field $\mathbb{K}$) equipped with a mapping $\langle\cdot,\cdot\rangle : \mathbf{X}\times\mathbf{X} \to \mathbb{K}$ which is called an inner product and satisfies the following properties:

1. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;
2. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

for all $x, y \in \mathbf{X}$ and $\alpha \in \mathbb{X}$. Every inner product induces a norm, by the following identity

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}, \quad \text{for all } x \in \mathbf{X}.$$

Jordan and Von Neumann [66] proved that in a normed space $(\mathbf{X}, \|\cdot\|)$, the norm $\|\cdot\|$ is induced by an inner product, if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \tag{2.1}$$

for all $x, y \in \mathbf{X}$. Equality (2.1) is referred to as the *parallelogram law*, or *Jordan-Von Neumann condition* (cf. Carlsson [19, p. 316]).

Every norm satisfying the parallelogram law is induced by the inner product

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2), \quad \text{for all } x, y \in \mathbf{X}.$$

Therefore, an inner product space is a normed space, but not conversely. For example, the space ℓ^p for $p \neq 2$ (cf. Example 2.1.1) is a normed space, but not an inner product space.

A Hilbert norm is a Banach norm that is induced by an inner product. An inner product space is a *Hilbert space* or complete inner product space if its norm is a Hilbert norm. An important example of a Hilbert space is the space of square-summable sequences ℓ^2 (cf. Example 2.1.1).

In the theory of operators in a Hilbert space, the space acts more as an inner-product space, rather than a particular Banach space [82]. In this sense, we cannot apply the theories that we have in Hilbert spaces to those in Banach spaces. Therefore, we need an inner-product type construction in order to carry the theory of Hilbert spaces into the theory of Banach spaces [82].

With the motivation above, Lumer [82] in 1961 introduced a type of generalized inner product on a vector space. This form has more general axioms than those of a Hilbert space [54], and is called the *semi-inner product*. In contrast to the inner product, the semi-inner product is linear in one component only, is strictly positive and satisfies a Cauchy-Schwarz type inequality [82]. In 1967, Giles [54] added another homogeneity property to the concept which had been stated by Lumer. Formally, it is defined as follows:

Definition 2.1.3. Let $\mathbf{X}$ be a normed space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). The mapping $[\cdot,\cdot] : \mathbf{X} \times \mathbf{X} \in \mathbb{K}$ is called the semi-inner product in the sense of Lumer-Giles, if the following properties are satisfied:

1. $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in \mathbf{X}$;
2. $[\lambda x, y] = \lambda[x, y]$ for all $x, y \in \mathbf{X}$ and λ a scalar in $\mathbb{K}$;
3. $[x, x] \geq 0$ for all $x \in \mathbf{X}$ and $[x, x] = 0$ implies that $x = 0$
4. $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in \mathbf{X}$;
5. $[x, \lambda y] = \overline{\lambda}[x, y]$ for all $x, y \in X$ and λ a scalar in $\mathbb{K}$ and $\overline{\lambda}$ is the conjugate of λ.

A vector space equipped with a semi-inner product is called a *semi-inner product space*. According to Lumer [54], the importance of this concept is that every normed space can be represented as a semi-inner product space, so that the theory of operators on Banach spaces may be penetrated by Hilbert spaces type arguments. As it has more general axioms, obviously there are some limitations on the theory of semi-inner product spaces in comparison to that of Hilbert spaces [54].

Dragomir mentioned some other types of semi-inner product which were considered by other mathematicians such as Miličić, Tapia, Pavel and Dincă [37]. In a normed linear space $(\mathbf{X}, \|\cdot\|)$, the mapping $f : \mathbf{X} \to \mathbb{R}$ defined by $f(x) = \frac{1}{2}\|x\|^2$ is convex and the following limits exist

$$\langle x, y\rangle_i = \lim_{t\to 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t} \text{ and } \langle x, y\rangle_s = \lim_{t\to 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t},$$

for any $x, y \in \mathbf{X}$ [37, 115]. The mappings $\langle\cdot,\cdot\rangle_s$ and $\langle\cdot,\cdot\rangle_i$ are called the *superior semi-inner product* and *inferior semi-inner product*, respectively, associated with the norm

$\|\cdot\|$. The properties of the superior and inferior semi-inner products can be summarized as follows:

1. $\langle x, x\rangle_p = \|x\|^2$, for all $x \in \mathbf{X}$;
2. $\langle ix, x\rangle_p = \langle x, ix\rangle_p = 0$, for all $x \in \mathbf{X}$;
3. $\langle \lambda x, y\rangle_p = \lambda\langle x, y\rangle_p$, for all nonnegative scalar λ and $x, y \in \mathbf{X}$;
4. $\langle x, \lambda y\rangle_p = \lambda\langle x, y\rangle_p$, for all nonnegative scalar λ and $x, y \in \mathbf{X}$;
5. $\langle \lambda x, y\rangle_p = \lambda\langle x, y\rangle_q$, for all negative scalar λ and $x, y \in \mathbf{X}$;
6. $\langle x, \lambda y\rangle_p = \lambda\langle x, y\rangle_q$, for all negative scalar λ and $x, y \in \mathbf{X}$;
7. $\langle ix, y\rangle_p = -\langle x, iy\rangle_p = 0$, for all $x \in \mathbf{X}$;

where $p, q \in \{s, i\}$ and $p \neq q$. For further properties of the superior and inferior semi-inner products, we refer to the book by Dragomir [37].

Semi-inner products have been applied in characterizing different classes of normed spaces, approximating continuous linear functionals, as well as extending the notion of orthogonality in general normed spaces. For further reading on the study of semi-inner products, we refer to the book "*Semi-Inner Products and Applications*" by Dragomir [37].

2.1.1 Dual space and reflexivity

A mapping f of an element in a normed space $\mathbf{X}$ to an element in its scalar field is called a *functional*. An obvious example of a functional is the given norm itself. A functional is called *linear* when it satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),$$

for all scalar α and β; and $x, y \in \mathbf{X}$. A functional f satisfying

$$|f(x)| \leq M\|x\|$$

for some scalar M, is called a *bounded* functional. It is well-known that the boundedness of a functional is equivalent to its continuity (cf. Theorem 1.4.2. of Megginson [85, p. 28]).

Definition 2.1.4. Let $\mathbf{X}$ be a normed space. The *dual space* of $\mathbf{X}$ is the space of all bounded linear functionals on $\mathbf{X}$ with the operator norm:

$$\|f\| = \sup_{x \in \mathbf{X} \setminus \{0\}} \frac{|f(x)|}{\|x\|}.$$

The dual space is commonly denoted by $\mathbf{X}^*$. The dual space of $\mathbf{X}^*$ is called the *bidual* of $\mathbf{X}$ and is denoted by $\mathbf{X}^{**}$. The dual space is always complete [85, p. 99].

Definition 2.1.5. A normed space is *reflexive* whenever it is isomorphic to its bidual.

It implies that any reflexive normed space is always complete. Thus, the completeness is a necessary condition for a normed space to be reflexive (cf. Theorem 1.11.7 of Megginson [85, p. 99]). The incomplete reflexive normed space is defined by the reflexivity of its completion [85, p. 99]. We also note that a Banach space $\mathbf{X}$ is reflexive if and only if the dual space $\mathbf{X}^*$ is reflexive [85, p. 104]. Every closed subspace of a reflexive normed space is reflexive [85, p. 104].

A normed space that is isomorphic to a reflexive space is itself reflexive (cf. Proposition 1.11.8 of Megginson [85, p. 99]). Moreover, a Banach space is reflexive if it is an image of a reflexive space under a bounded linear operator, regardless of whether it is an isomorphism or not [85, p. 105]. The following proposition is a direct consequence of this fact.

Proposition 2.1.6. *Let $(\mathbf{X}, \|\cdot\|)$ be a reflexive Banach space. If there exists a norm $|||\cdot|||$ on $\mathbf{X}$ which is equivalent to $\|\cdot\|$, then $(\mathbf{X}, |||\cdot|||)$ is also reflexive.*

Proof. Since $\|\cdot\|$ and $|||\cdot|||$ are equivalent, the identity operator, considered as a linear operator from $(\mathbf{X}, \|\cdot\|)$ onto $(\mathbf{X}, |||\cdot|||)$, is bounded. Therefore $(\mathbf{X}, |||\cdot|||)$ is reflexive, since $(\mathbf{X}, \|\cdot\|)$ is reflexive. $\square$

2.1.2 Geometrical properties of Banach spaces

The study of the geometrical properties of a normed space deals with the behaviour of its unit circle. One may start by visualizing the unit circle of 2-dimensional Euclidean space. Some unit circles may not be 'nicely shaped'. A familiar example is the unit circle of the 2-dimensional Euclidean space equipped with the maximum norm, which is the unit square. This unit circle is not 'round'; and it has sharp corners. Intuitively speaking, a normed space is smooth when it has no sharp corners. The notion of strict (also, uniform) convexity deals with the 'roundness' of the unit circle, in the sense that the unit circle contains no nontrivial line segments [85, p. 426].

Smoothness

As mentioned earlier, some unit circles are not 'smooth'. The notion of the smoothness of a normed space deals with the smoothness of its unit circle. It is well-known that the smoothness of a real-valued function has a close connection to its differentiability. Analogously, the smoothness of the unit circle of a normed space has a close connection to the Gâteaux differentiability of the norm [85, p. 483].

In any normed space $(\mathbf{X}, \|\cdot\|)$, the following limits

$$\begin{aligned} (\nabla_+\|\cdot\|(x))(y) &:= \lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t}, \\ \text{and} \quad (\nabla_-\|\cdot\|(x))(y) &:= \lim_{t\to 0^-} \frac{\|x+ty\|-\|x\|}{t} \end{aligned}$$

exist for all $y \in \mathbf{X}$ [85, p. 483–485] and are called the *Gâteaux lateral derivatives* of the norm $\|\cdot\|$ at a point $x \in \mathbf{X}\setminus\{0\}$. The norm $\|\cdot\|$ is *Gâteaux differentiable* at $x \in \mathbf{X}\setminus\{0\}$ if and only if

$$(\nabla_+\|\cdot\|(x))(y) = (\nabla_-\|\cdot\|(x))(y), \quad \text{for all } y \in \mathbf{X}.$$

The Gâteaux derivative of $\|\cdot\|$ at x in y direction is denoted by $(\nabla\|\cdot\|(x))(y)$.

Definition 2.1.7. A normed linear space $(\mathbf{X}, \|\cdot\|)$ is said to be *smooth* if and only if the norm $\|\cdot\|$ is Gâteaux differentiable on $\mathbf{X}\setminus\{0\}$.

The following identity gives a relationship between the semi-inner products and the Gâteaux lateral (one-sided) derivatives of the given norm [37, p. 43]:

$$\langle x, y\rangle_{s(i)} = \|y\|(\nabla_{+(-)}\|\cdot\|(y))(x), \quad \text{for all } x, y \in \mathbf{X}, \text{ where } y \neq 0. \tag{2.2}$$

Note that the following holds for any $x, y \in \mathbf{X}$:

$$\langle x, y\rangle_i \leq \langle x, y\rangle_s. \tag{2.3}$$

The following result provides a necessary and sufficient condition for a normed space to be smooth (cf. Dragomir [33, 34], Dragomir and Koliha [41]).

Proposition 2.1.8. *Equality holds in (2.3) if and only if* $\mathbf{X}$ *is smooth.*

The norm $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ is said to be *Fréchet differentiable* at $x \in \mathbf{X}$ if and only if there exists a continuous linear functional φ'_x on $\mathbf{X}$ such that

$$\lim_{\|h\|\to 0} \frac{|\|x+h\| - \|x\| - \varphi'_x(h)|}{\|h\|} = 0.$$

When this property holds for any $x \in \mathbf{X}$, then the normed space is said to be *Fréchet smooth* [85, p. 504].

We remark that every subspace of a (Fréchet) smooth normed space is itself a (Fréchet) smooth space [85, p. 488]. Note that Fréchet differentiability implies Gâteaux differentiability [85, p. 504], but not conversely. As an example (this example is due to Sova [112]), the mapping $f : L^1[0,\pi] \to \mathbb{R}$ defined by $f(x) = \int_0^\pi \sin x(t)dt$ is everywhere Gâteaux differentiable, but nowhere Fréchet differentiable.

Convexity

The strict convexity (or rotundity) can be intuitively described as the condition where any nontrivial straight line segment, whose endpoints lie in the unit sphere, has its midpoint in the interior of the closed unit ball [85, p. 441]. The notion of uniform convexity deals with the question of how far the midpoint (of such a segment) is into the interior of the closed unit ball [85, p. 441-442]. The formal definitions can be stated as follows:

Definition 2.1.9. Let $S_{\mathbf{X}}$ be the unit circle in $\mathbf{X}$, that is, $S_{\mathbf{X}} := \{x \in \mathbf{X} : \|x\| = 1\}$. Then,

1. The space $\mathbf{X}$ is *strictly convex* if for every $x, y \in S_{\mathbf{X}}$ with $x \neq y$, we have

$$\|\lambda x + (1-\lambda)y\| < 1, \quad \text{for all} \quad \lambda \in (0,1);$$

2. The space $\mathbf{X}$ is *uniformly convex* if for any positive ϵ, there exists a positive δ depending on ϵ such that

$$\left\| \frac{x+y}{2} \right\| \leq 1-\delta, \quad \text{whenever } x, y \in S_{\mathbf{X}} \text{ and } \|x-y\| > \epsilon.$$

Proposition 2.1.10. *The strict (uniform) convexity of a normed space is inherited by its subspaces.*

We refer to Megginson [85, p. 436, 454] for the proof of Proposition 2.1.10.

2.2 Banach sequence spaces

Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space. For a fixed positive integer n, consider the *Cartesian power* of $\mathbf{X}$,

$$\mathbf{X}^n = \mathbf{X} \times \cdots \times \mathbf{X} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n : x_i \in \mathbf{X}\}.$$

Under the usual addition and scalar multiplication, it becomes a normed space when equipped with any of the following p-norms:

$$\|\mathbf{x}\|_p = \begin{cases} (\|x_1\|^p + \cdots + \|x_n\|^p)^{1/p}, & 1 \leq p < \infty; \\ \max\{\|x_1\|, \ldots, \|x_n\|\}, & p = \infty, \end{cases}$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$. Note that these spaces are the vector-valued analogues of the ℓ_n^p spaces (cf. Example 2.1.2). The space $(\mathbf{X}^n, \|\cdot\|_p)$ is commonly denoted by $\ell_n^p(\mathbf{X})$. The p-norms are all equivalent in $\mathbf{X}^n$ by the following inequality:

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq n^{1/p}\|\mathbf{x}\|_\infty, \quad \mathbf{x} \in \mathbf{X}^n.$$

The following result is an immediate consequence of Hölder's inequality [14, p. 186].

Proposition 2.2.1. *The p-norm is decreasing as a function of p on $[1, \infty]$, that is, for any $1 \leq r < s \leq \infty$ and $\mathbf{x} \in \mathbf{X}^n$, we have*

$$\|\mathbf{x}\|_s \leq \|\mathbf{x}\|_r. \tag{2.4}$$

The p-norms preserve the completeness of the original normed space in $\mathbf{X}^n$, as shown in the following proposition.

Proposition 2.2.2. *Let $\mathbf{X}$ be a Banach space and $1 \leq p \leq \infty$. Then, $\mathbf{X}^n$ is also a Banach space when equipped with any of the p-norms.*

Proof. Since the p-norms are all equivalent on $\mathbf{X}^n$, it is sufficient to prove the proposition for $p = 1$. Let $(\mathbf{x}_j)_{j=1}^{\infty} = \big((x_j^1, \ldots, x_j^n)\big)_{j=1}^{\infty}$ be a Cauchy sequence in $\mathbf{X}^n$. Given $\epsilon > 0$, there exists a $K_0 \in \mathbb{N}$ such that for any $j, k \geq K_0$

$$\begin{aligned} \|\mathbf{x}_j - \mathbf{x}_k\|_1 &= \|(x_j^1 - x_k^1, \ldots, x_j^n - x_k^n)\|_1 \\ &= \|x_j^1 - x_k^1\| + \cdots + \|x_j^n - x_k^n\| < \epsilon, \end{aligned}$$

which implies that $\|x_j^i - x_k^i\| < \epsilon$ for any $i \in \{1, \ldots, n\}$. Thus, each $(x_j^i)_{j=1}^{\infty}$ is a Cauchy sequence in $\mathbf{X}$. Since $\mathbf{X}$ is a Banach space, then each $(x_j^i)_{j=1}^{\infty}$ converges to $x^i \in \mathbf{X}$. The convergence implies that there exist $K_i \in \mathbb{K}$, such that

$$\|x_j^i - x^i\| < \frac{\epsilon}{n}, \quad j \geq K_i.$$

Let $K = \max\{K_i\}$, then for $j \geq K$, we have

$$\begin{aligned} \|(x_j^1, \ldots, x_j^n) - (x^1, \ldots, x^n)\|_1 &= \|(x_j^1 - x^1, \ldots, x_j^n - x^n)\|_1 \\ &= \|x_j^1 - x^1\| + \cdots + \|x_j^n - x^n\| \\ &< \frac{\epsilon}{n} + \cdots + \frac{\epsilon}{n} = \epsilon. \end{aligned}$$

This completes the proof. □

In the next proposition, it is noted that when $\mathbf{X}$ is equipped with an inner product, the 2-norm is induced by an inner product in $\mathbf{X}^n$. Consequently, when $\mathbf{X}$ is a Hilbert space, then $\mathbf{X}^n$ together with the 2-norm is also a Hilbert space.

Proposition 2.2.3. *Suppose $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ is an inner product space. Then, the 2-norm is induced by the following inner product*

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 = \langle x_1, y_1 \rangle + \cdots + \langle x_n, y_n \rangle$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are in $\mathbf{X}^n$, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbf{X}$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. We have

$$\begin{aligned} &\|\mathbf{x} + \mathbf{y}\|_2^2 + \|\mathbf{x} - \mathbf{y}\|_2^2 \\ = \;& \|x_1 + y_1\|^2 + \cdots + \|x_n + y_n\|^2 + \|x_1 - y_1\|^2 + \cdots + \|x_n - y_n\|^2 \\ = \;& (\|x_1 + y_1\|^2 + \|x_1 - y_1\|^2) + \cdots + (\|x_n + y_n\|^2 + \|x_n - y_n\|^2). \end{aligned} \tag{2.5}$$

Since $\mathbf{X}$ is an inner product space, the parallelogram law holds and (2.5) becomes

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2+\|\mathbf{x}-\mathbf{y}\|_2^2 &= 2(\|x_1\|^2+\|y_1\|^2)+\cdots+2(\|x_n\|^2+\|y_n\|^2)\\ &= 2(\|x_1\|^2+\cdots+\|x_n\|^2)+2(\|y_1\|^2+\cdots+\|y_n\|^2)\\ &= 2(\|\mathbf{x}\|_2^2+\|\mathbf{y}\|_2^2).\end{aligned}$$

Therefore, the 2-norm is induced by an inner product. By the polarization identity, we have

$$\begin{aligned}\langle \mathbf{x},\mathbf{y}\rangle_2 &= \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|_2^2-\|\mathbf{x}-\mathbf{y}\|_2^2)\\ &= \frac{1}{4}[\|x_1+y_1\|^2+\cdots+\|x_n+y_n\|^2-(\|x_1-y_1\|^2+\|x_n-y_n\|^2)]\\ &= \frac{1}{4}(\|x_1+y_1\|^2-\|x_1-y_1\|^2)+\cdots+\frac{1}{4}(\|x_n+y_n\|^2-\|x_n-y_n\|^2)\\ &= \langle x_1,y_1\rangle+\cdots+\langle x_n,y_n\rangle,\end{aligned}$$

which completes the proof. □

Each of the p-norms extends in a natural way to a norm on a space of sequences in $\mathbf{X}$, giving the $\ell^p(\mathbf{X})$ spaces, which are the vector-valued analogues of the ℓ^p spaces (cf. Example 2.1.1). Despite their equivalence on $\mathbf{X}^n$ the norms on the $\ell^p(\mathbf{X})$ spaces are all inequivalent. Furthermore, the $\ell^p(\mathbf{X})$ spaces are all different for different values of p. In the following, we recall the definition and some results concerning the $\ell^p(\mathbf{X})$ spaces.

Let $(\mathbf{X},\|\cdot\|)$ be a Banach space. The Banach sequence spaces are defined as follows:

$$\ell^p(\mathbf{X})=\left\{\mathbf{x}|\mathbf{x}:\mathbb{N}\to\mathbf{X},\ \mathbf{x}=\{x_n\}_{n\geq 1},\ \mathbf{x}_n\in\mathbf{X},\ \sum_{n=1}^{\infty}\|\mathbf{x}_n\|^p<\infty\right\},\ 1\leq p<\infty,$$

$$\ell^\infty(\mathbf{X})=\left\{\mathbf{x}|\mathbf{x}:\mathbb{N}\to\mathbf{X},\ \mathbf{x}=\{x_n\}_{n\geq 1},\ \mathbf{x}_n\in\mathbf{X},\ \sup_{n\geq 1}\|\mathbf{x}_n\|<\infty\right\}.$$

It is well-known that these sequence spaces are Banach spaces when equipped with the following norms

$$\begin{aligned}\|\mathbf{x}\|_p &= \left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{1/p}, \quad \text{for } \mathbf{x}\in\ell^p(\mathbf{X}),\ 1\leq p<\infty,\\ \|\mathbf{x}\|_\infty &= \sup_{n\geq 1}\|x_n\|, \quad \text{for } \mathbf{x}\in\ell^\infty(\mathbf{X}).\end{aligned}$$

Note that when $\mathbf{X} = \mathbb{R}$, we obtain the classical Banach spaces $\ell^p(\mathbb{R}) = \ell^p$ for any $1 \leq p \leq \infty$ (cf. Example 2.1.1). Further, when $\mathbf{X}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, then $\ell^2(\mathbf{X})$ is also an inner product space with the following inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle + \dots$$

for any $\mathbf{x} = (x_1, x_2, \dots), \mathbf{y} = (y_1, y_2, \dots) \in \ell^2(\mathbf{X})$.

Leonard [78, p. 246–247] noted that the space $\ell^p(\mathbf{X})$, $1 < p < \infty$ are reflexive if and only if $\mathbf{X}$ is reflexive. He also remarked that the spaces $\ell^1(\mathbf{X})$ and $\ell^\infty(\mathbf{X})$ are never reflexive, since they contain, respectively, the classical spaces ℓ^1 and ℓ^∞ as closed subspaces [78, p. 247]. However, the 1-norm and the ∞-norm preserve the reflexivity of $\mathbf{X}$ in $\mathbf{X}^n$, due to the fact that all the p-norms are equivalent in $\mathbf{X}^n$ (cf. Proposition 2.1.6).

As for the geometrical properties, such as strict convexity, uniform convexity, smoothness and Fréchet smoothness, of $\ell^p(\mathbf{X})$, $1 < p < \infty$, they are inherited from the corresponding properties of $\mathbf{X}$. We refer to the works of Boas [67], Clarkson [26], Day [27,29], Leonard and Sundaresan [79], McShane [84], Smith and Turett [111], for the proofs.

2.3 Bochner function spaces

The other important examples of Banach spaces are the Lebesgue function spaces L^p. The $L^p[0,1]$ $(1 \leq p < \infty)$ spaces are the spaces of all measurable functions defined on the interval $[0,1]$ (or rather equivalent classes of measurable functions) in which $\int_0^1 |f(t)|^p \, dt < \infty$. They are Banach spaces together with the norm

$$\|f\|_{L^p} = \left(\int_0^1 |f(t)|^p \, dt \right)^{\frac{1}{p}}.$$

The $L^\infty[0,1]$ space is the space of all bounded measurable functions (or rather measurable functions which are bounded almost everywhere) on $[0,1]$. It is also a Banach space, together with the norm

$$\|f\|_{L^\infty} = \operatorname{ess\,sup} |f(t)|.$$

For further properties of these spaces, we refer to the books by Dunford and Schwarz [47] and Royden [109].

A definition of Lebesgue integral for functions on an interval of real numbers to a Banach space $(\mathbf{X}, \|\cdot\|)$ has been given by Bochner in [12], which is now referred to as the *Bochner integral*. Bochner introduced a generalization of Lebesgue function space L^p as follows: the space $L^p([0,1],\mathbf{X})$ is the class of functions f defined on the interval $[0,1]$, with values in $\mathbf{X}$ for which the norm

$$\|f\|_{L^p} := \left(\int_0^1 \|f(t)\|^p dt\right)^{\frac{1}{p}}$$

is finite [13, p. 914]. With this definition of norm, $L^p([0,1],\mathbf{X})$ is a Banach space [12,47]. This space is called the Lebesgue-Bochner (or sometimes, Bochner) function space [111].

The geometrical properties of $L^p([0,1],\mathbf{X})$ are closely connected to those of $\mathbf{X}$. The results are summarized in the following.

Lemma 2.3.1. *Let $1 < p < \infty$. The space $L^p([0,1],\mathbf{X})$ is a smooth (Fréchet smooth) Banach space whenever $\mathbf{X}$ is smooth (Fréchet smooth, respectively).*

We refer to McShane [84, p. 233-237, 404] for the proof of Lemma 2.3.1.

Lemma 2.3.2. *Let $1 < p < \infty$. The space $L^p([0,1],\mathbf{X})$ is a reflexive Banach space if $\mathbf{X}$ is reflexive.*

Bochner in [13, p. 930] stated that if $\mathbf{X}$ and its dual $\mathbf{X}^*$ are of (D)-property (namely, any function of bounded variation is differentiable almost everywhere [13, p. 914–915]) and $\mathbf{X}$ is reflexive, then $L^p([0,1],\mathbf{X})$ is reflexive. However, further studies have shown that these conditions could be reduced to a simpler condition. The argument is as follows: any reflexive space has the *Radon-Nikodym property*, namely, every absolutely continuous Banach-valued function is differentiable almost everywhere [7, p. 20]. Hence, any function of bounded variation is differentiable almost everywhere (cf. [109, Theorem 5.5]). By the fact that $\mathbf{X}$ is reflexive if and only if $\mathbf{X}^*$ is reflexive, we conclude that the reflexivity of $\mathbf{X}$ is necessary and sufficient for $L^p([0,1],\mathbf{X})$ to be reflexive.

Lemma 2.3.3. *Let $1 < p < \infty$. The space $L^p([0,1],\mathbf{X})$ is a strictly (uniformly) convex Banach space, whenever $\mathbf{X}$ is.*

The proof is implied by the strict (uniform) convexity of $\ell^p(\mathbf{X})$ [27,29], which follows by the embedding argument similar to Clarkson's argument in [26]. Consider a step function on a partition of $[0,1]$ into equal parts. Such a function can be identified as an element of $\ell^p(\mathbf{X})$. Since the set of all step functions on $[0,1]$ is a dense set in $L^p([0,1],\mathbf{X})$ [116, p.

132], and by the continuity of the norm, each function can be 'identified' by an element in $\ell^p(\mathbf{X})$. The proof is completed by the fact that $\ell^p(\mathbf{X})$ is strictly (uniformly) convex when $\mathbf{X}$ is a strictly (uniformly) convex space.

In a more general setting, the Bochner integral has been extended for functions defined on a finite measure space; and the similar results apply for this setting, which will be summarized in Lemma 2.3.4. To be precise, let $\Omega = (\Omega, \Sigma, \mu)$ be a finite measure space, $(\mathbf{X}, \|\cdot\|)$ be a Banach space and $1 \leq p < \infty$. We denote by $L^p(\Omega, \mathbf{X})$ the Banach space of all (classes of) $\mathbf{X}$-valued p-Bochner μ-integrable functions with the norm [103, p. 1109]:

$$\|f\|_{L^p(\Omega,\mathbf{X})} := \left(\int_\Omega \|f(\omega)\|^p d\mu(\omega) \right)^{\frac{1}{p}}.$$

Lemma 2.3.4. *Let* $1 < p < \infty$. *Then,*

1. *if* $\mathbf{X}$ *is a smooth (Fréchet smooth) space, then so is* $L^p(\Omega, \mathbf{X})$*;*
2. *if* $\mathbf{X}$ *is a strictly (uniformly) convex space, then so is* $L^p(\Omega, \mathbf{X})$*;*
3. *if* $\mathbf{X}$ *is a reflexive convex space, then so is* $L^p(\Omega, \mathbf{X})$*.*

We refer to the works by Leonard and Sundaresan [79, p. 233–237], Day [27,29], McShane [84], Smith and Turett [111], for the proof of Lemma 2.3.4.

Chapter 3

The p-HH-norms

A new family of norms on the Cartesian square of a normed space is introduced, which will be called the p-HH-norms. When the underlying space is the field of real numbers, this norm is the pth order generalized logarithmic mean of two positive numbers (cf. Chapter 1). The p-HH-norms preserve the completeness and the reflexivity of the underlying normed space, as an immediate consequence of their equivalency to the p-norms. The smoothness and convexity of the Cartesian square are inherited from the underlying normed space (with the p-HH-norms). However, the 1-HH-norm does not preserve the convexity of the original space. The results in this chapter are mainly taken from the author's research paper with Dragomir [71]. Throughout this dissertation, all the vector spaces considered are over the field of real numbers, unless told otherwise.

3.1 Hermite-Hadamard inequality in normed spaces

The following results are due to Dragomir [33,34] with regards to the Hermite-Hadamard inequality in linear spaces (cf. Pečarić and Dragomir [99]). For any pair of distinct vectors x and y in a (real) linear space $\mathbf{X}$, let

$$[x, y] := \{(1-t)x + ty,\ t \in [0, 1]\}$$

be the *segment* generated by x and y. We consider a function $f : [x, y] \to \mathbb{R}$ and the associated function $g_{x,y} : [0, 1] \to \mathbb{R}$ defined by

$$g_{x,y}(t) := f[(1-t)x + ty].$$

It is well-known that f is convex on the segment $[x,y]$ if and only if $g_{x,y}$ is convex on $[0,1]$ [99, p. 104]. When $g_{x,y}$ is convex, the Hermite-Hadamard inequality (1.1) gives us

$$g_{x,y}\left(\frac{1}{2}\right) \le \int_0^1 g_{x,y}(t)\ dt \le \frac{g_{x,y}(0)+g_{x,y}(1)}{2};$$

or, equivalently

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]\ dt \le \frac{f(x)+f(y)}{2}. \tag{3.1}$$

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. It is well-known that the norm $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ is a convex function. Therefore, we obtain the following refinement of the triangle inequality

$$\left\|\frac{x+y}{2}\right\| \le \int_0^1 \|(1-t)x+ty\|\ dt \le \frac{\|x\|+\|y\|}{2} \tag{3.2}$$

by (3.1). Furthermore, for any $1 < p < \infty$, the function $f : \mathbf{X} \to \mathbb{R}$ defined by $f(x) = \|x\|^p$, is also convex. Thus, we have

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x+ty\|^p\ dt \le \frac{\|x\|^p+\|y\|^p}{2} \tag{3.3}$$

by (3.1) (cf. Pečarić and Dragomir [99, p. 106]). Since $p > 0$, (3.3) can be rewritten as

$$\left\|\frac{x+y}{2}\right\| \le \left(\int_0^1 \|(1-t)x+ty\|^p\ dt\right)^{\frac{1}{p}} \le \left(\frac{\|x\|^p+\|y\|^p}{2}\right)^{\frac{1}{p}} \tag{3.4}$$

The right hand side of (3.4) resembles the p-norm of the pair $(x,y) \in \mathbf{X}^2$ (cf. Chapter 2). In the next section, we discuss a type of norm which is motivated by the integral mean in (3.3).

3.2 The p-HH-norm

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $1 \le p \le \infty$. For any $(x,y) \in \mathbf{X}^2$, define the quantity

$$\|(x,y)\|_{p-HH} := \begin{cases} \left(\int_0^1 \|(1-t)x+ty\|^p dt\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty; \\ \sup\limits_{t\in[0,1]} \|(1-t)x+ty\|, & \text{if } p = \infty. \end{cases} \tag{3.5}$$

The integral is finite by the Hermite-Hadamard inequality (3.3). We remark that $\|(\cdot,\cdot)\|_{p-HH}$ is symmetric, that is, $\|(x,y)\|_{p-HH} = \|(y,x)\|_{p-HH}$ for all $(x,y) \in \mathbf{X}^2$.

Remark 3.2.1. For any $x, y \in \mathbf{X}$, consider the function

$$f(t) = \|(1-t)x + ty\|, \quad t \in [0,1].$$

Since f is continuous and convex on $[0,1]$, the supremum of f on $[0,1]$ is exactly its maximum and is attained at one of the endpoints. In other words, for any $(x,y) \in \mathbf{X}^2$,

$$\|(x,y)\|_{\infty-HH} = \sup_{t\in[0,1]} \|(1-t)x + ty\| = \max\{\|x\|, \|y\|\} = \|(x,y)\|_\infty.$$

Thus, $\|(\cdot,\cdot)\|_{\infty-HH}$ is a norm. We will not distinguish these two norms and will refer to them as the ∞-norm.

Theorem 3.2.2 (Kikianty and Dragomir [71])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space and* $1 \le p < \infty$*. Then, the mapping* $\|(\cdot,\cdot)\|_{p-HH} : \mathbf{X}^2 \to \mathbb{R}$ *defined by*

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p dt\right)^{\frac{1}{p}}$$

is a norm.

Proof. The positive homogeneity of the norm follows directly by definition. The triangle inequality follows by the Minkowski inequality [47, p. 120]. The nonnegativity of the norm is trivial, by definition.

Suppose that $(x,y) = (0,0)$. Then, $\|(1-t)x+ty\| = 0$ for all $t \in [0,1]$, and therefore $\|(x,y)\|_{p-HH} = 0$. Conversely, let $(x,y) \in \mathbf{X}^2$ such that $\|(x,y)\|_{p-HH} = 0$. Therefore,

$$0 \le \left\|\frac{x+y}{2}\right\| \le \left(\int_0^1 \|(1-t)x + ty\|^p dt\right)^{\frac{1}{p}} = 0,$$

which implies that $\left\|\frac{x+y}{2}\right\| = 0$. Thus, $x = -y$ and

$$\left(\int_0^1 \|(1-t)x + ty\|^p dt\right)^{\frac{1}{p}} = \left(\int_0^1 |2t-1|^p \|y\|^p dt\right)^{\frac{1}{p}} = \|y\| \left(\frac{1}{p+1}\right)^{\frac{1}{p}}. \tag{3.6}$$

Since $\|(x,y)\|_{p-HH} = 0$ and $\frac{1}{p+1} \ne 0$, we conclude that $\|y\| = 0$ by (3.6). Hence, $x = y = 0$, as desired. $\square$

Although it is possible to define the quantity in (3.5) for $p < 1$, we are only interested in the case where $p \geq 1$, since $\|(\cdot,\cdot)\|_{p-HH}$ does not define a norm on $\mathbf{X}^2$ for $p < 1$, as shown in the next example.

Example 3.2.3. We consider the normed space $(\mathbb{R}, |\cdot|)$. Thus, for any $(x,y) \in \mathbb{R}^2$ and $p < 1$, we have

$$|(x,y)|_{p-HH} = \left(\int_0^1 |(1-t)x + ty|^p dt\right)^{\frac{1}{p}}.$$

We want to show that $|(\cdot,\cdot)|_{p-HH}$ is not a norm on $\mathbb{R}^2$.

Choose $(x,y) = (1,0)$ and $(u,v) = (0,1)$ and consider the following cases:

Case 1: $p \in (-1,1)$. We have

$$|(x,y)|_{p-HH} + |(u,v)|_{p-HH} = 2(p+1)^{-\frac{1}{p}} \quad \text{and} \quad |(x,y) + (u,v)|_{p-HH} = 1.$$

We claim that $(p+1)^{-\frac{1}{p}} < \frac{1}{2}$ for any $p \in (-1,1)$. Thus,

$$|(x,y)|_{p-HH} + |(u,v)|_{p-HH} = 2(p+1)^{-\frac{1}{p}} < 1 = |(x,y) + (u,v)|_{p-HH},$$

which fails the triangle inequality.

Proof of claim. Define

$$f(p) = \begin{cases} (p+1)^{-\frac{1}{p}}, & p \in (-1,1) \setminus \{0\}; \\ e^{-1}, & p = 0. \end{cases}$$

For any $0 < a < 1$, consider the pth order generalized logarithmic mean of $(1,a)$, that is,

$$\mathfrak{L}^{[p]}(1,a) = \left[\frac{1}{p+1}\left(\frac{1-a^{p+1}}{1-a}\right)\right]^{\frac{1}{p}}.$$

Since $\mathfrak{L}^{[p]}$ is strictly increasing as a function of p, we have the following for any $-1 \leq r < s \leq 1$ $(r, s \neq 0)$:

$$\left[\frac{1}{r+1}\left(\frac{1-a^{r+1}}{1-a}\right)\right]^{\frac{1}{r}} < \left[\frac{1}{s+1}\left(\frac{1-a^{s+1}}{1-a}\right)\right]^{\frac{1}{s}}.$$

By taking $a \to 0^+$, we get $(r+1)^{-\frac{1}{r}} < (s+1)^{-\frac{1}{s}}$, which shows that f is strictly increasing on $(-1,1) \setminus \{0\}$. Observe that f is continuous at $p = 0$ because $\lim_{p\to 0}(p+1)^{-\frac{1}{p}} = e^{-1}$.

It implies that f is continuous and strictly increasing on $(-1,1)$. Therefore,

$$\sup_{p\in(-1,1)}(p+1)^{-\frac{1}{p}}=\lim_{p\to 1^-}(p+1)^{-\frac{1}{p}}=\frac{1}{2}.$$

This completes the proof. □

Case 2: $p\in(-\infty,-1)$. We have

$$|(x,y)|_{p-HH}^{p}=\int_0^1(1-t)^p dt\to\infty,\ \text{and}\ |(u,v)|_{p-HH}^{p}=\int_0^1 t^p dt\to\infty.$$

Since $p<0$, $|(x,y)|_{p-HH}\to 0$ and $|(u,v)|_{p-HH}\to 0$, which imply that

$$|(x,y)|_{p-HH}+|(u,v)|_{p-HH}\to 0. \tag{3.7}$$

We also have $|(x,y)+(u,v)|_{p-HH}=1$. By (3.7), we can find $\epsilon>0$ such that

$$0<|(x,y)|_{p-HH}+|(u,v)|_{p-HH}<\epsilon<1=|(x,y)+(u,v)|_{p-HH},$$

which fails the triangle inequality.

In the next example, we consider the simplest form of this norm, that is, when $\mathbf{X}=\mathbb{R}$, to enable us in 'visualizing' it in 2-dimensional space.

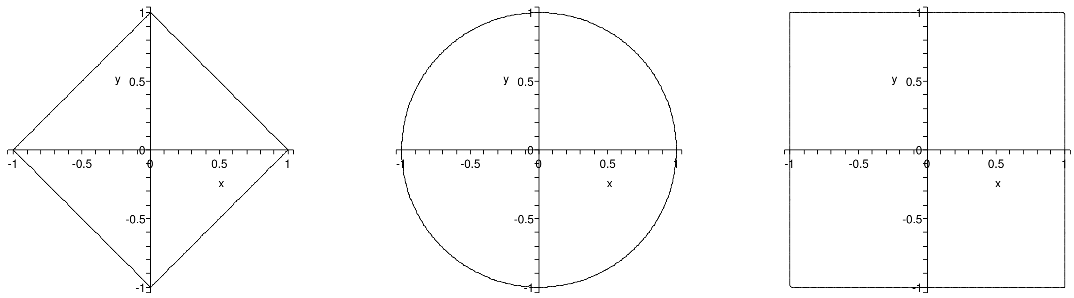
Example 3.2.4. In $\mathbb{R}^2$, we have the following norm:

$$|(x,y)|_{p-HH}:=\left(\int_0^1|(1-t)x+ty|^p dt\right)^{\frac{1}{p}},\ p\geq 1,$$

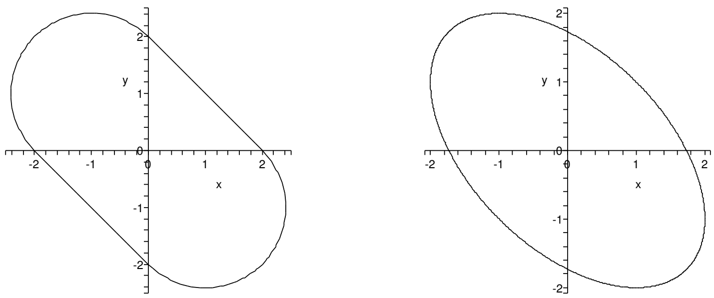
for any $(x,y)\in\mathbb{R}^2$. If $x=y$, then $|(x,y)|_{p-HH}=|x|$. Therefore, we may assume $x\neq y$ and without loss of generality (since the p-HH-norm is symmetric), $x<y$. Therefore,

$$|(x,y)|_{p-HH}=\begin{cases}\left[\frac{1}{p+1}\left(\frac{y^{p+1}-x^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } x,y\geq 0;\\ \left[\frac{1}{p+1}\left(\frac{(-x)^{p+1}+y^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } x<0 \text{ and } y\geq 0;\\ \left[\frac{1}{p+1}\left(\frac{(-x)^{p+1}-(-y)^{p+1}}{y-x}\right)\right]^{\frac{1}{p}}, & \text{if } x,y<0.\end{cases} \tag{3.8}$$

The unit circles in $\mathbb{R}^2$, associated to the 1-norm, the 2-norm, the ∞-norm, the 1-HH-norm and the 2-HH-norm, are shown in Figure 3.1 for comparison.



(a) Unit circle of the 1-norm (b) Unit circle of the 2-norm (c) Unit circle of the ∞-norm



(d) Unit circle of the 1-HH-norm (e) Unit circle of the 2-HH-norm

Figure 3.1: Unit circles in $\mathbb{R}^2$

Particularly, for positive real numbers x and y, it follows that $|(x,y)|_{p-HH} = \mathfrak{L}^{[p]}(x,y)$ $(1 \le p \le \infty)$, i.e. the pth order generalized logarithmic mean (cf. Chapter 1). Therefore, the p-HH-norms extend the generalized logarithmic mean of pairs of real numbers to pairs of vectors in normed spaces. The monotonicity remains to hold in this extension. The following result [14, p. 375–376] will be used to prove the monotonicity of the p-HH-norm as a function of p on $[1,\infty]$.

Lemma 3.2.5 (Bullen [14]). *Let $f : I = [a,b] \to \mathbb{R}$, $f \in L^p[a,b]$ $(-\infty \le p \le \infty)$, $f \ge 0$ almost everywhere on I and $f > 0$ almost everywhere on I if $p < 0$. The p-th power mean of f on $[a,b]$, which is defined by*

$$\mathfrak{M}^{[p]}_{[a,b]}(f) = \left(\frac{1}{b-a} \int_a^b f(x)^p dx \right)^{\frac{1}{p}},$$

is increasing on $\mathbb{R}$, that is, if $-\infty \le r < s \le \infty$, then, $\mathfrak{M}^{[r]}_{[a,b]}(f) \le \mathfrak{M}^{[s]}_{[a,b]}(f)$.

By utilizing Lemma 3.2.5, we obtain the following consequence.

Corollary 3.2.6 (Kikianty and Dragomir [71]). *The p-HH-norm is monotonically increasing as a function of p on $[1,\infty]$, that is, for any $1 \le r < s \le \infty$ and $(x,y) \in \mathbf{X}^2$, we have*

$$\|(x,y)\|_{r-HH} \le \|(x,y)\|_{s-HH}.$$

Proof. Consider the nonnegative function $f(t) = \|(1-t)x + ty\|$ on $[0,1]$. By the Hermite-Hadamard inequality (3.3), we conclude that $f \in L^p[0,1]$ for $1 \le p \le \infty$. We obtain the desired result by applying Lemma 3.2.5 to f for $1 \le p \le \infty$. $\square$

Remark 3.2.7. We have the following inequalities for $1 < p \le q < \infty$,

$$\begin{aligned} \|(x,y)\|_{1-HH} \le \|(x,y)\|_{p-HH} &\le \|(x,y)\|_{q-HH} \\ &\le \|(x,y)\|_{\infty} \le \|(x,y)\|_{q} \le \|(x,y)\|_{p} \le \|(x,y)\|_{1} \end{aligned}$$

for any $(x,y) \in \mathbf{X}^2$.

To end this section, we point out that if $\mathbf{X}$ is an inner product space, then the 2-HH-norm is induced by an inner product in $\mathbf{X}^2$.

Theorem 3.2.8 (Kikianty and Dragomir [71]). *Let $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ be an inner-product space, then $\|(\cdot,\cdot)\|_{2-HH}$ is induced by an inner product in $\mathbf{X}^2$, namely*

$$\langle (x,y),(u,v)\rangle_{HH} = \frac{1}{6}(2\langle x,u\rangle + \langle x,v\rangle + \langle u,y\rangle + 2\langle y,v\rangle),$$

and

$$\|(x,y)\|_{2-HH}^2 = \langle (x,y),(x,y)\rangle_{HH} = \frac{1}{3}\left(\|x\|^2 + \langle x,y\rangle + \|y\|^2\right).$$

Proof. Let $(x,y),(u,v) \in \mathbf{X}^2$. We want to show that the parallelogram law

$$\|(x,y)+(u,v)\|_{2-HH}^2 + \|(x,y)-(u,v)\|_{2-HH}^2 = 2\left(\|(x,y)\|_{2-HH}^2 + \|(u,v)\|_{2-HH}^2\right)$$

is satisfied. We have

$$\begin{aligned} &\|(x,y)+(u,v)\|_{2-HH}^2 + \|(x,y)-(u,v)\|_{2-HH}^2 \\ = &\int_0^1 \|(1-t)(x+u)+t(y+v)\|^2 dt + \int_0^1 \|(1-t)(x-u)+t(y-v)\|^2 dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \|(1-t)x+ty+(1-t)u+tv\|^2 + \|(1-t)x+ty-[(1-t)u+tv]\|^2 dt \\
&= \int_0^1 2\|(1-t)x+ty\|^2 + 2\|(1-t)u+tv\|^2 dt
\end{aligned}$$

since $\mathbf{X}$ is an inner product space. Note that the last identity is equivalent to

$$2\|(x,y)\|_{2-HH}^2 + 2\|(u,v)\|_{2-HH}^2,$$

which completes the proof. Therefore, the 2-HH-norm is induced by an inner product.

By the polarization identity, it can be shown that

$$\begin{aligned}
\langle (x,y),(u,v)\rangle_{HH} &= \frac{1}{4}(\|(x,y)+(u,v)\|_{2-HH}^2 - \|(x,y)-(u,v)\|_{2-HH}^2) \\
&= \int_0^1 \langle (1-t)x+ty,(1-t)u+tv\rangle dt \\
&= \frac{1}{6}(2\langle x,u\rangle + \langle x,v\rangle + \langle u,y\rangle + 2\langle y,v\rangle)
\end{aligned}$$

as desired. Note that $\frac{1}{4}(\|(x,y)+(u,v)\|_{2-HH}^2 - \|(x,y)-(u,v)\|_{2-HH}^2)$ is equal to

$$\int_0^1 \frac{1}{4}\|(1-t)x+ty+[(1-t)u+ty]\|^2 - \frac{1}{4}\|(1-t)x+ty-[(1-t)u+ty]\|^2 dt$$

and by the polarization identity,

$$\begin{aligned}
&\frac{1}{4}\left(\|(1-t)x+ty+[(1-t)u+ty]\|^2 - \|(1-t)x+ty-[(1-t)u+ty]\|^2\right) \\
= \; &\langle (1-t)x+ty,(1-t)u+tv\rangle.
\end{aligned}$$

The last part of the theorem follows by letting $(u,v)=(x,y)$. $\square$

Remark 3.2.9. For any $1 \le p < \infty$, the norm $\|(\cdot,\cdot)\|_{p-HH}$ in $\mathbf{X}^2$ does not induce an inner-product. To verify this, let $\mathbf{X}$ be an inner product space with the norm $\|\cdot\|$. Let $x \in \mathbf{X}$ be a nonzero vector and consider $(x,0),(0,x) \in \mathbf{X}$ and note that

$$\|(x,0)\|_{p-HH} = \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \|x\| = \|(0,x)\|_{p-HH}.$$

Observe that

$$2\left(\|(x,0)\|_{p-HH}^2 + \|(0,x)\|_{p-HH}^2\right) = 4\|x\|^2 \left(\frac{1}{p+1}\right)^{\frac{2}{p}}$$

and

$$
\begin{aligned}
& \|(x,0)+(0,x)\|_{p-HH}^2 + \|(x,0)-(0,x)\|_{p-HH}^2 \\
= & \left(\int_0^1 \|(1-t)x+tx\|^p dt\right)^{\frac{2}{p}} + \left(\int_0^1 \|(1-t)x-tx\|^p dt\right)^{\frac{2}{p}} \\
= & \|x\|^2 \left[1+\left(\int_0^1 |1-2t|^p \ dt\right)^{\frac{2}{p}}\right] = \|x\|^2 \left[1+\left(\frac{1}{p+1}\right)^{\frac{2}{p}}\right].
\end{aligned}
$$

For any $p \neq 2$,

$$
\|x\|^2 \left[1+\left(\frac{1}{p+1}\right)^{\frac{2}{p}}\right] \neq 4\|x\|^2 \left(\frac{1}{p+1}\right)^{\frac{2}{p}}.
$$

This shows that the parallelogram law does not hold in this case.

In general, the 2-HH-norm on $\mathbf{X}^2$ is not necessarily induced by an inner product. To verify this, let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$ be two distinct nonzero vectors. Then,

$$
\begin{aligned}
& \|(x,x)+(y,y)\|_{2-HH}^2 + \|(x,x)-(y,y)\|_{2-HH}^2 \\
= & \int_0^1 \|(1-t)(x+y)+t(x+y)\|^2 dt + \int_0^1 \|(1-t)(x-y)+t(x-y)\|^2 dt \\
= & \|x+y\|^2 + \|x-y\|^2 \\
\neq & 2\left(\|x\|^2+\|y\|^2\right) = 2\left(\|(x,x)\|_{2-HH}^2 + \|(y,y)\|_{2-HH}^2\right),
\end{aligned}
$$

unless $\mathbf{X}$ is an inner product space.

3.3 Completeness and reflexivity

Our main goal in this section is to show that the p-HH-norm is equivalent to the p-norm for any $1 \leq p < \infty$. To assist us in proving the equivalency, we employ the following lemma.

Lemma 3.3.1 (Kikianty and Sinnamon [75]). *Let* $\mathbf{X}$ *be a vector space and let* f *be a real-valued, even, convex function on* $\mathbf{X}$. *For any* $x, y \in \mathbf{X}$ *and any* $t \in [0,1]$ *we have the following inequality,*

$$
\begin{aligned}
f((1-2t)x)+f((2t-1)y) & \leq f((1-t)x+ty)+f((1-t)y+tx) \qquad (3.9) \\
& \leq f(x)+f(y).
\end{aligned}
$$

Proof. Since f is convex,

$$\begin{aligned} f((1-t)a+tb)+f((1-t)b+ta) &\leq (1-t)f(a)+tf(b)+(1-t)f(b)+tf(a) \\ &= f(a)+f(b) \end{aligned}$$

for any $a, b \in \mathbf{X}$. With $a = x$ and $b = y$ this proves the second inequality in (3.9).

To prove the first inequality in (3.9), we apply the above with $a = (1-t)x + ty$ and $b = -(1-t)y - tx$. Since

$$(1-t)a+tb = (1-t)((1-t)x+ty)+t(-(1-t)y-tx) = (1-2t)x,$$

$$(1-t)b+ta = (1-t)(-(1-t)y-tx)+t((1-t)x+ty) = (2t-1)y,$$

and since f is assumed to be even we have

$$\begin{aligned} f((1-2t)x)+f((2t-1)y) &= f((1-t)a+tb)+f((1-t)b+ta) \\ &\leq f(a)+f(b) \\ &= f((1-t)x+ty)+f(-(1-t)y-tx) \\ &= f((1-t)x+ty)+f((1-t)y+tx). \end{aligned}$$

This completes the proof. □

In the next theorem, we provide an inequality which establishes the equivalency of the p-norm and the p-HH-norm, together with the second part of the Hermite-Hadamard inequality (3.1).

Theorem 3.3.2 (Kikianty and Sinnamon [75]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. For any* $x, y \in \mathbf{X}$ *and* $1 \leq p < \infty$*, we have the following inequality*

$$\frac{\|x\|^p + \|y\|^p}{2(p+1)} \leq \int_0^1 \|(1-t)x+ty\|^p dt. \tag{3.10}$$

Equality holds if and only if $x = -y$*.*

Proof. If $1 \leq p < \infty$, then the map $f(x) = \|x\|^p$ is even and convex on $\mathbf{X}$. By Lemma 3.3.1, for each $t \in [0,1]$, we have

$$\|(1-2t)x\|^p + \|(2t-1)y\|^p \leq \|(1-t)x+ty\|^p + \|(1-t)y+tx\|^p. \tag{3.11}$$

Using the positive homogeneity of the norm and integrating from $t = 0$ to $t = 1$, we obtain

$$\int_0^1 |1-2t|^p(\|x\|^p + \|y\|^p)dt \leq \int_0^1 \|(1-t)x+ty\|^p dt + \int_0^1 \|(1-t)y+tx\|^p dt.$$

Evaluating the first integral and making the substitution $t \mapsto -t$ in the third yields

$$\frac{1}{p+1}\left(\|x\|^p + \|y\|^p\right) \leq 2\int_0^1 \|(1-t)x+ty\|^p dt,$$

as required.

If $x = -y$, we obtain equality in (3.3.2). Also, note that if equality holds in (3.3.2), then equality must hold in (3.11) for almost every $t \in [0,1]$. Since both sides of (3.11) are continuous (as functions of t), we may set $t = \frac{1}{2}$ to see that $\|x+y\| = 0$. Thus, equality holds if and only if $x = -y$. $\square$

Corollary 3.3.3. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space and* $1 \leq p < \infty$. *Then the p-HH-norm is equivalent to the p-norm on* $\mathbf{X}^2$. *If* $\mathbf{X}$ *is a Banach space, then* $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ *is also a Banach space. If* $\mathbf{X}$ *is a reflexive Banach space, then* $(\mathbf{X}^2, \|\cdot\|_{p-HH})$ *is also a reflexive Banach space.*

Proof. The Hermite-Hadamard inequality (3.3) gives us the upper bound and the Theorem 3.3.2 gives us the lower bound. If $\mathbf{X}$ is complete then $(\mathbf{X}^2, \|(\cdot,\cdot)\|_p)$ is complete. It implies that $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ is complete since the norms are equivalent. If $\mathbf{X}$ is reflexive then $(\mathbf{X}^2, \|(\cdot,\cdot)\|_p)$ is reflexive. Therefore, $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ is reflexive, since the norms are equivalent. $\square$

3.4 Convexity and smoothness

The convexity and smoothness of the p-HH-norms are not preserved under the norm equivalence. It is well-known that the convexity and smoothness of a normed space are inherited by its subspaces. Our approach is to embed the space $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ as a subspace of a suitable space which possess the convexity and the smoothness, such that these properties are inherited by $\mathbf{X}^2$. We choose the Lebesgue-Bochner space $L^p([0,1], \mathbf{X})$ (cf. Chapter 2), which inherits the convexity and smoothness from $\mathbf{X}$, provided that

p lies strictly between 1 and infinity. Such embedding exists as may be seen by the following arguments.

Consider a mapping Ψ on $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ to $L^p([0,1],\mathbf{X})$, defined by

$$\Psi(x,y) = g_{x,y}, \quad \text{where} \quad g_{x,y}(t) := (1-t)x + ty \quad (t \in [0,1]).$$

It is easy to verify that $g_{x,y}$ is measurable and integrable. Hence, $g_{x,y} \in L^p([0,1],\mathbf{X})$. The mapping Ψ is an isometric embedding. In other words, we view the space $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ as a subspace of $L^p([0,1],\mathbf{X})$ and we obtain the following consequence.

Corollary 3.4.1. *If $(\mathbf{X}, \|\cdot\|)$ is a strictly (uniformly) convex normed space, then the space $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ is also strictly (uniformly, respectively) convex, for any $1 < p < \infty$.*

Proof. Since $\mathbf{X}$ is strictly (uniformly) convex, the space $L^p([0,1],\mathbf{X})$ is strictly (uniformly, respectively) convex, by Lemma 2.3.3. It implies that all subspaces of $L^p([0,1],\mathbf{X})$ is also strictly (uniformly) convex. Since $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ is isometrically embedded in $L^p([0,1],\mathbf{X})$, then the strict (uniform) convexity is inherited by $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$. $\square$

Remark 3.4.2. The space $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{1-HH})$ is not always strictly (uniformly) convex, even if $\mathbf{X}$ is strictly (uniformly) convex. For example, take $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $(x,y) = (2,0)$ and $(u,v) = (0,2)$ in $\mathbb{R}^2$. Observe that

$$\|(x,y)\|_{1-HH} = \int_0^1 2(1-t)\ dt = 1 \text{ and } \|(u,v)\|_{1-HH} = \int_0^1 2t\ dt = 1,$$

but

$$\|(x,y) + (u,v)\|_{1-HH} = \int_0^1 2\ dt = 2,$$

which shows that this space is not strictly convex. Hence, it cannot be uniformly convex.

For Fréchet smoothness we exclude the case $p = 1$ and also require that $\mathbf{X}$ be complete.

Corollary 3.4.3. *If $(\mathbf{X}, \|\cdot\|)$ is Fréchet smooth, then the space $(\mathbf{X}^2, \|(\cdot,\cdot)\|_{p-HH})$ is also Fréchet smooth, for any $1 < p < \infty$.*

Proof. The norm in the Banach space $\mathbf{X}$ is Fréchet differentiable away from zero so, according to Lemma 2.3.4 (cf. Theorem 2.5 of Leonard and Sundaresan [79]), the norm in $L^p([0,1],\mathbf{X})$ is also Fréchet differentiable away from zero. In particular, the norm in $L^p([0,1],\mathbf{X})$ is Fréchet differentiable at each nonzero point of the isometric image of $(\mathbf{X}^2, \|\cdot\|_{p-HH})$ in $L^p([0,1],\mathbf{X})$. It follows that $(\mathbf{X}^2, \|\cdot\|_{p-HH})$ is Fréchet smooth. $\square$

In contrast to the 1-norm, the 1-HH-norm preserves the smoothness of the underlying space in the Cartesian square. We employ the superior (inferior) semi-inner product to assist us in proving the smoothness of the p-HH-norm.

Theorem 3.4.4. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $1 \le p < \infty$. For any $(x,y),(u,v) \in \mathbf{X}^2$ with $(u,v) \neq (0,0)$,*

$$\begin{aligned}(\nabla_+ \|\cdot\|_{p-HH}(u,v))(x,y) &= \|(u,v)\|_{p-HH}^{1-p} \int_0^1 \|(1-t)u+tv\|^{p-1} \\ &\quad \times (\nabla_+ \|\cdot\|((1-t)u+tv))((1-t)x+ty)\,dt;\end{aligned}$$

and

$$\begin{aligned}&\langle (x,y),(u,v)\rangle_{p-HH,s} \\ = \ &\|(u,v)\|_{p-HH}^{2-p} \int_0^1 \|(1-t)u+tv\|^{p-2} \langle (1-t)x+ty, (1-t)u+tv\rangle_s \,dt.\end{aligned}$$

Corresponding formulas hold for the left-hand derivative and the inferior semi-inner product.

Proof. If $(u,v) \neq (0,0)$, then the set $\{t \in [0,1] : (1-t)u+tv = 0\}$ has measure zero. Therefore the expressions $\|(1-t)u+tv\|^{p-1}$ and $\|(1-t)u+tv\|^{p-2}$ appearing above are well-defined and finite almost everywhere.

Fix $(x,y),(u,v) \in \mathbf{X}^2$ with $(u,v) \neq (0,0)$ and define

$$f_s = f_s(t) = \|(1-t)(u+sx)+t(v+sy)\|$$

for all $s \in (0,1)$ and for all $t \in [0,1]$ satisfying $(1-t)u+tv \neq 0$. The triangle inequality shows that $|f_s| \le \|(u,v)\|_1 + \|(x,y)\|_1$ for all t and that

$$\tfrac{1}{s}(f_s - f_0) \le \|(1-t)x+ty\| \le \|(x,y)\|_1 \le \|(u,v)\|_1 + \|(x,y)\|_1.$$

By the mean value theorem,

$$\begin{aligned}|\tfrac{1}{s}(f_s^p - f_0^p)| &\le p(\|(u,v)\|_1 + \|(x,y)\|_1)^{p-1} |\tfrac{1}{s}(f_s - f_0)| \\ &\le p(\|(u,v)\|_1 + \|(x,y)\|_1)^p.\end{aligned}$$

Thus, $\frac{1}{s}(f_s^p - f_0^p)$ is dominated by a constant independent of s and t.

For almost every $t \in [0,1]$, $f_0 = \|(1-t)u + tv\| \neq 0$, so by chain rule,

$$\lim_{s\to 0^+} \tfrac{1}{s}(f_s^p - f_0^p) = p f_0^{p-1}(\nabla_+\|\cdot\|((1-t)u+tv))((1-t)x+ty)$$

and Lebesgue's dominated convergence theorem implies

$$\begin{aligned} & \lim_{s\to 0^+} \frac{1}{s}\left(\int_0^1 f_s^p\,dt - \int_0^1 f_0^p\,dt\right) \\ = & \int_0^1 p f_0^{p-1}(\nabla_+\|\cdot\|((1-t)u+tv)((1-t)x+ty)\,dt. \end{aligned}$$

Applying the chain rule again gives

$$\begin{aligned} & \lim_{s\to 0^+} \tfrac{1}{s}(\|(u,v)+t(x,y)\|_{p-HH} - \|(u,v)\|_{p-HH}) \\ = & \|(u,v)\|_{p-HH}^{1-p} \int_0^1 \|(1-t)u+tv\|^{p-1} \\ & \times(\nabla_+\|\cdot\|((1-t)u+tv)((1-t)x+ty)\,dt, \end{aligned}$$

the first formula of the theorem.

The second formula follows from the first by applying (2.2). With obvious minor modifications the proof will apply to the left-hand derivative and the inferior semi-inner product. □

These formulas imply that if the superior and inferior semi-inner products of $\mathbf{X}$ agree, then the superior and inferior semi-inner products of $(\mathbf{X}^2, \|\cdot\|_{p-HH})$ agree, giving the following corollary.

Corollary 3.4.5. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space and* $1 \le p < \infty$. *If* $\mathbf{X}$ *is smooth then so is* $(\mathbf{X}^2, \|\cdot\|_{p-HH})$.

Proof. Since $\mathbf{X}$ is smooth, $\langle x,y\rangle_s = \langle x,y\rangle_i$ for all $x,y \in \mathbf{X}$. It follows that for all $(x,y),(u,v) \in \mathbf{X}^2$ with $(u,v) \neq (0,0)$ and for almost all $t \in [0,1]$,

$$\begin{aligned} & \|(1-t)u+tv\|^{2-p}\langle (1-t)x+ty, (1-t)u+tv\rangle_s \\ = & \|(1-t)u+tv\|^{2-p}\langle (1-t)x+ty, (1-t)u+tv\rangle_i. \end{aligned}$$

Theorem 3.4.4 implies that $\langle (x,y),(u,v)\rangle_{p-HH,s} = \langle (x,y),(u,v)\rangle_{p-HH,i}$ for all $(u,v) \neq (0,0)$. It also holds when $(u,v) = (0,0)$, from the definition of the semi-inner products. Equality of these two semi-inner products for the p-HH-norm implies that $(\mathbf{X}^2, \|\cdot\|_{p-HH})$ is smooth. $\square$

Chapter 4

Ostrowski type inequality involving the p-HH-norms

The results in this chapter are mainly taken from the author's research papers with Dragomir and Cerone [72, 73]. We establish some Ostrowski type inequalities to give quantitative comparison between the p-norm and the p-HH-norm, for a fixed real number $1 \leq p < \infty$. In the first section, we recall the classical Ostrowski inequality and some extensions that have been considered in the literature.

Section 4.2 discusses an Ostrowski type inequality for absolutely continuous functions on segments of (real) linear spaces. Some particular cases are provided which recapture earlier results (see for example the paper by Dragomir [38]) along with the results for trapezoidal type inequalities and the classical Ostrowski inequality. In particular, some norm inequalities are obtained to estimate the absolute difference of $\|(\cdot,\cdot)\|_{p-HH}$ and $\frac{1}{2^{1/p}}\|(\cdot,\cdot)\|_p$, that is, the counterpart of the second inequality in (3.3) (cf. Chapter 3). Some of these inequalities are proven to be sharp.

In Section 4.3, an Ostrowski type inequality for convex functions defined on linear spaces is generalized. The results in normed linear spaces are used to obtain some sharp inequalities which are related to the given norm and associated semi-inner products. These inequalities are then utilized to estimate the absolute difference of $\|(\cdot,\cdot)\|_{p-HH}$ and $\frac{1}{2^{1/p}}\|(\cdot,\cdot)\|_p$.

The last section of this chapter discusses the comparison of these two types of inequalities. Although the results in Section 4.3 are not more general than those in Section 4.2, they are proven to be better, in some particular cases. It is conjectured that this statement holds in general.

Throughout this chapter, we will denote p' as the conjugate pair of a real number $p > 1$, that is, p and p' satisfy

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

4.1 Ostrowski inequality

In 1938, Ostrowski [95, p. 226] considered the problem of estimating the deviation of a function from its integral mean. If a function g defined on an interval $[a, b] \subset \mathbb{R}$ is continuous, then the deviation of g at a point $x \in [a, b]$ from its integral mean $\frac{1}{b-a}\int_a^b g(x)dx$ can be approximated by the difference between its maximum and minimum value. Furthermore, if g is differentiable on (a, b) and the derivative is bounded on (a, b), then the difference between the maximum and minimum value does not exceed $(b-a)M$ (however, it may reach this value). Moreover, the absolute deviation of $g(x)$ from its integral mean does not exceed $\frac{1}{2}(b-a)M$. If x is the midpoint of the interval, that is $x = \frac{a+b}{2}$, then the absolute deviation is bounded by the value $\frac{1}{4}(b-a)M$.

The above statements are formulated in the following arguments. Let $g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Suppose that there exists a real number M such that $|g'(x)| \leq M$ for all $x \in (a, b)$. Then, the following inequality

$$\left| g(x) - \frac{1}{b-a}\int_a^b g(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M, \tag{4.1}$$

holds for every $x \in [a, b]$ [95, p. 226–227]. Inequality (4.1) is known in the literature as the *Ostrowski inequality* [90, p. 468]. The first factor on the right hand side of (4.1) reaches the value of $\frac{1}{4}$ at the midpoint and monotonically increases to $\frac{1}{2}$ which is attained at both endpoints [95, p. 226]. It implies that the constant $\frac{1}{4}$ is best possible, that is, it cannot be replaced by a smaller quantity. Anastassiou [6, p. 3775–3776] gave an alternative proof for the best constant in this inequality.

Numerous developments, extensions and generalizations of Ostrowski inequality have been carried out in various directions. One way to extend this result is to consider other classes of integrable functions. The case for absolutely continuous functions has been considered by Dragomir and Wang [44–46] (cf. Dragomir [31, 32, 39]; Dragomir and Rassias [43, p. 2]). The result is summarized in the following lemma.

Lemma 4.1.1 (Dragomir and Wang). *Let g be a real-valued, absolutely continuous function defined on $[a, b]$. Then, for all $x \in [a, b]$,*

$$\left|g(x) - \frac{1}{b-a}\int_a^b g(t)dt\right|$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|g'\|_{L^\infty}, & \text{if } g' \in L^\infty[a,b]; \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[\left(\frac{x-a}{b-a}\right)^{p'+1} + \left(\frac{b-x}{b-a}\right)^{p'+1}\right]^{\frac{1}{p'}}(b-a)^{\frac{1}{p'}}\|g'\|_{L^p}, & \\ & \text{if } g' \in L^p[a,b],\ p>1; \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|g'\|_{L^1}. & \end{cases} \tag{4.2}$$

The constants $\frac{1}{4}$, $\frac{1}{(p'+1)^{\frac{1}{p'}}}$ and $\frac{1}{2}$ are sharp.

Note that $\|\cdot\|_{L^p}$ ($p \in [1, \infty]$) are the Lebesgue norms (cf. Section 2.3).

The following is a generalization of Ostrowki inequality for absolutely continuous functions, which provides upper bounds for the absolute difference of a linear combination of values of a function at $k+1$ partition points and its integral mean (cf. Dragomir [31, 32, 35, 38], Dragomir and Rassias [43]). Lemma 4.1.1 is a particular case of this inequality [35, p. 378–381]. We refer to Dragomir [35] for the proof.

Lemma 4.1.2 (Dragomir [35]). *Let $I_k : a = s_0 < s_1 < \cdots < s_{k-1} < s_k = b$ be a partition of the interval $[a, b]$ and α_i $(i = 0, \ldots, k+1)$ be $k+2$ points such that $\alpha_0 = a$, $\alpha_i \in [s_{i-1}, s_i]$ $(i = 1, \ldots, k)$ and $\alpha_{k+1} = b$. If $g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$, then*

$$\left|\int_a^b g(t)dt - \sum_{i=0}^{k}(\alpha_{i+1} - \alpha_i)g(s_i)\right|$$

$$\leq \begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1} - \frac{s_i + s_{i+1}}{2}\right)^2\right]\|g'\|_{L^\infty}, & \text{if } g' \in L^\infty[a,b]; \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1} - s_i)^{p'+1} + (s_{i+1} - \alpha_{i+1})^{p'+1}\right]\right]^{\frac{1}{p'}}\|g'\|_{L^p}, & \\ & \text{if } g' \in L^p[a,b],\ p>1; \\ \left[\frac{1}{2}\nu(h) + \max_{i\in\{0,\ldots,k-1\}}\left|\alpha_{i+1} - \frac{s_i+s_{i+1}}{2}\right|\right]\|g'\|_{L^1}. & \end{cases}$$

The constants $\frac{1}{4}$, $\frac{1}{(p'+1)^{\frac{1}{p'}}}$ *and* $\frac{1}{2}$ *are sharp.*

Another possibility of generalizing the Ostrowski inequality is to consider the case of convex functions. The results can be summarized in the next two lemmas.

Lemma 4.1.3 (Dragomir [39]). *Let* $g : [a,b] \to \mathbb{R}$ *be a convex function. Then, for any* $x \in [a,b]$ *we have*

$$\begin{aligned}\frac{1}{2}\left[(b-x)^2 g'_+(x) - (x-a)^2 g_-(x)\right] &\leq \int_a^b g(t)dt - (b-a)g(x)\\ &\leq \frac{1}{2}\left[(b-x)^2 g'_-(b) - (x-a)^2 g_+(a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*

Lemma 4.1.4 (Dragomir [35]). *Let* $I_k : a = s_0 < s_1 < \cdots < s_{k-1} < s_k = b$ *be a partition of the interval* $[a,b]$ *and* α_i $(i = 0, \ldots, k+1)$ *be* $k+2$ *points such that* $\alpha_0 = a$, $\alpha_i \in [s_{i-1}, s_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = b$. *If* $g : [a,b] \to \mathbb{R}$ *is convex on* $[a,b]$, *then*

$$\begin{aligned}&\frac{1}{2}\sum_{i=0}^{k-1}[(s_{i+1}-\alpha_{i+1})^2 g'_+(\alpha_{i+1}) - (\alpha_{i+1}-s_i)^2 g'_-(\alpha_{i+1})]\\ \leq\ &\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)g(s_i) - \int_0^1 g(t)dt \qquad (4.3)\\ \leq\ &\frac{1}{2}\sum_{i=0}^{k-1}[(s_{i+1}-\alpha_{i+1})^2 g'_-(s_{i+1}) - (\alpha_{i+1}-s_i)^2 g'_+(s_i)].\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*

Some extensions for other classes of functions have been considered in the literature. Due to the large amount of literature, some results are omitted. We refer to Dragomir [30,35], Dragomir and Rassias [43, p. 3–4] for functions of bounded variation; Dragomir and Rassias [43, p. 3] for Hölder continuous functions and Lipschitzian functions; and Barnett, Buşe, Cerone and Dragomir [8] for Banach-valued functions. For other possible directions, we refer to the results by Cerone [20–22], Cerone and Dragomir [23, 24]; Cerone, Dragomir and Roumeliotis [25].

4.2 Ostrowski inequality for absolutely continuous functions on linear spaces

Let $x, y \in \mathbf{X},\ x \neq y$ and define the segment $[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}$. Let $f : [x, y] \to \mathbb{R}$ and the associated function

$$h = g_{x,y} : [0,1] \to \mathbb{R},$$

where

$$h(t) = g_{x,y}(t) := f[(1-t)x + ty],\ t \in [0,1].$$

It is well-known that the function h is absolutely continuous on $[0,1]$ if and only if h is differentiable almost everywhere; the derivative h' is Lebesgue integrable; and $h(t) = \int_0^t h'(s)ds + h(0)$ (cf. Aliprantis and Burkinshaw [2, p. 263] and Royden [109, p. 106–107]). Therefore, h is absolutely continuous if and only if f satisfies the following properties:

1. $\big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x)$ exists almost everywhere on $[0,1]$;
2. $\big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x)$ is Lebesgue integrable on $[0,1]$;
3. $f[(1-t)x + ty] = \displaystyle\int_0^t \big(\nabla f[(1-s)x + sy]\big)(y-x)ds + f(x).$

Definition 4.2.1 (Kikianty, Dragomir and Cerone [72])**.** Let f be a real-valued function defined on a segment $[x, y]$ of a linear space $\mathbf{X}$. We say that f is *absolutely continuous on segment* $[x, y]$ if f satisfies conditions 1–3 above.

Therefore, f is absolutely continuous on segment $[x, y]$ if and only if h is absolutely continuous on $[0,1]$.

Example 4.2.2. The function $f_r : [x, y] \to \mathbb{R}$ defined by $f_r(w) = \|w\|^r$ is convex on $[x, y]$. It implies that the function

$$g_{x,y}(t) = f_r[(1-t)x + ty] = \|(1-t)x + ty\|^r$$

is also convex on $[0,1]$. Hence $g_{x,y}$ is absolutely continuous on $[0,1]$, which implies that f_r is absolutely continuous on $[x, y]$. Thus,

$$\big(\nabla f_r[(1-t)x + ty]\big)(y-x) = r\|(1-t)x + ty\|^{r-1}\big(\nabla\|\cdot\|[(1-t)x + ty]\big)(y-x)$$

exists almost everywhere on $[0,1]$.

Note that for any $y \neq 0$,

$$(\nabla_+ \|\cdot\|(x))(y) = \langle y, x\rangle_s \quad \text{and} \quad (\nabla_- \|\cdot\|(x))(y) = \langle y, x\rangle_i,$$

where $\langle\cdot,\cdot\rangle_s$ and $\langle\cdot,\cdot\rangle_i$ are the superior and inferior, respectively, semi-inner products associated with the norm $\|\cdot\|$. The superior and inferior semi-inner products, $\langle x, y\rangle_s$ and $\langle x, y\rangle_i$, are equal almost everywhere for fixed $x, y \in \mathbf{X}$. Therefore,

$$\big(\nabla f_r[(1-t)x + ty]\big)(y-x) = r\|(1-t)x+ty\|^{r-2}\langle y-x, (1-t)x+ty\rangle_{s(i)} \tag{4.4}$$

exists almost everywhere on $[0,1]$ for any $x, y \in \mathbf{X}$, whenever $r \geq 2$; otherwise we need to assume that x and y are linearly independent in $\mathbf{X}$.

An Ostrowski type inequality for functions defined on segments in linear spaces has been established by Dragomir [38] in 2005 along with its application for semi-inner products [38, p. 95–99]. The result is summarized in the following lemma.

Lemma 4.2.3 (Dragomir [38])**.** *Let $\mathbf{X}$ be a linear space, $x, y \in \mathbf{X}$, $x \neq y$ and $f : [x,y] \subset \mathbf{X} \to \mathbb{R}$ be a function defined on the segment $[x,y]$ and such that the Gâteaux derivative $\big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x)$ exists almost everywhere on $[0,1]$ and is Lebesgue integrable on $[0,1]$. Then for any $s \in [0,1]$ we have*

$$\left|\int_0^1 f[(1-t)x+ty]dt - f[(1-s)x+sy]\right| \tag{4.5}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(s-\frac{1}{2}\right)^2\right] \|\big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x)\|_{L^\infty}, \\ \qquad\qquad \textit{if } \big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x) \in L^\infty[0,1]; \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}[s^{p'+1} + (1-s)^{p'+1}]^{\frac{1}{p'}} \|\big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x)\|_{L^p}, \\ \qquad\qquad \textit{if } \big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x) \in L^p[0,1],\ p>1; \\ \left[\frac{1}{2} + \left|s-\frac{1}{2}\right|\right] \|\big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x)\|_{L^1}. \end{cases} \tag{4.6}$$

However, the sharpness of the constants of these inequalities has not been considered.

In this section, we suggest an Ostrowski type inequality for estimating deviation of the integral mean of an absolutely continuous function and the linear combination of its values at $k+1$ partition points on a segment of a linear space. This result is essentially an extension for the previous results, namely,

1. Ostrowski type inequality for estimating the absolute difference between the linear combination of values of a function at $k+1$ partition points from its integral mean (Lemma 4.1.2) ;

2. Ostrowski type inequality for functions defined on segments of a linear space (Lemma 4.2.3);

3. Ostrowski inequality for absolutely continuous function (cf. Dragomir [31,32,35,38], Dragomir and Rassias [43]).

Our main result is presented in the following theorem.

Theorem 4.2.4 (Kikianty, Dragomir and Cerone [72]). *Let* $\mathbf{X}$ *be a linear space,* $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ *be a partition of the interval* $[0,1]$ *and* α_i $(i = 0, \ldots, k+1)$ *be* $k+2$ *points such that* $\alpha_0 = 0,\ \alpha_i \in [s_{i-1}, s_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = 1$.

If $f : [x,y] \subset \mathbf{X} \to \mathbb{R}$ *is absolutely continuous on segment* $[x,y]$*, then we have*

$$\left| \int_0^1 f[(1-t)x+ty]dt - \sum_{i=0}^{k} (\alpha_{i+1}-\alpha_i) f[(1-s_i)x+s_i y] \right| \tag{4.7}$$

$$\leq \begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1} \left(\alpha_{i+1} - \dfrac{s_i+s_{i+1}}{2} \right)^2 \right] \|\big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x)\|_{L^\infty}, \\ \qquad\qquad \text{if } \big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x) \in L_\infty[0,1]; \\ \frac{1}{(p'+1)^{\frac{1}{p'}}} \left[\sum_{i=0}^{k-1} \left[(\alpha_{i+1}-s_i)^{p'+1} + (s_{i+1}-\alpha_{i+1})^{p'+1} \right] \right]^{\frac{1}{p'}} \\ \qquad \times \|\big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x)\|_{L^p}, \\ \qquad\qquad \text{if } \big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x) \in L_p[0,1],\ p>1; \\ \left[\frac{1}{2}\nu(h) + \max\limits_{i\in\{0,\ldots,k-1\}} \left| \alpha_{i+1} - \frac{s_i+s_{i+1}}{2} \right| \right] \|\big(\nabla f[(1-\cdot)x+\cdot y]\big)(y-x)\|_{L^1}, \end{cases}$$

where $\nu(h) := \max\{h_i | i = 0, \ldots, k-1\},\ h_i := s_{i+1} - s_i$ $(i = 0, \ldots, k-1)$.
The constants $\frac{1}{4}$, $\frac{1}{(q+1)^{\frac{1}{q}}}$ *and* $\frac{1}{2}$ *are sharp.*

Proof. Consider the auxiliary function $g(t) = f[(1-t)x+ty]$ defined on $[0,1]$. Since f is absolutely continuous on the segment $[x,y]$, it follows that g is an absolutely continuous function on $[0,1]$ and we may apply Lemma 4.1.2. We omit the details of the proof. The sharpness of the constants follows by the particular cases which are given in Corollary 4.2.5. $\square$

Note that Lemma 4.2.3 is a particular case of Theorem 4.2.4. The following lemma is a special case of Theorem 4.2.4. In particular, the following lemma gives bounds for the trapezoidal type functional.

Corollary 4.2.5 (Kikianty, Dragomir and Cerone [72]). *Let $\mathbf{X}$ be a linear space, $x, y \in \mathbf{X}$, $x \neq y$ and $f : [x, y] \subset \mathbf{X} \to \mathbb{R}$ be an absolutely continuous function on segment $[x, y]$. Then for any $s \in [0, 1]$ we have the inequalities*

$$\left| \int_0^1 f[(1-t)x + ty]dt - sf(x) - (1-s)f(y) \right| \tag{4.8}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(s - \frac{1}{2}\right)^2\right] \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^\infty}, \\ \qquad \text{if } \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \in L_\infty[0,1]; \\ \frac{1}{(p'+1)^{\frac{1}{p'}}} \left[s^{p'+1} + (1-s)^{p'+1}\right]^{\frac{1}{p'}} \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^p}, \\ \qquad \text{if } \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \in L_p[0,1],\ p > 1; \\ \left[\frac{1}{2} + \left|s - \frac{1}{2}\right|\right] \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^1}. \end{cases} \tag{4.9}$$

Particularly, we have

$$\left| \int_0^1 f[(1-t)x + ty]dt - \frac{f(x) + f(y)}{2} \right|$$

$$\leq \begin{cases} \frac{1}{4} \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^\infty}, \\ \frac{1}{2(p'+1)^{\frac{1}{p'}}} \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^p}, \quad p > 1; \\ \frac{1}{2} \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^1}. \end{cases} \tag{4.10}$$

The constants in (4.9) and (4.10) are sharp.

Proof. We obtain (4.9) by choosing $s_0 = 0$, $s_1 = 1$ and $0 = \alpha_0 < \alpha_1 = s < \alpha_2 = 1$ in Theorem 4.2.4. Let $s = \frac{1}{2}$ in (4.9) to obtain (4.10). To prove the sharpness of the constants in (4.10), suppose α and β are real positive constants such that

$$\left| \int_0^1 f[(1-t)x + ty]dt - \frac{f(x) + f(y)}{2} \right| \leq \begin{cases} \alpha \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^\infty}, \\ \beta \left(\frac{1}{(p'+1)^{\frac{1}{p'}}} \right) \left\| \big(\nabla f[(1-\cdot)x + \cdot y]\big)(y-x) \right\|_{L^p}, \end{cases}$$

where $p > 1$. We choose $\mathbf{X} = \mathbb{R}$, $[x, y] = [a, b] \subset \mathbb{R}$ $(a \neq b)$ and $f(x) = \left|x - \frac{b+a}{2}\right|$. Note that f is a convex function on the closed interval $[a, b]$, thus, it is an absolutely continuous function [109, Proposition 5.16]. Therefore,

$$\frac{1}{4}(b-a) \leq \begin{cases} \alpha(b-a), \\ \beta\left(\frac{1}{(p'+1)^{\frac{1}{p'}}}\right)(b-a), \end{cases}$$

From the first case, we obtain $\alpha \geq \frac{1}{4}$ since $b - a \neq 0$, which proves the sharpness of $\frac{1}{4}$ in the first case of (4.10). Now, let $p' \to 1$ in the second case, we obtain $\frac{1}{4}(b-a) \leq \frac{1}{2}\beta(b-a)$, that is, $\beta \geq \frac{1}{2}$, since $b - a \neq 0$, which shows that $\frac{1}{2}$ is sharp in the second case of (4.10).

Now, suppose that

$$\left|\int_0^1 f[(1-t)x + ty]dt - \frac{f(x) + f(y)}{2}\right| \leq \gamma \|\big(\nabla f[(1-\cdot)x + \cdot y]\big)(y - x)\|_1,$$

for a real constant $\gamma > 0$. By choosing $\mathbf{X} = \mathbb{R}$ and the absolutely continuous function $f(x) = \frac{C}{C^2+x^2} - \tan^{-1}\left(\frac{1}{C}\right)$ $(C > 0)$ on the interval $[0, 1]$ (the proof of this part is due to Peachey, McAndrew and Dragomir [97, p. 99–100]), we obtain

$$\frac{1}{2C} - \tan^{-1}\left(\frac{1}{C}\right) + \frac{C}{2(C^2+1)} \leq \gamma\left[\frac{1}{C(C^2+1)}\right].$$

Thus,

$$\gamma \geq (C^2+1)\left[\frac{1}{2} - C\tan^{-1}\left(\frac{1}{C}\right) + \frac{C^2}{2(C^2+1)}\right],$$

and by taking $C \to 0^+$, we obtain $\gamma \geq \frac{1}{2}$ and the proof for the sharpness of the constants in (4.10) is completed. This implies that all constants in (4.7) and (4.9) are sharp. □

Remark 4.2.6. It is important to note that the upper bounds in Corollary 4.2.5 are the same to those of Lemma 4.2.3. Cerone [22, Remark 1] stated that there is a strong relationship between the Ostrowski functional (4.5) and trapezoidal functional (4.8) which is highlighted by the symmetric transformations amongst their kernels. Particularly, the bounds in the Ostrowski (cf. (4.6)) and trapezoidal type inequalities (cf. (4.9)) are the same [22, p. 317].

Remark 4.2.7. If f is convex in (4.10), then

$$\frac{f(x) + f(y)}{2} - \int_0^1 f[(1-t)x + ty]dt \geq 0,$$

by the Hermite-Hadamard inequality [34, p. 2].

Example 4.2.8 (Example of a non-convex function)**.** Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space and consider the absolutely continuous function $f(x) = \ln(\|x\|)$, $x \in \mathbf{X} \setminus \{0\}$. Applying this to (4.10) we obtain the following for any linearly independent $x, y \in \mathbf{X}$:

$$\left| \int_0^1 \ln \|(1-t)x + ty\| dt - \frac{\ln \|x\| + \ln \|y\|}{2} \right|$$
$$\leq \begin{cases} \frac{1}{4} \sup_{u\in[0,1]} \left| \frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2} \right|, \\ \frac{1}{2(p'+1)^{\frac{1}{p'}}} \left(\int_0^1 \left| \frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2} \right|^p du \right)^{\frac{1}{p}}, \quad p > 1; \\ \frac{1}{2} \int_0^1 \left| \frac{\langle y-x,(1-u)x+uy\rangle_{s(i)}}{\|(1-u)x+uy\|^2} \right| du. \end{cases}$$

Using the Cauchy-Schwarz inequality for superior (inferior) semi-inner products [37, p. 29], we obtain

$$\left| \int_0^1 \ln(\|(1-t)x + ty\|) dt - \ln \sqrt{\|x\|\|y\|} \right|$$
$$\leq \|y-x\| \begin{cases} \frac{1}{4} \sup_{u\in[0,1]} \|(1-u)x + uy\|^{-1}, \\ \frac{1}{2(p'+1)^{\frac{1}{p'}}} \left(\int_0^1 \|(1-u)x + uy\|^{-p} du \right)^{\frac{1}{p}}, \quad p > 1; \\ \frac{1}{2} \int_0^1 \|(1-u)x + uy\|^{-1} du. \end{cases}$$

4.2.1 Application for semi-inner products

In this subsection, we consider a particular case of Theorem 4.2.4. Recall from Chapter 2 that every normed space can be equipped with the superior and inferior semi inner products.

The following result holds in any normed linear space with the semi-inner products $\langle \cdot, \cdot \rangle_{s(i)}$.

Corollary 4.2.9 (Kikianty, Dragomir and Cerone [72])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed linear space,* $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ *be a partition of the interval* $[0, 1]$ *and* α_i $(i = 0, \ldots, k+1)$ *be* $k + 2$ *points such that* $\alpha_0 = 0$, $\alpha_i \in [s_{i-1}, s_i]$ $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = 1$*. If* $1 \leq r < \infty$ *then*

$$\left|\int_0^1 \|(1-t)x+ty\|^r dt - \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\|(1-s_i)x+s_iy\|^r\right|$$

$$\leq \begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right)^2\right] \\ \quad \times \sup_{u\in[0,1]} [r\|(1-u)x+uy\|^{r-2}|\langle y-x,(1-u)x+uy\rangle_{s(i)}|], \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-s_i)^{p'+1}+(s_{i+1}-\alpha_{i+1})^{p'+1}\right]\right]^{\frac{1}{p'}} \\ \quad \times \left[\int_0^1 |r\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|^p du\right]^{\frac{1}{p}}, \ p>1; \\ \left[\frac{1}{2}\nu(h)+\max_{i\in\{0,\ldots,k-1\}}\left|\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right|\right] \\ \quad \times \int_0^1 |r\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|du, \end{cases} \tag{4.11}$$

hold for any $x, y \in \mathbf{X}$, for $r \geq 2$, otherwise they hold for any linearly independent $x, y \in \mathbf{X}$. Here, $\nu(h) := \max\{h_i | i = 0, \ldots, k-1\}$ and $h_i := s_{i+1} - s_i$ $(i = 0, \ldots, k-1)$.

Proof. Let $f(x) = \|x\|^r$, where $x \in \mathbf{X}$ and $1 \leq r < \infty$. Since f is convex on $\mathbf{X}$, $g_{x,y}(\cdot) = f((1-\cdot)x + \cdot\, y)$ is convex on $[0,1]$ for any $1 \leq r < \infty$ and $x, y \in \mathbf{X}$. It follows that $g_{x,y}(\cdot) = \|(1-\cdot)x + \cdot\, y\|^r$ is an absolutely continuous function. Therefore, we may apply Theorem 4.2.4 for f and obtained the desired result. Note the use of identity (4.4) of Example 4.2.2. □

Remark 4.2.10. The result we obtain in Corollary 4.2.9 is 'complicated' in the sense that the upper bounds are not practical to apply. Here, we suggest simpler, although coarser, upper bounds using the Cauchy-Schwarz inequality for semi-inner products [38, p. 97–98]. Under the assumptions of Corollary 4.2.9, and by the Cauchy-Schwarz inequality for superior (inferior) semi-inner products [37, p. 29], we obtain

$$\begin{aligned} &\sup_{u\in[0,1]} [r\|(1-u)x+uy\|^{r-2}|\langle y-x,(1-u)x+uy\rangle_{s(i)}| \\ \leq\ & r\|y-x\| \sup_{u\in[0,1]} \|(1-u)x+uy\|^{r-1} = r\|y-x\|\max\{\|x\|^{r-1},\|y\|^{r-1}\}, \end{aligned} \tag{4.12}$$

for all $x, y \in \mathbf{X}$.

We also have the following for any $x, y \in \mathbf{X}$

$$\begin{aligned} & r\left(\int_0^1 |\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|^p du\right)^{\frac{1}{p}} \\ \leq\ & r\|y-x\|\left(\int_0^1 \|(1-u)x+uy\|^{p(r-1)}du\right)^{\frac{1}{p}} \\ \leq\ & r\|y-x\|\left(\frac{\|x\|^{p(r-1)}+\|y\|^{p(r-1)}}{2}\right)^{\frac{1}{p}}, \end{aligned}$$

by the Hermite-Hadamard inequality [99, p. 106], and

$$\begin{aligned} & r\left(\int_0^1 |\|(1-u)x+uy\|^{r-2}\langle y-x,(1-u)x+uy\rangle_{s(i)}|du\right) \\ \leq\ & r\|y-x\|\left(\int_0^1 \|(1-u)x+uy\|^{r-1}du\right) \\ \leq\ & \frac{1}{2}r\|y-x\|(\|x\|^{r-1}+\|y\|^{r-1}), \end{aligned}$$

again, by the Hermite-Hadamard inequality [99, p. 106]. Therefore, we have the following inequalities

$$\left|\int_0^1 \|(1-t)x+ty\|^r dt - \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\|(1-s_i)x+s_iy\|^r\right|$$

$$\leq r\|y-x\|\begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1}h_i^2+\sum_{i=0}^{k-1}\left(\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right)^2\right] \\ \qquad \times \sup_{u\in[0,1]}\|(1-u)x+uy\|^{r-1}, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-s_i)^{p'+1}+(s_{i+1}-\alpha_{i+1})^{p'+1}\right]\right]^{\frac{1}{p'}} \\ \qquad \times\left[\int_0^1\|(1-u)x+uy\|^{p(r-1)}du\right]^{\frac{1}{p}}, \quad p>1; \\ \left[\frac{1}{2}\nu(h)+\max_{i\in\{0,\ldots,k-1\}}\left|\alpha_{i+1}-\frac{s_i+s_{i+1}}{2}\right|\right] \\ \qquad \times\int_0^1\|(1-u)x+uy\|^{r-1}du, \end{cases} \tag{4.13}$$

$$
\leq r\|y-x\| \begin{cases} \left[\frac{1}{4}\sum_{i=0}^{k-1} h_i^2 + \sum_{i=0}^{k-1}\left(\alpha_{i+1} - \frac{s_i + s_{i+1}}{2}\right)^2\right] \max\{\|x\|^{r-1}, \|y\|^{r-1}\}, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1} - s_i)^{p'+1} + (s_{i+1} - \alpha_{i+1})^{p'+1}\right]\right]^{\frac{1}{p'}} \\ \qquad \times \left[\frac{\|x\|^{p(r-1)} + \|y\|^{p(r-1)}}{2}\right]^{\frac{1}{p}}, \quad p > 1; \\ \frac{1}{2}\left[\frac{1}{2}\nu(h) + \max_{i\in\{0,\ldots,k-1\}}\left|\alpha_{i+1} - \frac{s_i+s_{i+1}}{2}\right|\right](\|x\|^{r-1} + \|y\|^{r-1}), \end{cases} \tag{4.14}
$$

which hold for any $x, y \in \mathbf{X}$. The constants in the first and second cases of (4.13) and (4.14) are sharp. The proof follows by its particular cases which are mentioned in Corollary 4.2.12.

In the next few results, we consider the particular cases of Theorem 4.2.9 (with the upper bounds as stated in Remark 4.2.10), that is, the case of trapezoidal functional. The trapezoidal functional will be employed in estimating the absolute difference of $\|(\cdot,\cdot)\|_{p-HH}$ and $\frac{1}{2^{1/p}}\|(\cdot,\cdot)\|_p$, in the next subsection. We remark that the same upper bounds hold for the case of the Ostrowski functional.

Corollary 4.2.11 (Kikianty, Dragomir and Cerone [72]). *Let* $\mathbf{X}$ *be a normed linear space,* $s \in [0,1]$ *and* $1 \leq r < \infty$. *Then, we have the inequalities*

$$
\left|\int_0^1 \|(1-t)x + ty\|^r dt - (1-s)\|x\|^r - s\|y\|^r\right|
$$

$$
\leq \quad r\|y-x\| \begin{cases} \left[\frac{1}{4} + (s-\frac{1}{2})^2\right] \sup_{u\in[0,1]} \|(1-u)x + uy\|^{r-1}, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[s^{p'+1} + (1-s)^{p'+1}\right]^{\frac{1}{p'}} \\ \qquad \times \left(\int_0^1 \|(1-u)x + uy\|^{p(r-1)} du\right)^{\frac{1}{p}}, \quad p > 1; \\ \left[\frac{1}{2} + \left|s - \frac{1}{2}\right|\right]\left(\int_0^1 \|(1-u)x + uy\|^{r-1} du\right). \end{cases} \tag{4.15}
$$

$$
\leq \quad r\|y-x\| \begin{cases} \left[\frac{1}{4} + (s-\frac{1}{2})^2\right] \max\{\|x\|^{r-1}, \|y\|^{r-1}\}, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[s^{p'+1} + (1-s)^{p'+1}\right]^{\frac{1}{p'}} \\ \qquad \times \left(\frac{\|x\|^{p(r-1)} + \|y\|^{p(r-1)}}{2}\right)^{\frac{1}{p}}, \quad p > 1; \\ \frac{1}{2}\left[\frac{1}{2} + \left|s - \frac{1}{2}\right|\right](\|x\|^{r-1} + \|y\|^{r-1}). \end{cases} \tag{4.16}
$$

for any $x, y \in \mathbf{X}$. The constants in the first and second cases of (4.15) and (4.16) are sharp.

Proof. Choose $s_0 = 0$, $s_1 = 1$ and $0 = \alpha_0 < \alpha_1 = s < \alpha_2 = 1$ in (4.13) and (4.14). The sharpness of the constants follows by the particular case which is pointed out in Corollary 4.2.12. □

Corollary 4.2.12 (Kikianty, Dragomir and Cerone [72]). *Particularly,*

$$\left|\int_0^1 \|(1-t)x+ty\|^2 dt - (1-s)\|x\|^2 - s\|y\|^2\right| \\ \leq 2\|y-x\| \begin{cases} \left[\frac{1}{4}+(s-\frac{1}{2})^2\right] \sup\limits_{u\in[0,1]} \|(1-u)x+uy\|, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[s^{p'+1}+(1-s)^{p'+1}\right]^{\frac{1}{p'}} \\ \qquad \times \left(\int_0^1 \|(1-u)x+uy\|^p du\right)^{\frac{1}{p}}, \quad p>1; \\ \left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right]\left(\int_0^1 \|(1-u)x+uy\| du\right), \end{cases} \tag{4.17}$$

$$\leq 2\|y-x\| \begin{cases} \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right] \max\{\|x\|,\|y\|\}, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\left[s^{p'+1}+(1-s)^{p'+1}\right]^{\frac{1}{p'}} \\ \qquad \times \left(\frac{\|x\|^p+\|y\|^p}{2}\right)^{\frac{1}{p}}, \quad p>1; \\ \frac{1}{2}\left[\frac{1}{2}+\left|s-\frac{1}{2}\right|\right](\|x\|+\|y\|), \end{cases} \tag{4.18}$$

for any $x, y \in \mathbf{X}$. The constants in the first and second cases of (4.17) and (4.18) are sharp.

We also have

$$\left|\int_0^1 \|(1-t)x+ty\| dt - (1-s)\|x\| - s\|y\|\right| \leq \left[\frac{1}{4}+\left(s-\frac{1}{2}\right)^2\right]\|y-x\|. \tag{4.19}$$

The constant $\frac{1}{4}$ in (4.19) is sharp.

Proof. We obtain (4.17) and (4.18) by choosing $r = 2$ in (4.15) and (4.16), respectively. The proof for the sharpness of the constants is implied by Corollary 4.2.14. By choosing

$r = 1$ in (4.15), we obtain

$$\begin{aligned} &\left| \int_0^1 \|(1-t)x + ty\| dt - (1-s)\|x\| - s\|y\| \right| \\ \leq \quad & \|y - x\| \begin{cases} \frac{1}{4} + \left(s - \frac{1}{2}\right)^2, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}} \left[s^{p'+1} + (1-s)^{p'+1}\right]^{\frac{1}{p'}}, \\ \frac{1}{2} + \left|s - \frac{1}{2}\right|, \end{cases} \end{aligned} \tag{4.20}$$

for any $x, y \in \mathbf{X}$. Note that for all $1 < p' < \infty$ and $s \in [0, 1]$,

1. $\frac{1}{4} + \left(s - \frac{1}{2}\right)^2 = \frac{s^2 + (1-s)^2}{2} = \displaystyle\int_0^1 |t - s| dt,$

2. $\frac{1}{(p'+1)^{\frac{1}{p'}}} \left[s^{p'+1} + (1-s)^{p'+1}\right]^{\frac{1}{p'}} = \left(\displaystyle\int_0^1 |t - s|^{p'} dt\right)^{\frac{1}{p'}},$

3. $\frac{1}{2} + \left|s - \frac{1}{2}\right| = \max\{s, 1-s\} = \displaystyle\sup_{t \in [0,1]} |t - s|,$

and

$$\int_0^1 |t - s| dt \leq \left(\int_0^1 |t - s|^{p'} dt\right)^{\frac{1}{p'}} \leq \sup_{t \in [0,1]} |t - s|$$

by the Hölder inequality. Thus,

$$\begin{aligned} \frac{1}{4} + \left(s - \frac{1}{2}\right)^2 &\leq \frac{1}{(p'+1)^{\frac{1}{p'}}} [s^{p'+1} + (1-s)^{p'+1}]^{\frac{1}{p'}} \\ &\leq \frac{1}{2} + \left|s - \frac{1}{2}\right|. \end{aligned} \tag{4.21}$$

We conclude that the constant $\frac{1}{4} + \left(s - \frac{1}{2}\right)^2$ is best possible amongst the constants in all cases of (4.20) and we obtain (4.19). The proof of the sharpness of the constant will be given in Corollary 4.2.14. □

4.2.2 Inequalities involving the *p*-HH-norm and the *p*-norm

In the next two results, we employ Corollaries 4.2.11 and 4.2.12 to estimate the absolute difference of $\|(\cdot, \cdot)\|_{p-HH}$ and $\frac{1}{2^{1/p}} \|(\cdot, \cdot)\|_p$.

Corollary 4.2.13 (Kikianty, Dragomir and Cerone [72]). *Let* $\mathbf{X}$ *be a normed linear space and* $1 \le r < \infty$. *Then*

$$\begin{aligned} 0 &\le \frac{\|(x,y)\|_r^r}{2} - \|(x,y)\|_{r-HH}^r \\ &\le r\|y-x\| \begin{cases} \frac{1}{4}\|(x,y)\|_\infty^{r-1}, \\ \frac{1}{2(p'+1)^{\frac{1}{p'}}}\|(x,y)\|_{p(r-1)-HH}^{r-1}, & p > 1; \\ \frac{1}{2}\|(x,y)\|_{(r-1)-HH}^{r-1}, \end{cases} \qquad (4.22) \\ &\le r\|y-x\| \begin{cases} \frac{1}{4}\|(x,y)\|_\infty^{r-1}, \\ \frac{1}{2^{\frac{1}{p}+1}(p'+1)^{\frac{1}{p'}}}\|(x,y)\|_{p(r-1)}^{r-1}, & p > 1; \\ \frac{1}{4}\|(x,y)\|_{r-1}^{r-1}, \end{cases} \qquad (4.23) \end{aligned}$$

hold for any $x, y \in \mathbf{X}$. *The constants in the first and second cases of (4.22) and (4.23) are sharp.*

Proof. Choose $s = \frac{1}{2}$ in (4.15) and (4.16). The sharpness of the constants follows by the particular case which is pointed out in Corollary 4.2.14. □

Corollary 4.2.14 (Kikianty, Dragomir and Cerone [72]). *Particularly,*

$$\begin{aligned} 0 &\le \frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 \\ &\le \|y-x\| \begin{cases} \frac{1}{2}\|(x,y)\|_\infty, \\ \frac{1}{(p'+1)^{\frac{1}{p'}}}\|(x,y)\|_{p-HH}, & p > 1; \\ \|(x,y)\|_{1-HH}, \end{cases} \qquad (4.24) \\ &\le \|y-x\| \begin{cases} \frac{1}{2}\|(x,y)\|_\infty, \\ \frac{1}{2^{\frac{1}{p}}(p'+1)^{\frac{1}{p'}}}\|(x,y)\|_p, & p > 1; \\ \frac{1}{2}\|(x,y)\|_1, \end{cases} \qquad (4.25) \end{aligned}$$

hold for any $x, y \in \mathbf{X}$. *The constants in the first and second cases of (4.24) and (4.25) are sharp. We also have*

$$0 \le \frac{\|(x,y)\|_1}{2} - \|(x,y)\|_{1-HH} \le \frac{1}{4}\|y-x\|. \qquad (4.26)$$

The constant $\frac{1}{4}$ in (4.26) is the best possible constant.

Proof. We obtain (4.24) and (4.25) by choosing $r = 2$ in (4.22) and (4.23), respectively. To prove the sharpness of the constants in the first case of (4.24) and (4.25), suppose that the inequality holds for a constant $A > 0$ instead of $\frac{1}{2}$, that is,

$$\frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 \leq A\|y-x\|\|(x,y)\|_\infty.$$

Note that it is sufficient for us to prove the sharpness of the constant in the first case of (4.24), since both quantities are equal. Choose $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$, $x = (\frac{1}{n}, n)$ and $y = (-\frac{1}{n}, n)$ for $n \in \mathbb{N}$, then we have

$$\frac{3n^2+2}{3n^2} \leq A\left(\frac{2n^2+2}{n^2}\right).$$

Taking $n \to \infty$, we obtain $1 \leq 2A$, that is, $A \geq \frac{1}{2}$.

Note that the constants in the second case of (4.24) and (4.25) are also sharp. Suppose that the inequality holds for the constants $B, C > 0$ instead of the multiplicative constant 1, that is,

$$\begin{aligned}\frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 &\leq B\frac{\|y-x\|}{(p'+1)^{\frac{1}{p'}}}\|(x,y)\|_{p-HH} \\ &\leq C\frac{\|y-x\|}{2^{\frac{1}{p}}(p'+1)^{\frac{1}{p'}}}\|(x,y)\|_p.\end{aligned}$$

Choose $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$, $x = (\frac{1}{n}, n)$ and $y = (-\frac{1}{n}, n)$ for $n \in \mathbb{N}$, then we have

$$\frac{3n^2+2}{3n^2} \leq 2B\left(\frac{(n^2(n^2+1)^p + (n^2+1)^p - n^{2p+2})^{\frac{1}{p}}}{n^2(p'+1)^{\frac{1}{p'}}(p+1)^{\frac{1}{p}}}\right) \leq C\left(\frac{2(n^2+1)}{n^2(p'+1)^{\frac{1}{p'}}}\right).$$

Taking $p' \to 1$ and $n \to \infty$, we obtain $B \geq 1$ and $C \geq 1$.

By choosing $r = 1$ in (4.22) (or (4.23)), we obtain

$$0 \leq \frac{\|(x,y)\|_1}{2} - \|(x,y)\|_{1-HH} \leq \begin{cases} \frac{1}{4}\|y-x\|, \\ \frac{1}{2(p'+1)^{\frac{1}{p'}}}\|y-x\|, \\ \frac{1}{2}\|y-x\|, \end{cases} \tag{4.27}$$

for any $x, y \in \mathbf{X}$.

Note that for any $1 < p' < \infty$, we have

$$\frac{1}{4} \leq \frac{1}{2(p'+1)^{\frac{1}{p'}}} \leq \frac{1}{2}$$

(the proof follows by choosing $s = \frac{1}{2}$ in (4.21)). Therefore, $\frac{1}{4}$ is the best possible among the constants of all cases in (4.27) and we get (4.26). Now, suppose that the inequality holds for any constant $D > 0$ instead of $\frac{1}{4}$, that is,

$$\frac{\|(x,y)\|_1}{2} - \|(x,y)\|_{1-HH} \leq D\|y-x\|.$$

Choose $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$, $x = (2,1)$ and $y = (2,-1)$ to obtain $\frac{1}{2} \leq 2D$, that is, $D \geq \frac{1}{4}$. Thus, the constant $\frac{1}{4}$ is sharp. $\square$

Remark 4.2.15 (The case of inner product space)**.** If $\mathbf{X}$ is an inner product space, the constant in the first case of (4.25) is not sharp, since

$$\frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 = \frac{1}{6}\|y-x\|^2,$$

and the fact that

$$\frac{1}{6}\|y-x\|^2 \leq \frac{1}{6}\|y-x\|(\|x\|+\|y\|) \leq \frac{1}{3}\|y-x\|\|(x,y)\|_\infty.$$

The sharpness of the constant in the second case of (4.25) is not preserved in this case, since we have the fact that

$$\frac{1}{6}\|y-x\|^2 \leq \frac{1}{6}\|y-x\|(\|x\|+\|y\|) \leq \frac{1}{3}\|y-x\|\|(x,y)\|_p,$$

and that $\frac{1}{3} \leq \frac{1}{2^{\frac{1}{p}}(q+1)^{\frac{1}{q}}}$.

The constant in the third case of (4.25) is not sharp, since

$$\frac{1}{6}\|y-x\|^2 \leq \frac{1}{6}\|y-x\|\|(x,y)\|_1.$$

The constant $\frac{1}{4}$ in (4.26) remains sharp in this case. The proof follows by choosing $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, $x = 1$ and $y = -1$.

4.3 Ostrowski inequality for convex functions on linear spaces

A generalization of the classical Ostrowski inequality by considering the class of real convex functions has been obtained in [35, 39]. The following result is a generalization of an Ostrowski type inequality in [35] for convex functions defined on linear spaces.

Theorem 4.3.1 (Kikianty, Dragomir and Cerone [73])**.** *Let* $\mathbf{X}$ *be a vector space,* $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ *be a partition of the interval* $[0,1]$*,* α_i $(i = 0, \ldots, k+1)$ *be* $k+2$ *points such that* $\alpha_0 = 0$*,* $\alpha_i \in [s_{i-1}, s_i]$*,* $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = 1$*. If* $f : [x,y] \subset \mathbf{X} \to \mathbb{R}$ *is a convex function on the segment* $[x,y]$*, then we have*

$$\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{k-1}\Big\{(s_{i+1}-\alpha_{i+1})^2\big(\nabla_+ f[(1-\alpha_{i+1})x+\alpha_{i+1}y]\big)(y-x)\\
&\qquad -(\alpha_{i+1}-s_i)^2\big(\nabla_- f[(1-\alpha_{i+1})x+\alpha_{i+1}y]\big)(y-x)\Big\}\\
\le\ & \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f[(1-s_i)x+s_iy]-\int_0^1 f[(1-t)x+ty]dt \qquad (4.28)\\
\le\ & \frac{1}{2}\sum_{i=0}^{k-1}\Big\{(s_{i+1}-\alpha_{i+1})^2\big(\nabla_- f[(1-s_{i+1})x+s_{i+1}y]\big)(y-x)\\
&\qquad -(\alpha_{i+1}-s_i)^2\big(\nabla_+ f[(1-s_i)x+s_iy]\big)(y-x)\Big\}.
\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*

Proof. Consider the function $g(t) = f[(1-t)x+ty]$ defined on $[0,1]$. Since f is a convex function on $[x,y]$, g is also convex on $[0,1]$. Therefore, we may apply Lemma 4.1.4. The right-sided and left-sided derivatives of g can be computed as follows:

$$g'_\pm(t) = \big(\nabla_\pm f[(1-t)x+ty]\big)(y-x), \quad t \in [0,1].$$

We obtained the desired result by employing (4.3) of Lemma 4.1.4 for g. The sharpness of the constants follows by some particular cases which will be considered later in Corollary 4.3.6. □

The following result is a particular case of Theorem 4.3.1 for a trapezoidal type functional.

Corollary 4.3.2 (Kikianty, Dragomir and Cerone [73]). *Let* $\mathbf{X}$ *be a vector space,* $x, y \in \mathbf{X}$, $x \neq y$ *and* $f : [x, y] \subset \mathbf{X} \to \mathbb{R}$ *be a convex function on the segment* $[x, y]$*. Then for any* $s \in (0, 1)$ *one has the inequality*

$$\begin{aligned}
& \frac{1}{2}[(1-s)^2\big(\nabla_+ f[(1-s)x+sy]\big)(y-x) - s^2\big(\nabla_- f[(1-s)x+sy]\big)(y-x)] \\
\leq\ & sf(x) + (1-s)f(y) - \int_0^1 f[(1-t)x+ty]dt \qquad (4.29) \\
\leq\ & \frac{1}{2}[(1-s)^2(\nabla_- f(y))(y-x) - s^2(\nabla_+ f(x))(y-x)].
\end{aligned}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*

Proof. The result can be obtained by choosing $k = 1$ and $s_0 = \alpha_0 = 0$, $\alpha_1 = s \in (0, 1)$ and $s_1 = \alpha_2 = 1$ in Theorem 4.3.1. The sharpness of the constants will be proven later in Corollary 4.3.6. □

An alternative proof of Corollary 4.3.2 can be found in Theorem 2.4 of Dragomir [34]. The following result provides an improvement for the second Hermite-Hadamard inequality [34].

Remark 4.3.3. By letting $s = \frac{1}{2}$ in (4.29) we have

$$\begin{aligned}
& \frac{1}{8}\left[\left(\nabla_+ f\left(\frac{x+y}{2}\right)\right)(y-x) - \left(\nabla_- f\left(\frac{x+y}{2}\right)\right)(y-x)\right] \\
\leq\ & \frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]dt \qquad (4.30) \\
\leq\ & \frac{1}{8}[(\nabla_- f(y))(y-x) - (\nabla_+ f(x))(y-x)].
\end{aligned}$$

The constant $\frac{1}{8}$ is sharp, which proof can be found in Corollary 4.3.6.

4.3.1 Application for semi-inner products

Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space. We obtain the following inequalities for the semi-inner products $\langle\cdot,\cdot\rangle_s$ and $\langle\cdot,\cdot\rangle_i$.

Corollary 4.3.4 (Kikianty, Dragomir and Cerone [73]). *Let* $I_k : 0 = s_0 < s_1 < \cdots < s_{k-1} < s_k = 1$ *be a partition of the interval* $[0, 1]$ *and* α_i $(i = 0, \ldots, k+1)$ *be* $k+2$ *points such that* $\alpha_0 = 0$, $\alpha_i \in [s_{i-1}, s_i]$, $(i = 1, \ldots, k)$ *and* $\alpha_{k+1} = 1$*. Assume that* $1 \leq p < \infty$*.*

Then,

$$
\begin{aligned}
& \frac{1}{2}p\sum_{i=0}^{k-1}\|(1-\alpha_{i+1})x+\alpha_{i+1}y\|^{p-2}[(s_{i+1}-\alpha_{i+1})^2\langle y-x,(1-\alpha_{i+1})x+\alpha_{i+1}y\rangle_s \\
& \qquad -(\alpha_{i+1}-s_i)^2\langle y-x,(1-\alpha_{i+1})x+\alpha_{i+1}y\rangle_i] \\
\leq\ & \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\|(1-s_i)x+s_iy\|^p-\int_0^1\|(1-t)x+ty\|^p dt \qquad (4.31) \\
\leq\ & \frac{1}{2}p\sum_{i=0}^{k-1}[(s_{i+1}-\alpha_{i+1})^2\|(1-s_{i+1})x+s_{i+1}y\|^{p-2}\langle y-x,(1-s_{i+1})x+s_{i+1}y\rangle_i \\
& \qquad -(\alpha_{i+1}-s_i)^2\|(1-s_i)x+s_iy\|^{p-2}\langle y-x,(1-s_i)x+s_iy\rangle_s],
\end{aligned}
$$

holds for any $x, y \in \mathbf{X}$ whenever $p \geq 2$; otherwise, it holds for linearly independent $x, y \in \mathbf{X}$.
The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. Apply Theorem 4.3.1 to the convex function $f_p(x) = \|x\|^p$, where $x \in \mathbf{X}$ and $1 \leq p < \infty$. Note the use of identity (4.4) of Example 4.2.2. The sharpness of the constants will be proven later in Corollary 4.3.6. $\square$

The following result is a particular case of Corollary 4.3.4 for a trapezoidal type functional.

Corollary 4.3.5 (Kikianty, Dragomir and Cerone [73]). *Let x and y be any two vectors in $\mathbf{X}$, $\sigma \in (0,1)$ and $1 \leq p < \infty$. Then*

$$
\begin{aligned}
& \frac{1}{2}p\|(1-\sigma)x+\sigma y\|^{p-2}[(1-\sigma)^2\langle y-x,(1-\sigma)x+\sigma y\rangle_s \\
& \qquad -\sigma^2\langle y-x,(1-\sigma)x+\sigma y\rangle_i] \\
\leq\ & \sigma\|x\|^p+(1-\sigma)\|y\|^p-\int_0^1\|(1-t)x+ty\|^p dt \qquad (4.32) \\
\leq\ & \frac{1}{2}p[(1-\sigma)^2\|y\|^{p-2}\langle y-x,y\rangle_i-\sigma^2\|x\|^{p-2}\langle y-x,x\rangle_s],
\end{aligned}
$$

holds for any $x, y \in \mathbf{X}$ whenever $p \geq 2$; otherwise, it holds for linearly independent $x, y \in \mathbf{X}$.
The constant $\frac{1}{2}$ is sharp in both inequalities.

We also have two particular cases that are of interest, namely

$$\begin{aligned}
&(1-\sigma)^2\langle y-x,(1-\sigma)x+\sigma y\rangle_s-\sigma^2\langle y-x,(1-\sigma)x+\sigma y\rangle_i\\
\leq\ &\sigma\|x\|^2+(1-\sigma)\|y\|^2-\int_0^1\|(1-t)x+ty\|^2dt \qquad (4.33)\\
\leq\ &(1-\sigma)^2\langle y-x,y\rangle_i-\sigma^2\langle y-x,x\rangle_s,
\end{aligned}$$

for any $x, y \in \mathbf{X}$ and

$$\begin{aligned}
&\frac{1}{2}\left[(1-\sigma)^2\left\langle y-x,\frac{(1-\sigma)x+\sigma y}{\|(1-\sigma)x+\sigma y\|}\right\rangle_s-\sigma^2\left\langle y-x,\frac{(1-\sigma)x+\sigma y}{\|(1-\sigma)x+\sigma y\|}\right\rangle_i\right]\\
\leq\ &\sigma\|x\|+(1-\sigma)\|y\|-\int_0^1\|(1-t)x+ty\|dt \qquad (4.34)\\
\leq\ &\frac{1}{2}\left[(1-\sigma)^2\left\langle y-x,\frac{y}{\|y\|}\right\rangle_i-\sigma^2\left\langle y-x,\frac{x}{\|x\|}\right\rangle_s\right],
\end{aligned}$$

for any linearly independent $x, y \in \mathbf{X}$. The constants in (4.33) and (4.34) are sharp.

Proof. Choose $k = 1$, $s_0 = \alpha_0 = 0$, $\alpha_1 = \sigma \in (0, 1)$ and $s_1 = \alpha_2 = 1$ in Proposition 4.3.4. As an alternative proof, this result can be obtained by choosing $f(x) = \|x\|^p$, $(1 \leq p < \infty)$ and $s = \sigma$ in Corollary 4.3.2 (note the use of identity (4.4) of Example 4.2.2). Take $p = 2$ and $p = 1$ in (4.32) to obtain (4.33) and (4.34). The sharpness of the constants will be proven later in Corollary 4.3.6 by considering some particular cases. □

We note that (4.33) and (4.34) recapture the results of Dragomir [34, Proposition 3.1 and Proposition 3.2]. However, the sharpness of the constants was not considered in the paper.

4.3.2 Inequalities involving the *p*-HH-norm and the *p*-norm

The main result of this section follows from Section 4.3.1. Furthermore, it provides an improvement for the Hermite-Hadamard inequalities (1.11).

The next corollary gives a generalization of the results by Dragomir [34]. We also state some particular cases which recapture the results by Dragomir [34]. However, the sharpness of the constants in these inequalities were not considered. Here, we provide the proof for the sharpness of these inequalities.

Corollary 4.3.6 (Kikianty, Dragomir and Cerone [73]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space and $1 \le p < \infty$. Then,*

$$\begin{aligned} 0 &\le \frac{1}{8}p\left\|\frac{y+x}{2}\right\|^{p-2}\left[\left\langle y-x, \frac{y+x}{2}\right\rangle_s - \left\langle y-x, \frac{y+x}{2}\right\rangle_i\right] \\ &\le \frac{\|(x,y)\|_p^p}{2} - \|(x,y)\|_{p-HH}^p \\ &\le \frac{1}{8}p[\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s], \end{aligned} \tag{4.35}$$

holds for any $x, y \in \mathbf{X}$ whenever $p \ge 2$; otherwise, it holds for linearly independent $x, y \in \mathbf{X}$.

In particular, we have

$$\begin{aligned} 0 &\le \frac{1}{8}[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i] \\ &\le \frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 \\ &\le \frac{1}{4}[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s], \end{aligned} \tag{4.36}$$

for any $x, y \in \mathbf{X}$ and

$$\begin{aligned} 0 &\le \frac{1}{8}\left[\left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_s - \left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_i\right] \\ &\le \frac{\|(x,y)\|_1}{2} - \|(x,y)\|_{1-HH} \\ &\le \frac{1}{8}\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right], \end{aligned} \tag{4.37}$$

for any linearly independent $x, y \in \mathbf{X}$.
The constants in (4.35), (4.36) and (4.37) are sharp.

Proof. We obtain (4.35) by taking $\sigma = \frac{1}{2}$ in (4.32). We may also obtain (4.35) by taking $f(x) = \|x\|^p$ $(1 \le p < \infty)$ in Remark 4.3.3. We get (4.36) and (4.37) by taking $p = 2$ and $p = 1$, respectively, in (4.35).

Note that we may also obtain (4.36) from (4.33) of Corollary 4.3.5 and (4.37) from (4.34) of Corollary 4.3.5, by letting $\sigma = \frac{1}{2}$. The sharpness of the constants in (4.35) follows by the sharpness of the constants in (4.36) and (4.37) as its particular cases.

To prove the sharpness of the constants in (4.36), we assume that the above inequality holds for constants $E, F > 0$ instead of $\frac{1}{8}$ and $\frac{1}{4}$, respectively. We have

$$\begin{aligned} 0 \le E[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i] &\le \frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 \\ &\le F[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s]. \end{aligned}$$

Miličić [87] (cf. Dragomir [37]) computed the superior and inferior semi-inner products in the space ℓ^1, as follows:

$$\langle x, y\rangle_{s(i)} = \|y\|_{\ell^1}\left(\sum_{y_i\neq 0}\frac{y_i}{|y_i|}x_i \pm \sum_{y_i=0}|x_i|\right),$$

for any $x = (x_i) \in \ell^1$ and $y = (y_i) \in \ell^1$. By taking $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$, we have the inequality

$$\begin{aligned} &2E\|y+x\|_1 \sum_{y_i+x_i=0}|y_i - x_i| \\ \le\ & \frac{\|x\|_{\ell^1}^2 + \|y\|_{\ell^1}^2}{2} - \int_0^1 \|(1-t)x + ty\|_{\ell^1}^2 dt \\ \le\ & F\Bigg[\|y\|_1\left(\sum_{y_i\neq 0}\frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i=0}|y_i - x_i|\right) \\ &\quad -\|x\|_1\left(\sum_{x_i\neq 0}\frac{x_i}{|x_i|}(y_i - x_i) + \sum_{x_i=0}|y_i - x_i|\right)\Bigg]. \end{aligned}$$

By taking $x = \left(-\frac{1}{n}, n\right)$ and $y = \left(\frac{1}{n}, n\right)$, for any $n \in \mathbb{N}$, then we have the following:

$$8E \le \frac{3n^2+2}{3n^2} \le 4F\left(\frac{1}{n^2}+1\right).$$

Allowing $n \to \infty$, we get

$$8E \le 1 \le 4F,$$

that is, $E \le \frac{1}{8}$ in the first inequality and $F \ge \frac{1}{4}$ in the second inequality. Thus, the constants $\frac{1}{8}$ and $\frac{1}{4}$ are sharp in the first and second inequality, respectively.

To prove the sharpness of the constants in (4.37), we assume that the above inequality holds for constants $G, H > 0$ instead of $\frac{1}{8}$, that is

$$\begin{aligned} 0 &\le G\left[\left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_s - \left\langle y-x, \frac{\frac{y+x}{2}}{\|\frac{y+x}{2}\|}\right\rangle_i\right] \\ &\le \frac{\|(x,y)\|_1}{2} - \|(x,y)\|_{1-HH} \\ &\le H\left[\left\langle y-x, \frac{y}{\|y\|}\right\rangle_i - \left\langle y-x, \frac{x}{\|x\|}\right\rangle_s\right]. \end{aligned}$$

Again, we choose $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$. Therefore we have the following inequalities

$$\begin{aligned} 0 &\le 2G \sum_{y_i+x_i=0} |y_i - x_i| \\ &\le \frac{\|x\|_{\ell^1} + \|y\|_{\ell^1}}{2} - \int_0^1 \|(1-t)x + ty\|_{\ell^1} dt \\ &\le H\left[\sum_{y_i \neq 0} \frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i=0} |y_i - x_i| - \sum_{x_i \neq 0} \frac{x_i}{|x_i|}(y_i - x_i) - \sum_{x_i=0} |y_i - x_i|\right], \end{aligned}$$

for any linearly independent x and y. Let $x = (1, 0)$ and $y = (-1, 1)$. Clearly x and y are linearly independent. Therefore the above inequality holds for these vectors. We have

$$4G \le \frac{1}{2} \le 4H,$$

that is, $G \le \frac{1}{8}$ in the first inequality and $H \ge \frac{1}{8}$ in the second inequality. Thus, the constant $\frac{1}{8}$ is sharp in both inequalities. □

Remark 4.3.7 (The case of inner product spaces)**.** Let $\mathbf{X}$ be an inner product space, with the inner product $\langle \cdot, \cdot \rangle$. Then, by Corollary 4.3.6, we have

$$\begin{aligned} 0 &\le \frac{\|(x,y)\|_p^p}{2} - \|(x,y)\|_{p-HH}^p \\ &\le \frac{1}{8} p \langle y - x, y\|y\|^{p-2} - x\|x\|^{p-2} \rangle, \end{aligned} \tag{4.38}$$

holds for any $x, y \in \mathbf{X}$ whenever $p \ge 2$; otherwise, it holds for nonzero $x, y \in \mathbf{X}$.

Particularly, for $p = 2$, we have

$$0 \le \frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 \le \frac{1}{4}[\langle y - x, y - x \rangle] = \frac{1}{4}\|y - x\|^2,$$

for any $x, y \in \mathbf{X}$. The constant $\frac{1}{4}$ is not the best possible constant in this case, since we always have

$$\frac{\|(x,y)\|_2^2}{2} - \|(x,y)\|_{2-HH}^2 = \frac{1}{6}\|y-x\|^2.$$

If $p = 1$, then

$$0 \leq \frac{\|(x,y)\|_1}{2} - \|(x,y)\|_{1-HH} \leq \frac{1}{8}\left\langle y - x, \frac{y}{\|y\|} - \frac{x}{\|x\|}\right\rangle,$$

for any nonzero $x, y \in \mathbf{X}$. We obtain a nontrivial equality by choosing $\mathbf{X} = \mathbb{R}$ and multiplication as its inner product (which induces the absolute value for its norm), $x = 1$ and $y = -1$. Thus, the constant $\frac{1}{8}$ is sharp.

Conjecture 4.3.8. We conjecture that the constant $\frac{1}{8}$ in (4.38) is not sharp for any $p > 1$. Utilizing MAPLE for the real-valued functions

$$\begin{aligned} F_p(x,y) &:= \frac{|x|^p + |y|^p}{2} - \int_0^1 |(1-t)x + ty|^p dt, \\ G_p(x,y) &:= \frac{1}{8}p(y-x)(y|y|^{p-2} - x|x|^{p-2}), \end{aligned}$$

for $(x, y) \in \mathbb{R}^2$, we observe that for several values of $p > 1$, the equation $F_p(x, y) = G_p(x, y) = k \neq 0$ has no solution in $\mathbb{R}^2$. Again, we utilize MAPLE to plot these functions with the choice of $p = 3$, as may be seen in Figure 4.1. One may see that the two surfaces in Figure 4.1 intersect on the level set $k = 0$. Therefore, the constant $\frac{1}{8}$ is not sharp for these values of p, since we have no nontrivial equality. However, we do not have an analytical proof for this claim.

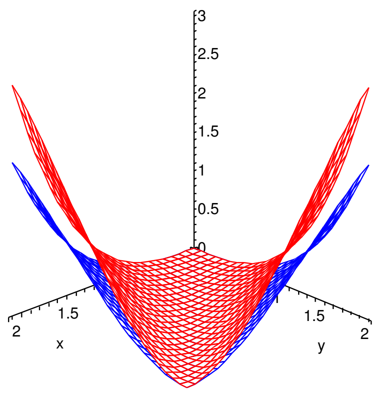


Figure 4.1: Plot of F_3 and G_3

4.4 Comparison analysis

We want to compare the two bounds that have been obtained in Sections 4.2 and 4.3, that is,

$$\frac{1}{8}r(\langle y-x, y\|y\|^{r-2}\rangle_i - \langle y-x, x\|x\|^{r-2}\rangle_s) \text{ and } \frac{1}{4}r\|y-x\|\max\{\|x\|^{r-1}, \|y\|^{r-1}\}.$$

The bound that is obtained in Section 4.2 is simpler in the sense that it only involves the given norm, while the bound in Section 4.3 involves not only the given norm, but also the superior and inferior semi-inner products associated with the norm. However, the bound in Section 4.3 is proven to be better than that of Section 4.2, when $\mathbf{X}$ is an inner product space for $r=1$ and $r=2$. The verification is presented in the following. Let $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ be an inner product space.

Case of r = 1. We wish to compare

$$\frac{1}{8}\left\langle y-x, \frac{y}{\|y\|}-\frac{x}{\|x\|}\right\rangle \quad \text{and} \quad \frac{1}{4}\|y-x\|,$$

for nonzero $x, y \in \mathbf{X}$. To do so, we employ the Dunkl-Williams inequality [49, p. 53] (cf. Kirk and Smiley [76, p. 890] and Mercer [86, p. 448])

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|}, \tag{4.39}$$

which holds for nonzero x and y in an inner product space $\mathbf{X}$. Now, for $x, y \in \mathbf{X}$ where $x, y \neq 0$, we have

$$\begin{aligned}
\frac{1}{8}\left\langle y-x, \frac{y}{\|y\|}-\frac{x}{\|x\|}\right\rangle &\leq \frac{1}{8}\|y-x\|\left\|\frac{y}{\|y\|}-\frac{x}{\|x\|}\right\| \\
&\leq \frac{1}{4}\|y-x\|\frac{\|x-y\|}{\|x\|+\|y\|} \\
&\leq \frac{1}{4}\|y-x\|\frac{\|x\|+\|y\|}{\|x\|+\|y\|} = \frac{1}{4}\|y-x\|.
\end{aligned}$$

We conclude that the bound in Section 4.3 is sharper.

Case of r = 2. We want to compare

$$\frac{1}{4}\langle y-x, y-x\rangle = \frac{1}{4}\|y-x\|^2 \quad \text{and} \quad \frac{1}{2}\|y-x\|\max\{\|x\|, \|y\|\}.$$

For all $x, y \in \mathbf{X}$, we have

$$\frac{1}{4}\|y-x\|^2 \leq \frac{1}{4}\|y-x\|(\|x\|+\|y\|) \leq \frac{1}{2}\|y-x\| \max\{\|x\|, \|y\|\}.$$

We conclude that the bound in Section 4.3 is sharper.

Case of $\mathbf{1 < r < \infty, r \neq 2}$. We conjecture that:

Conjecture 4.4.1. In an inner product space $(\mathbf{X}, \langle\cdot,\cdot\rangle)$, the following inequality

$$\frac{1}{8}r\langle y-x, y\|y\|^{r-2} - x\|x\|^{r-2}\rangle \leq \frac{1}{4}r\|y-x\| \max\{\|x\|^{r-1}, \|y\|^{r-1}\}$$

holds for any $x, y \in \mathbf{X}$ whenever $r \geq 2$; otherwise it holds for any nonzero $x, y \in \mathbf{X}$.

We observe that the above statement is true in some cases. Taking $\mathbf{X} = \mathbb{R}$ and multiplication as its inner product and utilizing MAPLE for the following functions

$$\Phi(x,y) \ := \ \frac{1}{4}r|y-x| \max\{|x|^{r-1}, |y|^{r-1}\} - \frac{1}{8}r(y-x)(y|y|^{r-2} - x|x|^{r-2}),$$

for $x, y \in \mathbb{R}$, we observe that for several values of r, we have $\Phi(x,y) \geq 0$ for any $x, y \in \mathbf{X}$. Figure 4.2 gives us the plot of Φ with the choice of $r = 3$. However, we have no analytical proof for this statement.

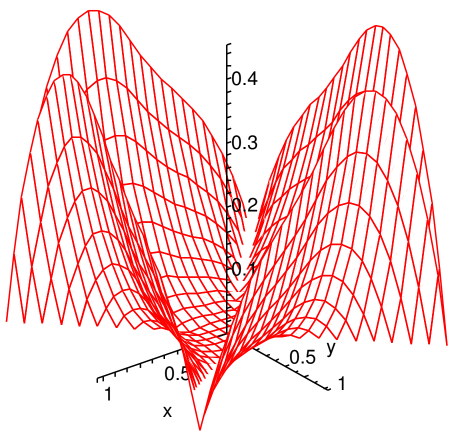


Figure 4.2: Plot of Φ for $r = 3$.

Conjecture 4.4.2. In a normed linear space $(\mathbf{X}, \|\cdot\|)$, the following inequality

$$\frac{1}{8}r(\langle y-x, y\|y\|^{r-2}\rangle_i - \langle y-x, x\|x\|^{r-2}\rangle_s) \leq \frac{1}{4}r\|y-x\| \max\{\|x\|^{r-1}, \|y\|^{r-1}\}$$

holds for any $x, y \in \mathbf{X}$ whenever $r \geq 2$; otherwise it holds for any nonzero $x, y \in \mathbf{X}$ (here, $\langle\cdot,\cdot\rangle_{s(i)}$ is the superior (inferior) semi-inner product with respect to the norm $\|\cdot\|$).

We observe that the above statement is true in some cases. In $(\mathbf{X}, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$, consider the case of $r = 1$, we have the following functions

$$f(x,y) = \frac{1}{8}\left[\sum_{y_i \neq 0} \frac{y_i}{|y_i|}(y_i - x_i) - \sum_{y_i=0} |y_i - x_i| - \sum_{x_i \neq 0} \frac{x_i}{|x_i|}(y_i - x_i) - \sum_{x_i=0} |y_i - x_i|\right],$$

and

$$g(x,y) = \frac{1}{4}\|y-x\|_1,$$

for $x, y \in \mathbb{R}^2$. We observe that $f(x,y) \leq g(x,y)$ for some $x, y \in \mathbb{R}^2$. We choose $x = (1, 0)$ and $y = (a, b)$ $(a, b \neq 0)$ and plot the nonnegative function $\Psi(a,b) := g(x,y) - f(x,y) = \frac{1}{4}(|a-1| + |b|) - \frac{1}{8}\left(\frac{a(a-1)}{|a|} + \frac{b^2}{|b|} - (a-1) - |b|\right)$ in Figure 4.3. However, we do not have an analytical proof for this statement.

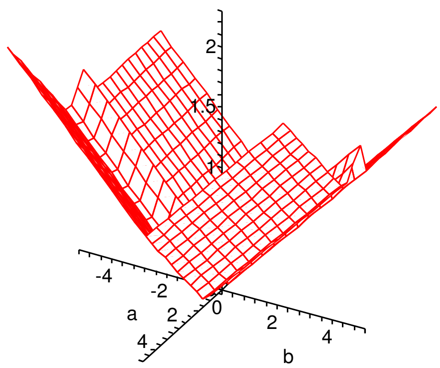


Figure 4.3: Plot of Ψ

Chapter 5

Grüss type inequality involving the p-HH-norms

In Chapter 4, we provide some inequalities of Ostrowski type, which involve the p-HH-norms and the p-norms. In the same spirit, we continue the work in this chapter, in considering some bounds to estimate the difference of $\|(\cdot,\cdot)\|_{p+q-HH}^{p+q}$ and the product $\|(\cdot,\cdot)\|_{p-HH}^{p}\|(\cdot,\cdot)\|_{q-HH}^{q}$ for any $p,q \geq 1$. This difference, however, is a particular type of *Čebyšev functional*. The results in this chapter are mainly taken from the author's research paper with Dragomir and Cerone [74].

In Section 5.1, we recall some known results regarding the Čebyšev functional. These results are then applied to obtain upper bounds in estimating the Čebyšev difference of $\|(x,y)\|_{p+q-HH}^{p+q}$ and $\|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q}$ $(p,q \geq 1)$, in Section 5.2. Some of these inequalities are proven to be sharp.

In Section 5.3, we establish some sharp bounds for the generalized Čebyšev functional in order to approximate the Riemann-Stieltjes integral for differentiable convex integrand and monotonically increasing integrator. The result follows by utilizing an Ostrowski type inequality.

We apply this result in Section 5.4, for the Čebyšev functional; and the obtained bounds are sharp. A similar result is established for a general convex function with the obtained bounds are shown to be also sharp. By applying the result for the p-HH-norms, we also procure some upper and lower bounds for the difference of $\|(x,y)\|_{p+q-HH}^{p+q}$ and $\|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q}$ $(p,q \geq 1)$. These bounds are proven to be sharp.

5.1 Grüss inequality and Čebyšev functional

We start by considering the following definition.

Definition 5.1.1 (Hardy, Littlewood and Pólya [58])**.** Let f and g be two real-valued functions defined on an interval I. We say that f and g are *similarly ordered*, if and only if

$$[f(t) - f(s)][g(t) - g(s)] \geq 0$$

for any $t, s \in I$; and *oppositely ordered* if the inequality is always reversed.

Let $f, g : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions. If f and g are similarly ordered, then

$$\int_a^b f(t)g(t)dt \geq \frac{1}{b-a} \int_a^b f(t)dt \int_a^b g(t)dt. \tag{5.1}$$

The inequality is reversed when f and g are oppositely ordered. Equality occurs when f and g are constants. Inequality (5.1) is known in the literature as the *Čebyšev inequality* [91, p. 239].

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the *Čebyšev functional* (the Čebyšev difference)

$$T(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{b-a} \int_a^b f(t)dt \cdot \frac{1}{b-a} \int_a^b g(t)dt.$$

In 1935, Grüss proved the following inequality which bounds the Čebyšev functional [91, p. 295–296]:

$$|T(f, g)| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{5.2}$$

provided that f and g satisfy the conditions

$$\phi \leq f(t) \leq \Phi \quad \text{and} \quad \gamma \leq g(t) \leq \Gamma \quad \text{for all } t \in [a, b].$$

The constant $\frac{1}{4}$ is best possible and is achieved for

$$f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right).$$

Some related results regarding the sharp upper bounds for this functional can be summarized as follows:

1. Čebyšev (1882): If f, g are continuously differentiable functions on $[a, b]$, then
$$|T(f,g)| \leq \frac{1}{12}\|f'\|_{L^\infty}\|g'\|_{L^\infty}(b-a)^2. \tag{5.3}$$
Equality holds if and only if f' and g' are constants [91, p. 297]. It is also valid for absolutely continuous functions f, g where $f', g' \in L^\infty[a, b]$.

2. Ostrowski (1970): Suppose that f is Lebesgue integrable on $[a, b]$ and $m, M \in \mathbb{R}$ such that $-\infty \leq m \leq f \leq M \leq \infty$. If g is absolutely continuous and $g' \in L^\infty[a, b]$, then
$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{L^\infty}, \tag{5.4}$$
and the constant $\frac{1}{8}$ is the best possible [91, p. 300].

3. Lupaş (1973): If f, g are absolutely continuous, $f', g' \in L^2[a, b]$, then
$$|T(f,g)| \leq \frac{1}{\pi^2}(b-a)\|f'\|_{L^2}\|g'\|_{L^2}. \tag{5.5}$$
Equality valid if and only if
$$f(x) = A + B\sin\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right]$$
and
$$g(x) = C + D\sin\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right],$$
where A, B, C and D are constants [91, p. 301].

These results are applied in Section 5.2 to obtain upper bounds in estimating the Čebyšev difference $\|(x,y)\|_{p+q-HH}^{p+q}$ and $\|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q}$, where $p, q \geq 1$ and (x, y) in the Cartesian space $\mathbf{X}^2$ of the normed space $\mathbf{X}$. We refer to the work by Dragomir and Fedotov [40] for more results regarding the Čebyšev functional.

In order to approximate the Riemann-Stieltjes integral, the generalized Čebyšev functional
$$D(f,u) := \int_a^b f(t)du(t) - \frac{1}{b-a}[u(b)-u(a)]\int_a^b f(s)ds$$

is employed, where f is Riemann integrable and Stieltjes integrable with respect to a function u. Some bounds for D, when u is monotonically non-decreasing, were obtained by Dragomir in 2004 [36]. The results are summarized as follows:

1. If $f : [a, b] \to \mathbb{R}$ is L-Lipschitzian on $[a, b]$, then

$$\begin{aligned} |D(f, u; a, b)| &\leq \frac{1}{2} L(b-a)[u(b) - u(a) - K(u; a, b)] \\ &\leq \frac{1}{2} L(b-a)[u(b) - u(a)], \end{aligned}$$

   where
$$K(u; a, b) := \frac{4}{(b-a)^2} \int_a^b u(t) \left(t - \frac{a+b}{2}\right) dt \geq 0.$$

   The constant $\frac{1}{2}$ is best possible in both inequalities.

2. If $f : [a, b] \to \mathbb{R}$ is a function of bounded variation on $[a, b]$ and $\int_a^b f(t) du(t)$ exists, then

$$\begin{aligned} |D(f, u; a, b)| &\leq [u(b) - u(a) - Q(u; a, b)] \bigvee_a^b (f) \\ &\leq [u(b) - u(a)] \bigvee_a^b (f), \end{aligned}$$

   where $\bigvee_a^b (f)$ is the total variation of f on $[a, b]$ and

$$Q(u; a, b) := \frac{1}{b-a} \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \geq 0.$$

   The first inequality is sharp.

In Section 5.3, we establish some sharp bounds for D for the case of differentiable convex integrand and monotonically increasing integrator. Furthermore, the results is employed to bound the Čebyšev difference $\|(x, y)\|_{p+q-HH}^{p+q} - \|(x, y)\|_{p-HH}^{p} \|(x, y)\|_{q-HH}^{q}$.

5.2 Inequalities involving the p-HH-norms

In this section, we obtain some norm inequalities involving the p-HH-norms. The following lemma is a norm inequality of Čebyšev type.

Lemma 5.2.1 (Kikianty, Dragomir and Cerone [74]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space,* $x, y \in \mathbf{X}$ *and* $p, q \geq 1$. *Then,*

$$\|(x,y)\|_{p+q-HH}^{p+q} \geq \|(x,y)\|_{p-HH}^{p} \|(x,y)\|_{q-HH}^{q}. \tag{5.6}$$

Equality holds in (5.6) for $x = y$.

Proof. Define $f_p(t) := \|(1-t)x + ty\|^p$, where $t \in [0,1]$. We claim that for any $p, q \geq 1$, f_p and f_q are similarly ordered on $[0,1]$. The proof is as follows: let $t, s \in [0,1]$ and assume that $f_1(t) \leq f_1(s)$ (as for the other case, the proof follows similarly). Since $f_1(t) \geq 0$ for any $t \in [0,1]$, it implies that $f_p(t) \leq f_p(s)$, for any $p \geq 1$. Thus, for any $t, s \in [0,1]$ and $p, q \geq 1$, we have

$$[f_p(t) - f_p(s)]\,[f_q(t) - f_q(s)] \geq 0.$$

Since f and g are similarly ordered, the Čebyšev inequality holds (cf. Hardy, Littlewood and Polya [58, p. 43]), that is,

$$\int_0^1 f_p(t) f_q(t) dt \geq \int_0^1 f_p(t) dt \int_0^1 f_q(t) dt,$$

or, equivalently,

$$\|(x,y)\|_{p+q-HH}^{p+q} \geq \|(x,y)\|_{p-HH}^{p} \|(x,y)\|_{q-HH}^{q},$$

as desired. It is easily shown that equality holds for $x = y$. □

In the next result, we employ the result by Čebyšev's (5.3) in bounding the difference of $\|(\cdot,\cdot)\|_{p+q-HH}^{p+q}$ and $\|(\cdot,\cdot)\|_{p-HH}^{p} \|(\cdot,\cdot)\|_{q-HH}^{q}$.

Theorem 5.2.2 (Kikianty, Dragomir and Cerone [74]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed linear space,* $x, y \in \mathbf{X}$, $p, q \geq 1$ *and set*

$$T_{p,q}(x,y) := \|(x,y)\|_{p+q-HH}^{p+q} - \|(x,y)\|_{p-HH}^{p} \|(x,y)\|_{q-HH}^{q} \geq 0.$$

Then,

$$0 \leq T_{p,q}(x,y) \leq \frac{1}{12} pq \|y - x\|^2 \max\{\|x\|, \|y\|\}^{p+q-2} =: C_{p,q}(x,y). \tag{5.7}$$

The constant $\frac{1}{12}$ *in (5.7) is sharp.*

Proof. Let $x, y \in \mathbf{X}$ and define

$$f(t) = \|(1-t)x + ty\|^p \quad \text{and} \quad g(t) = \|(1-t)x + ty\|^q,$$

for $t \in [0,1]$. Since both f and g are convex, they are absolutely continuous; also f' and g' exist almost everywhere on $[0,1]$. Therefore,

$$\begin{aligned} f'(t) &= \big(\nabla_{\pm}\|\cdot\|^p[(1-t)x + ty]\big)(y-x) \\ &= p\|(1-t)x + ty\|^{p-2}\langle y-x, (1-t)x + ty\rangle_{s(i)}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{s(i)}$ is the superior (inferior) semi-inner product (cf. identity (4.4) of Example 4.2.2) and by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|f'\|_{L^\infty} &= \sup_{t\in[0,1]} p\|(1-t)x + ty\|^{p-2}|\langle y-x, (1-t)x + ty\rangle_{s(i)}| \\ &\leq p\|y-x\| \sup_{t\in[0,1]} \|(1-t)x + ty\|^{p-1} \\ &= p\|y-x\| \max\{\|x\|, \|y\|\}^{p-1}. \end{aligned}$$

Similarly for g, we have $\|g'\|_{L^\infty} \leq q\|y-x\| \max\{\|x\|, \|y\|\}^{q-1}$. Due to Čebyšev's result (5.3), we thus conclude

$$\begin{aligned} T_{p,q}(x,y) &\leq \frac{1}{12}\|f'\|_{L^\infty}\|g'\|_{L^\infty} \\ &\leq \frac{1}{12}pq\|y-x\|^2 \max\{\|x\|, \|y\|\}^{p+q-2}. \end{aligned}$$

To prove the sharpness of the constant, we assume that inequality (5.7) holds for a constant $A > 0$ instead of $\frac{1}{12}$, that is,

$$\begin{aligned} &\|(x,y)\|_{p+q-HH}^{p+q} - \|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q} \\ \leq\ & A\, pq\|y-x\|^2 \max\{\|x\|, \|y\|\}^{p+q-2}. \end{aligned}$$

Choose $p = 1$, $q = 1$, $\mathbf{X} = \mathbb{R}$ and $0 < x < y$, to obtain

$$\frac{1}{3}\left(x^2 + xy + y^2\right) - \left(\frac{y+x}{2}\right)^2 = \frac{1}{12}(y-x)^2 \leq A(y-x)^2.$$

Since $x \neq y$, $A \geq \frac{1}{12}$ which completes the proof. $\square$

Similarly to Theorem 5.2.2, we employ the results by Grüss, Ostrowski and Lupaş to obtain more norm inequalities. We start by setting the following quantities for $x, y \in (\mathbf{X}, \|\cdot\|)$ and $p, q \geq 1$:

$$\begin{aligned}
G_{p,q}(x,y) &:= \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q} \\
O_{p,q}(x,y) &:= \frac{1}{8}q\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1}, \quad \text{and} \\
L_{p,q}(x,y) &:= \frac{1}{\pi^2}pq\|y-x\|^2\|(x,y)\|^{p-1}_{(2p-2)-HH}\|(x,y)\|^{q-1}_{(2q-2)-HH}.
\end{aligned}$$

The following proposition is due to the results by Grüss, Ostrowski and Lupaş. However, the sharpness of these inequalities are yet to be proven.

Proposition 5.2.3 (Kikianty, Dragomir and Cerone [74])**.** *Under the assumptions of Theorem 5.2.2 and the above notation, we have*

$$\begin{aligned}
0 \leq T_{p,q}(x,y) &\leq G_{p,q}(x,y), \\
0 \leq T_{p,q}(x,y) &\leq O_{p,q}(x,y), \ \textit{and} \\
0 \leq T_{p,q}(x,y) &\leq L_{p,q}(x,y),
\end{aligned}$$

for any $p, q \geq 1$ and $x, y \in \mathbf{X}$.

Proof. Let $x, y \in \mathbf{X}$ and define $f(t) = \|(1-t)x + ty\|^p$ and $g(t) = \|(1-t)x + ty\|^q$, for $t \in [0,1]$. Since $p, q \geq 1$, we have

$$0 \leq f(t) \leq \max\{\|x\|,\|y\|\}^p \quad \text{and} \quad 0 \leq g(t) \leq \max\{\|x\|,\|y\|\}^q.$$

Then, due to the result by Grüss (5.2), we have

$$T_{p,q}(x,y) \leq \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q} = G_{p,q}(x,y).$$

Since g is absolutely continuous, $\|g'\|_{L^\infty} \leq q\|y-x\|\max\{\|x\|,\|y\|\}^{q-1}$ (cf. proof of Theorem 5.2.2). Then, due to Ostrowski's result (5.4), we have

$$\begin{aligned}
T_{p,q}(x,y) &\leq \frac{1}{8}\max\{\|x\|,\|y\|\}^p\|g'\|_{L^\infty} \\
&\leq \frac{1}{8}q\|y-x\|\max\{\|x\|,\|y\|\}^{p+q-1} \\
&= O_{p,q}(x,y).
\end{aligned}$$

Note that for any $p \geq 1$, we have

$$\begin{aligned}\|f'\|_{L^2} &= \left[\int_0^1 |p\|(1-t)x+ty\|^{p-2}\langle y-x,(1-t)x+ty\rangle_{s(i)}|^2 dt\right]^{\frac{1}{2}} \\ &\leq p\,\|y-x\| \left[\int_0^1 \|(1-t)x+ty\|^{2p-2}dt\right]^{\frac{1}{2}} \\ &= p\,\|y-x\|\ \|(x,y)\|^{p-1}_{(2p-2)-HH}\end{aligned}$$

by the Cauchy-Schwarz inequality; and similarly for $q \geq 1$, we have

$$\|g'\|_{L^2} \leq q\|y-x\|\ \|(x,y)\|^{q-1}_{(2q-2)-HH}.$$

Therefore, by the result by Lupaş (5.5), we obtain

$$\begin{aligned}T_{p,q}(x,y) &\leq \frac{1}{\pi^2}\|f'\|_{L^2}\|g'\|_{L^2} \\ &\leq \frac{1}{\pi^2}pq\|y-x\|^2\|(x,y)\|^{p-1}_{(2p-2)-HH}\|(x,y)\|^{q-1}_{(2q-2)-HH} \\ &= L_{p,q}(x,y).\end{aligned}$$

This completes the proof. □

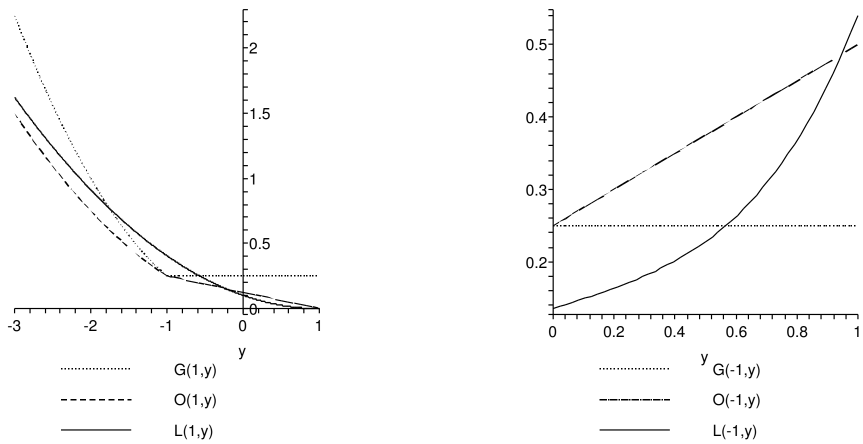
Remark 5.2.4. We note that none of the upper bounds for $T_{p,q}(x,y)$ that we have obtained in Proposition 5.2.3 is better than the other ones, for each $x,y \in \mathbf{X}$. For example, choose $\mathbf{X} = \mathbb{R}$, $p = q = 1$ and $x = 1$.

By utilizing MAPLE, we obtain (cf. Figure 5.1(a))

$$\begin{aligned}G(1,y) &\geq O(1,y) \geq L(1,y), \quad y \in [0,1], \\ G(1,y) &\geq L(1,y) \geq O(1,y), \quad y \in [-3,-2], \\ L(1,y)) &\geq G(1,y) \geq O(1,y), \quad y \in [-\tfrac{3}{2},-1].\end{aligned}$$

Again, by employing MAPLE, for $p = q = 2$ and $x = -1$, we have (cf. Figure 5.1(b))

$$\begin{aligned}O(-1,y) &\geq L(-1,y) \geq G(-1,y), \quad y \in [\tfrac{3}{5},\tfrac{4}{5}], \\ O(-1,y) &\geq G(-1,y)(x,y) \geq L(-1,y), \quad y \in [0,\tfrac{2}{5}], \\ L(-1,y) &\geq O(-1,y)(x,y) \geq G(-1,y), \quad y \in [\tfrac{19}{20},1].\end{aligned}$$



(a) Case of $p = q = 1$ (b) Case of $p = q = 2$

Figure 5.1: Upper bounds for $T_{p,q}$

Open Problem 5.2.5. Are the constants $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{\pi^2}$ in Proposition 5.2.3 the best possible?

5.3 New bounds for the generalized Čebyšev functional D

The following result gives upper and lower bounds for the generalized Čebyšev functional $D(\cdot,\cdot)$ in order to approximate the Riemann-Stieltjes integral.

Theorem 5.3.1 (Kikianty, Dragomir and Cerone [74])**.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function and $u : [a,b] \to \mathbb{R}$ be a monotonically increasing function. Then,*

$$\begin{aligned}\frac{(b-a)}{2}[f'(a)u(b) + f'(b)u(a)] &- \int_a^b u(t)\left[\frac{t-a}{b-a}f'(a) + \frac{b-t}{b-a}f'(b)\right]dt \\ &\leq D(f,u) \\ &\leq \int_a^b \left(t - \frac{b+a}{2}\right) f'(t)du(t).\end{aligned} \tag{5.8}$$

The constants $\frac{1}{2}$ and 1 in (5.8) are sharp.

Proof. Since f is a differentiable convex function on $[a,b]$, we have the following Ostrowski type inequality [35]

$$\begin{aligned} \frac{1}{2}\left[(b-t)^2-(t-a)^2\right]f'(t) &\le \int_a^b f(s)ds-(b-a)f(t) \\ &\le \frac{1}{2}\left[(b-t)^2 f'(b)-(t-a)^2 f'(a)\right], \end{aligned} \tag{5.9}$$

for any $t\in[a,b]$. By assumption, u is increasing on $[a,b]$, which enables us to integrate (5.9), in the Riemann-Stieltjes sense, with respect to u, that is,

$$\begin{aligned} &\frac{1}{2}\int_a^b \left[(b-t)^2-(t-a)^2\right]f'(t)du(t) \\ \le\ & \int_a^b \left[\int_a^b f(s)ds-(b-a)f(t)\right]du(t) \\ \le\ & \frac{1}{2}\int_a^b \left[(b-t)^2 f'(b)-(t-a)^2 f'(a)\right]du(t). \end{aligned} \tag{5.10}$$

Now, the three terms in (5.10) may be developed as,

$$\frac{1}{2}\int_a^b \left[(b-t)^2-(t-a)^2\right]f'(t)du(t)=(b-a)\int_a^b\left(\frac{b+a}{2}-t\right)f'(t)du(t),$$

and

$$\begin{aligned} &\int_a^b \left[\int_a^b f(s)ds-(b-a)f(t)\right]du(t) \\ =\ & \int_a^b f(s)ds\int_a^b du(t)-(b-a)\int_a^b f(t)du(t) \\ =\ & [u(b)-u(a)]\int_a^b f(s)ds-(b-a)\int_a^b f(t)du(t), \end{aligned}$$

and also, using integration by parts

$$\begin{aligned} &\frac{1}{2}\int_a^b \left[(b-t)^2 f'(b)-(t-a)^2 f'(a)\right]du(t) \\ =\ & \frac{1}{2}(b-a)^2[-f'(b)u(a)-f'(a)u(b)]+\int_a^b u(t)[(b-t)f'(b)+(t-a)f'(a)]dt. \end{aligned}$$

Therefore, by (5.10) we get

$$\begin{aligned}
&(b-a)\int_a^b \left(\frac{b+a}{2}-t\right) f'(t)du(t) \\
\leq\ & [u(b)-u(a)]\int_a^b f(s)ds-(b-a)\int_a^b f(t)du(t) \qquad (5.11)\\
\leq\ & \frac{1}{2}(b-a)^2[-f'(b)u(a)-f'(a)u(b)]+\int_a^b u(t)[(b-t)f'(b)+(t-a)f'(a)]dt.
\end{aligned}$$

The proof follows on multiplying (5.11) by $\left(-\frac{1}{b-a}\right)$. The sharpness of the constants follows by a particular case which will be stated in Corollary 5.4.1. □

Corollary 5.3.2 (Kikianty, Dragomir and Cerone [74]). *Under the assumptions of Theorem 5.3.1, if $f'(b)=-f'(a)$, then*

$$\begin{aligned}
&f'(a)\left[\frac{b-a}{2}(u(b)-u(a))-\frac{1}{b-a}\int_a^b u(t)\left(2t-(a+b)\right)dt\right] \\
\leq\ & D(f,u)\leq \int_a^b \left(t-\frac{b+a}{2}\right) f'(t)du(t). \qquad (5.12)
\end{aligned}$$

The proof of Corollary 5.3.2 follows directly from Theorem 5.3.1; and the details are omitted.

Remark 5.3.3. A common example of such a function is the function defined on an interval $[a,b]$ which is symmetric with respect to the midpoint $\frac{a+b}{2}$, for example, $f(t)=\left|t-\frac{a+b}{2}\right|^p$, where $p\geq 1$.

Corollary 5.3.4 (Kikianty, Dragomir and Cerone [74]). *Under the assumptions of Theorem 5.3.1, if $f'(a)=-f'(b)$ and f'' exists, then*

$$\begin{aligned}
&f'(a)\left[\frac{b-a}{2}(u(b)-u(a))-\frac{1}{b-a}\int_a^b u(t)\left(2t-(a+b)\right)dt\right] \\
\leq\ & D(f,u) \qquad (5.13)\\
\leq\ & \left(\frac{b-a}{2}\right) f'(b)[u(b)-u(a)]-\int_a^b u(t)\left[f'(t)+\left(t-\frac{b+a}{2}\right)f''(t)\right]dt.
\end{aligned}$$

Proof. This is a particular case of Corollary 5.3.2. Note that

$$\begin{aligned}
&\int_a^b \left(t-\frac{b+a}{2}\right) f'(t)du(t) \\
=\ & \left(\frac{b-a}{2}\right) f'(b)[u(b)-u(a)]-\int_a^b u(t)\left[f'(t)+\left(t-\frac{b+a}{2}\right)f''(t)\right]dt;
\end{aligned}$$

and the details are omitted. $\square$

Open Problem 5.3.5. Are the inequalities in Corollaries 5.3.2 and 5.3.4 sharp?

5.4 Application to the Čebyšev functional

In this section, we apply the result of Section 5.3 to obtain bounds for the classical Čebyšev functional.

Corollary 5.4.1 (Kikianty, Dragomir and Cerone [74]). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function and $g : [a, b] \to \mathbb{R}$ be a nonnegative Lebesgue integrable function. Then,*

$$\begin{aligned} & \frac{1}{2}\int_a^b \left[\left(\frac{t-a}{b-a}\right)^2 f'(a) - \left(\frac{b-t}{b-a}\right)^2 f'(b)\right] g(t)dt \\ \leq\ & T(f, g) \qquad (5.14) \\ \leq\ & \frac{1}{b-a}\int_a^b \left(t - \frac{b+a}{2}\right) f'(t)g(t)dt. \end{aligned}$$

The constants $\frac{1}{2}$ and 1 in (5.14) are sharp.

Proof. Recall that Theorem 5.3.1 gives us

$$\begin{aligned} & \frac{1}{2(b-a)}\int_a^b [(t-a)^2 f'(a) - (b-t)^2 f'(b)]du(t) \\ \leq\ & D(f, u) \qquad (5.15) \\ \leq\ & \int_a^b \left(t - \frac{b+a}{2}\right) f'(t)du(t). \end{aligned}$$

Since g is positive on $[a, b]$, $u(t) = \int_a^t g(s)ds$ is monotonically increasing on $[a, b]$, by the Fundamental Theorem of Calculus. Thus, inequality (5.14) follows by applying (5.15) to u on multiplying the obtained inequality by $\frac{1}{b-a}$. The sharpness of the constants in (5.14) is demonstrated by choosing $f(t) = g(t) = t$ on $[a, b]$. We omit the details. $\square$

Example 5.4.2. Let $f(t) = g(t) = \frac{1}{t}$ defined on $[x, y]$, where $x, y > 0$. Then by Corollary 5.4.1, we obtain

$$0 \leq \left(\frac{1}{G(x,y)}\right)^2 - \left(\frac{1}{L(x,y)}\right)^2 \leq \left(\frac{y-x}{2xy}\right)^2, \qquad (5.16)$$

where $G(x,y)$ and $L(x,y)$ are the geometric mean and logarithmic mean of x and y, respectively (cf. Chapter 1). Note that equality holds when $x=y$. We omit the lower bound in this example as it is not always positive.

5.4.1 Čebyšev functional for convex functions

In Corollary 5.4.1, we assume that f is a differentiable convex function. However, we may 'drop' the assumption of differentiability and get a similar result for any convex functions, in which the derivative exists almost everywhere.

Proposition 5.4.3 (Kikianty, Dragomir and Cerone [74]). *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $g:[a,b]\to\mathbb{R}$ be a nonnegative Lebesgue integrable function. Then,*

$$\begin{aligned}
&\frac{1}{2}\int_a^b\left[\left(\frac{t-a}{b-a}\right)^2 f'(a)-\left(\frac{b-t}{b-a}\right)^2 f'(b)\right]g(t)dt\\
\le\ & T(f,g) \qquad (5.17)\\
\le\ & \frac{1}{b-a}\int_a^b\left(t-\frac{b+a}{2}\right)f'(t)g(t)dt.
\end{aligned}$$

The constants 1 *and* $\frac{1}{2}$ *in (5.17) are sharp.*

Proof. Since f is a convex function on $[a,b]$, we have the following Ostrowski type inequality for any $t\in[a,b]$ (cf. Lemma 4.1.3)

$$\begin{aligned}
&\frac{1}{2}\left[(b-t)^2 f'_+(t)-(t-a)^2 f'_-(t)\right]\\
\le\ & \int_a^b f(s)ds-(b-a)f(t) \qquad (5.18)\\
\le\ & \frac{1}{2}\left[(b-t)^2 f'_-(b)-(t-a)^2 f'_+(a)\right].
\end{aligned}$$

We multiply (5.18) by $g(t)$, take the integral over $[a,b]$ and multiply it by $-\frac{1}{(b-a)^2}$ to obtain

$$\begin{aligned}
&\frac{1}{2}\int_a^b\left[\left(\frac{t-a}{b-a}\right)^2 f'_+(a)-\left(\frac{b-t}{b-a}\right)^2 f'_-(b)\right]g(t)dt\\
\le\ & T(f,g)\\
\le\ & \frac{1}{2}\int_a^b\left[\left(\frac{t-a}{b-a}\right)^2 f'_-(t)-\left(\frac{b-t}{b-a}\right)^2 f'_+(t)\right]g(t)dt.
\end{aligned}$$

Since f is convex, f' exists almost everywhere. Thus, we may write $f'(t) = f'_{\pm}(t)$, for almost every $t \in [a,b]$; and we omit the details of the proof. The sharpness of the constants follows by Remark 5.4.7. □

Similarly, we have the generalized version of Theorem 5.3.1 in the next proposition; and the proof is omitted.

Proposition 5.4.4. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $u : [a,b] \to \mathbb{R}$ be a monotonically increasing function. Then,*

$$\begin{aligned}
& \frac{b-a}{2}\,[f'(a)u(b) + f'(b)u(a)] - \int_a^b u(t)\left[\frac{t-a}{b-a}\,f'(a) + \frac{b-t}{b-a}\,f'(b)\right]dt \\
\leq\ & D(f,u) \qquad (5.19)\\
\leq\ & \int_a^b \left(t - \frac{b+a}{2}\right) f'(t)\ du(t).
\end{aligned}$$

The constants $\frac{1}{2}$ and 1 in (5.19) are sharp.

The following result is a consequence of Proposition 5.4.3 for convex functions on linear spaces.

Corollary 5.4.5 (Kikianty, Dragomir and Cerone [74])**.** *Let $\mathbf{X}$ be a linear space and x, y be two distinct vectors in $\mathbf{X}$. Let g be a nonnegative functional on $[x,y]$ such that $\int_0^1 g[(1-t)x+ty]dt < \infty$. Then, for any convex function f defined on the segment $[x,y]$ and $t \in (0,1)$, we have*

$$\begin{aligned}
& \frac{1}{2}\int_0^1 \left[t^2(\nabla f(x))(y-x) - (1-t)^2(\nabla f(y))(y-x)\right] g[(1-t)x+ty]dt \\
\leq\ & \int_0^1 f[(1-t)x+ty]g[(1-t)x+ty]dt \qquad (5.20)\\
& - \int_0^1 f[(1-t)x+ty]dt \int_0^1 g[(1-t)x+ty]dt \\
\leq\ & \int_0^1 \left(t - \frac{1}{2}\right)(\nabla f[(1-t)x+ty])(y-x)g[(1-t)x+ty]dt.
\end{aligned}$$

The constants $\frac{1}{2}$ and 1 in (5.20) are sharp.

Proof. Consider the functions h, k defined on $[0,1]$ by

$$h(t) = f[(1-t)x+ty] \quad \text{and} \quad k(t) = g[(1-t)x+ty].$$

Since f is convex on the segment $[x, y]$, h is also convex on $[0, 1]$. We apply Proposition 5.4.3 to h and k. Firstly, $h'_{\pm}(t) = \big(\nabla_{\pm} f[(1-t)x + ty]\big)(y - x)$; and since h is convex, then

$$h'(t) = h'_{\pm}(t) = \big(\nabla_{\pm} f[(1-t)x + ty]\big)(y - x) = \big(\nabla f[(1-t)x + ty]\big)(y - x)$$

exists almost everywhere on $[0, 1]$. We get a similar identity for k. The proof for the sharpness of the constants follows by the particular case given later in Corollary 5.4.6. $\square$

5.4.2 Application to the *p*-HH-norms

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Recall from Lemma 5.2.1 that

$$T_{p,q}(x, y) := \|(x, y)\|^{p+q}_{p+q-HH} - \|(x, y)\|^{p}_{p-HH} \|(x, y)\|^{q}_{q-HH} \geq 0,$$

for any $x, y \in \mathbf{X}$ and $p, q \geq 1$.

Corollary 5.4.6 (Kikianty, Dragomir and Cerone [74]). *Under the above notation and assumptions, we have*

$$\begin{aligned} & \frac{1}{2} p \int_0^1 \left[t^2 \|x\|^{p-2}(y - x, x) - (1-t)^2 \|y\|^{p-2}(y - x, y)\right] \|(1-t)x + ty\|^q dt \\ \leq\ & T_{p,q}(x, y) \qquad\qquad (5.21) \\ \leq\ & p \int_0^1 \left(t - \frac{1}{2}\right) \|(1-t)x + ty\|^{p+q-2}(y - x, (1-t)x + ty) dt, \end{aligned}$$

for any $x, y \in \mathbf{X}$ whenever $p \geq 2$ and $(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{s(i)}$. If $1 \leq p < 2$, then the inequality (5.21) holds for any nonzero $x, y \in \mathbf{X}$. The constants $\frac{1}{2}$ and 1 are sharp in (5.21).

Proof. Define

$$f[(1-t)x + ty] = \|(1-t)x + ty\|^p \quad \text{and} \quad g[(1-t)x + ty] = \|(1-t)x + ty\|^q$$

for $t \in [0, 1]$. Note that for any $x, y \in \mathbf{X}$,

$$\big(\nabla_{\pm} \|\cdot\|^p [(1-t)x + ty]\big)(y - x) = p \|(1-t)x + ty\|^{p-2} \langle y - x, (1-t)x + ty \rangle_{s(i)},$$

provided that $p \geq 2$; otherwise, it holds for any linearly independent x and y (cf. Example 4.2.2).

Since $\left(\nabla\|\cdot\|^p[(1-\cdot)x+\cdot y]\right)(y-x)$ exist almost everywhere on $[0,1]$, and by denoting $(\cdot,\cdot):=\langle\cdot,\cdot\rangle_{s(i)}$, we have

$$\left(\nabla\|\cdot\|^p[(1-t)x+ty]\right)(y-x)=p\|(1-t)x+ty\|^{p-2}(y-x,(1-t)x+ty).$$

We obtain the similar identity for g. Therefore, by Corollary 5.4.5,

$$\begin{aligned}&\frac{1}{2}p\int_0^1\left[t^2\|x\|^{p-2}(y-x,x)-(1-t)^2\|y\|^{p-2}(y-x,y)\right]\|(1-t)x+ty\|^q dt\\ \le\ & T_{p,q}(x,y)\\ \le\ & p\int_0^1\left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}(y-x,(1-t)x+ty)dt,\end{aligned}$$

for any $x,y\in\mathbf{X}$ whenever $p\ge 2$; otherwise, it holds for any nonzero $x,y\in\mathbf{X}$. The proof for the sharpness of the constants follows by a particular case which will be stated in Remark 5.4.7. □

Remark 5.4.7 (Case of inner product spaces)**.** Let $(\mathbf{X},\langle\cdot,\cdot\rangle)$ be an inner product space and x,y be two distinct vectors in $\mathbf{X}$. Then, for any $p,q\ge 1$, we have

$$\begin{aligned}&\frac{1}{2}p\int_0^1\langle y-x,t^2\|x\|^{p-2}x-(1-t)^2\|y\|^{p-2}y\rangle\|(1-t)x+ty\|^q dt\\ \le\ & T_{p,q}(x,y) \qquad (5.22)\\ \le\ & p\int_0^1\left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}\langle y-x,(1-t)x+ty\rangle dt.\end{aligned}$$

If $p=q=1$, then

$$(5.23)$$

$$\begin{aligned}\textit{nonumber}\qquad&\frac{1}{2}\int_0^1\|(1-t)x+ty\|\left\langle y-x,\frac{t^2}{\|x\|}x-\frac{(1-t)^2}{\|y\|}y\right\rangle dt \qquad (5.24)\\ \le\ & \|(x,y)\|_{2-HH}^2-\|(x,y)\|_{1-HH}^2 \qquad (5.25)\\ \le\ & \frac{1}{12}\|y-x\|^2.\end{aligned}$$

Note that when $\mathbf{X}=\mathbb{R}$ and $x,y>0$, then from (5.25)

$$\begin{aligned}&\frac{1}{2}\int_0^1((1-t)x+ty)(y-x)\left(t^2-(1-t)^2\right)dt\\ =\ & \frac{y-x}{2}\int_0^1(2t-1)[(1-t)x+ty]dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{y-x}{2}\int_0^1 \left(-2t^2+3t-1\right)x+\left(2t^2-t\right)y\ dt \\
&= \frac{y-x}{2}\left(\frac{y-x}{6}\right) \\
&= \frac{1}{12}(y-x)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\|(x,y)\|_{2-HH}^2-\|(x,y)\|_{1-HH}^2 &= \frac{y^3-x^3}{3(y-x)}-\left(\frac{y+x}{2}\right)^2 \\
&= \frac{x^2+xy+y^2}{3}-\frac{x^2+2xy+y^2}{4} \\
&= \frac{1}{12}(y-x)^2,
\end{aligned}
$$

to produce the equalities.

Remark 5.4.8. Although the inequalities that we obtain in Corollary 5.4.6 are sharp, the bounds are complicated to compute. We remark that the lower bound is not always positive. For example, if we take $\mathbf{X}=\mathbb{R}$, $p=q=1$, $x=-1$, $y=1$, then we have

$$
\frac{1}{2}p\int_0^1 \left(t^2\,|x|^{p-2}\,(y-x)\,x-(1-t)^2\,|y|^{p-2}\,(y-x)\,y\right)(|(1-t)\,x+ty|)^q\,dt=-\frac{3}{8}.
$$

In this case, the lower bound cannot be used to improve the Čebyšev inequality. We obtain coarser but simpler upper bounds for $T_{p,q}(x,y)$, as follows:

$$
\begin{aligned}
0 &\le T_{p,q}(x,y) \\
&\le p\int_0^1\left(t-\frac{1}{2}\right)\|(1-t)x+ty\|^{p+q-2}(y-x,(1-t)x+ty)dt, \\
&\le p\|y-x\|\int_0^1\left|t-\frac{1}{2}\right|\|(1-t)x+ty\|^{p+q-1}dt, \\
&\le p\|y-x\|\begin{cases}
\sup\limits_{t\in[0,1]}\left(\left|t-\frac{1}{2}\right|\right)\displaystyle\int_0^1\|(1-t)x+ty\|^{p+q-1}dt \\
\left(\displaystyle\int_0^1\left|t-\frac{1}{2}\right|^{s'}dt\right)^{\frac{1}{s'}}\left(\displaystyle\int_0^1\|(1-t)x+ty\|^{(p+q-1)s}dt\right)^{\frac{1}{s}}, \\
\qquad\qquad\qquad\qquad s>1,\ \frac{1}{s}+\frac{1}{s'}=1; \\
\sup\limits_{t\in[0,1]}\left(\|(1-t)x+ty\|^{p+q-1}\right)\displaystyle\int_0^1\left|t-\frac{1}{2}\right|dt.
\end{cases}
\end{aligned}
$$

$$\leq \quad p\|y-x\| \begin{cases} \frac{1}{2}\|(x,y)\|_{(p+q-1)-HH}^{p+q-1} \\ \left(\frac{1}{2^{s'}(s'+1)}\right)^{\frac{1}{s'}} \|(x,y)\|_{(p+q-1)s-HH}^{p+q-1}, s>1, \frac{1}{s}+\frac{1}{s'}=1; \\ \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q-1}. \end{cases} \tag{5.26}$$

Remark 5.4.9. Although, in general, these upper bounds are not always better than those obtained in Section 5.2, we remark that under certain conditions, they are better. For example, when $p \leq \frac{1}{2}q$, we have

$$\frac{1}{4}p\|y-x\| \max\{\|x\|,\|y\|\}^{p+q-1} \leq O_{p,q}(x,y)$$

(recall that $O_{p,q}(x,y) := \frac{1}{8}q\|y-x\| \max\{\|x\|,\|y\|\}^{p+q-1}$). Also, when $p \leq 1$ and $\|y-x\| \leq \max\{\|x\|,\|y\|\}$, we have

$$\frac{1}{4}p\|y-x\| \max\{\|x\|,\|y\|\}^{p+q-1} \leq G_{p,q}(x,y)$$

(recall that $G_{p,q}(x,y) := \frac{1}{4}\max\{\|x\|,\|y\|\}^{p+q}$).

Open Problem 5.4.10. Are the constants $\frac{1}{2}$, $\left(\frac{1}{2^{s'}(s'+1)}\right)^{\frac{1}{s'}}$ and $\frac{1}{4}$ in (5.26) the best possible?

Chapter 6

Orthogonality in normed spaces

In an inner product space, two vectors are orthogonal when their inner product vanishes. The study of orthogonality in normed space deals with the matter of extending the notion of orthogonality, without necessarily having an inner product construction. In the first part of this chapter, we recall several classical definitions of orthogonality in normed spaces which were introduced by Roberts, James, Birkhoff and Carlsson. By utilizing the 2-HH-norm (cf. Chapter 3), some new notions of orthogonality are introduced and investigated. These orthogonalities are closely connected to the classical ones, namely Pythagorean, Isosceles and Carlsson's orthogonalities. The homogeneity, as well as the additivity, of these orthogonalities is a necessary and sufficient condition for the space to be an inner product space.

6.1 Notions of orthogonality in normed spaces

Let $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ be an inner product space and $x, y \in \mathbf{X}$. We say that x is *orthogonal* to y (denoted by $x \perp y$) if and only if their inner product $\langle x, y\rangle$ is zero. It is a well-known fact that a normed space is not necessarily an inner product space. Hence, we cannot define orthogonality in a normed space, in the same manner as to that of an inner product space. Numerous notions of orthogonality in normed spaces have been introduced via equivalent propositions to the usual orthogonality in inner product spaces. An example of these propositions is the so-called Pythagorean law for two orthogonal vectors. In this section, we recall several classical notions of orthogonality which were introduced by Roberts, James, Birkhoff and Carlsson, together with their properties. Due to the large amount of literature, some results are omitted. For more results concerning other notions of orthogonality and their main properties, we refer to the survey papers by Alonso and Benitez [3, 4].

The following are the main properties of orthogonality in inner product space (cf. Alonso and Benitez [3], James [61] and Partington [96]). In the study of orthogonality in normed spaces, these properties are investigated to observe how the new definitions 'mimic' the usual one.

Let $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ be an inner product space. The following statements hold for all $x, y, z \in \mathbf{X}$.

1. If $x \perp x$, then $x = 0$ (*Non-degeneracy*).
2. If $x \perp y$, then $\lambda x \perp \lambda y$ for all $\lambda \in \mathbb{R}$ (*Simplification*).
3. If $(x_n), (y_n) \subset \mathbf{X}$ such that $x_n \perp y_n$ for every $n \in \mathbb{N}$, $x_n \to x$ and $y_n \to y$, then $x \perp y$ (*Continuity*).
4. If $x \perp y$, then $\lambda x \perp \mu y$ for all $\lambda, \mu \in \mathbb{R}$ (*Homogeneity*).
5. If $x \perp y$ then $y \perp x$ (*Symmetry*).
6. If $x \perp y$ and $x \perp z$ then $x \perp (y + z)$ (*Additivity*).
7. If $x \neq 0$, then there exists $\alpha \in \mathbb{R}$ such that $x \perp (\alpha x + y)$ (*Existence*).
8. The above α is unique (*Uniqueness*).

Alonso and Benitez [3, p. 2] defined the existence (and the uniqueness) as follows:

"For every oriented plane P, every $x \in P \setminus \{0\}$ and every $\rho > 0$, there exists (a unique) $y \in P$ such that the pair $[x, y]$ is in the given orientation, $\|y\| = \rho$ and $x \perp y$."

The definition of uniqueness as stated in this section is due to James [61, p. 292]. In the paper by Partington [96], this property is referred to as *resolvability*. Alonso and Benitez noted that when the orthogonality is nonhomogeneous, the existence in the James sense is not equivalent to their definition. It was also noted that the existence implies that for any nonzero vector x, the set $\{\alpha : x \perp \alpha x + y\}$ is a nonempty compact interval. Therefore, in investigating the uniqueness, for nonhomogeneous orthogonalities, they refer to James' result as the α-uniqueness property, where the (above) interval is reduced to a point [3, p. 8]. However, in this chapter, we use James' definition and we refer to the α-uniqueness as uniqueness as initially stated above.

6.1.1 Roberts' orthogonality

The following definition is due to Roberts [108] (cf. Alonso and Benitez [3]).

Definition 6.1.1 (Roberts [108]). Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. Then, x is said to be *orthogonal in the sense of Roberts*, or R-orthogonal for short, to y (denoted by $x \perp y$ (R)) if and only if $\|x + \lambda y\| = \|x - \lambda y\|$ for any $\lambda \in \mathbb{R}$.

In any normed space, R-orthogonality satisfies non-degeneracy, simplification, continuity, homogeneity and symmetry [3, p. 4]. Alonso and Benitez [3, p. 4] remarked that R-orthogonality is nonadditive.

James [61, p. 292] noted that the existence property is the most important one; since it would keep the concept of orthogonality from being vacuous. However, R-orthogonality is not existent. The following is an example of a space in which one of the two R-orthogonal vectors is always zero, which fails the existence criterion.

Example 6.1.2 (James [61]). Let $\mathbf{X}$ be the normed linear space consisting of all continuous functions of the form $f = ax + bx^2$, where $\|ax + bx^2\|$ is the maximum of $|ax + bx^2|$ in the interval $(0, 1)$. Then, two elements of $\mathbf{X}$ are R-orthogonal if and only if one is zero, that is, $\|f + kg\| = \|f - kg\|$ for all k only if $f = 0$ or $g = 0$. This also implies that R-orthogonality is trivially additive in such a space.

The existence of R-orthogonality characterizes inner product spaces, giving the following proposition.

Proposition 6.1.3 (James [61]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, R-orthogonality is existent if and only if the norm is induced by an inner product.*

6.1.2 Birkhoff's orthogonality

The following definition is due to Birkhoff [11] (cf. Partington [96]).

Definition 6.1.4 (Birkhoff [11]). Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. Then, x is said to be *orthogonal in the sense of Birkhoff*, or B-orthogonal, to y (denoted by $x \perp y$ (B)) if and only if $\|x\| \leq \|x + \lambda y\|$ for any $\lambda \in \mathbb{R}$.

In any normed spaces, B-orthogonality satisfies non-degeneracy, simplification, continuity and homogeneity [3, p. 4–5]. The following example shows that B-orthogonality is not symmetric.

Example 6.1.5. In the space ℓ_2^1, if $x = (2, -1)$ and $y = (1, 1)$, then $x \perp y$ (B), but $y \not\perp x$ (B). This shows that B-orthogonality is not always symmetric [55].

Therefore, it is important to distinguish the existence (also, the additivity) to the left and to the right.

The following result was proven by Birkhoff [11], James [62, 63] and Day [28] (cf. Alonso and Benitez [3]).

Proposition 6.1.6. *Let* $\mathbf{X}$ *be a normed space with* $\dim(\mathbf{X}) \geq 3$. *Then,* $\mathbf{X}$ *is an inner product space if and only if* B*-orthogonality is symmetric.*

In the view of the last proposition, the similar statement does not hold in 2-dimensional normed spaces, as shown in the following example.

Example 6.1.7 (James [62])**.** In $\mathbb{R}^2$ with the following norm

$$\|(x, y)\| = \begin{cases} (|x|^p + |y|^p)^{\frac{1}{p}}, & \text{if } xy \geq 0; \\ (|x|^{p'} + |y|^{p'})^{\frac{1}{p'}}, & \text{if } xy < 0, \end{cases}$$

where $p, p' > 0$, $\frac{1}{p} + \frac{1}{p'} = 1$, B-orthogonality is symmetric, but the norm is not induced by an inner product. Day [28] proved that every possible example is, as the above, a suitable combination (a kind of 'mixed norm'), on even and odd quadrants, of an arbitrary norm and its dual (cf. Alonso and Benitez [3, p. 6]).

Example 6.1.8. In ℓ_2^1, if $x = (2, 0)$, $y = (1, 1)$ and $z = (0, -1)$, then $x \perp y$ (B) and $x \perp z$ (B) , but $x \not\perp (y + z)$ (B). This shows that B-orthogonality is not always additive (to the right) [55].

The additivity property, however, holds in particular normed spaces, as shown in the next proposition.

Proposition 6.1.9 (James [63])**.** *Let* $\mathbf{X}$ *be a normed space. Then, the following statements are true.*

1. B*-orthogonality is additive to the right if and only if* $\mathbf{X}$ *is smooth.*
2. *If* $\dim(\mathbf{X}) = 2$*, then* B*-orthogonality is additive to the left if and only if* $\mathbf{X}$ *is strictly convex.*
3. *If* $\dim(\mathbf{X}) \geq 3$*, then* B*-orthogonality is additive to the left if and only if* $\mathbf{X}$ *is an inner product space.*

Existence to the right can be viewed as a consequence of the following (cf. James [63], Alonso and Benitez [3]):
"*For every $x \in \mathbf{X}$, there exists a closed and homogeneous hyperplane H such that $x \perp H$ (B);*"
or equivalently,
"*A point $x \in \mathbf{X}$ is B-orthogonal to other point $y \in \mathbf{X}$ if and only if there exists a continuous linear functional $f \in \mathbf{X}^* \setminus \{0\}$ such that $f(x) = \|f\|\|x\|$, $f(y) = 0$.*"
The existence to the left of B-orthogonality follows from the homogeneity of this orthogonality and the convexity of the function

$$\mathbb{R} \ni \lambda \mapsto \|\lambda x + y\|$$

(cf. James [63], Alonso and Benitez [3]). Therefore, we have the following proposition.

Proposition 6.1.10 (James [63]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, B-orthogonality is existent to the left and to the right.*

With regard to uniqueness, we have the following proposition.

Proposition 6.1.11 (James [63]). *Let $\mathbf{X}$ be a normed space. Then, B-orthogonality is unique to the right if and only if $\mathbf{X}$ is smooth; and is unique to the left if and only if $\mathbf{X}$ is strictly convex.*

This orthogonality is closely connected to the smoothness of the normed space, as shown in the next result.

Proposition 6.1.12 (James [63]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. If the norm $\|\cdot\|$ is Gâteaux differentiable, then x is B-orthogonal to y if and only if the Gâteaux derivative at x in y direction is zero, that is, $(\nabla\|\cdot\|(x))(y) = 0$.*

6.1.3 Carlsson type orthogonality

In an inner product space, two vectors are orthogonal if and only if they satisfy the so-called Pythagorean law.

Definition 6.1.13 (Partington [96]). Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. A vector x is said to be *orthogonal in the sense of Pythagoras*, or P-orthogonal, to y (denoted by $x \perp y$ (P)) if and only if

$$\|x\|^2 + \|y\|^2 = \|x + y\|^2.$$

Remark 6.1.14. Pythagorean orthogonality is initially defined as follows (cf. James [61]):

$$x \perp y \ (P) \quad \text{if and only if} \quad \|x\|^2 + \|y\|^2 = \|x-y\|^2.$$

However, the results remain the same with any of these definitions.

It is also well-known that the diagonals of the parallelogram spanned by two orthogonal vectors are of the same length, in an inner product space. This fact gives the following definition.

Definition 6.1.15 (James [61])**.** Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. A vector x is said to be *orthogonal in the sense of Isosceles*, or I-orthogonal, to y (denoted by $x \perp y \ (I)$) if and only if

$$\|x+y\| = \|x-y\|.$$

The following definition is due to Carlsson [19] and is a generalization of Pythagorean and Isosceles orthogonalities.

Definition 6.1.16 (Carlsson [19])**.** Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $x, y \in \mathbf{X}$ and m be a positive integer. Then, x is said to be *orthogonal in the sense of Carlsson*, or C-orthogonal for short (denoted by $x \perp y \ (C)$), if and only if

$$\sum_{i=1}^{m} \alpha_i \|\beta_i x + \gamma_i y\|^2 = 0,$$

where α_i, β_i, γ_i are real numbers such that

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0, \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1.$$

Remark 6.1.17. Other types of the Carlsson orthogonality [3]:

1. $x \perp y \ (S)$ if and only if $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp \|y\|^{-1}y \ (I)$ (*Singer/Unitary Isosceles*, 1957);

2. $x \perp y \ (UP)$ if and only if $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp \|y\|^{-1}y \ (P)$ (*Unitary Pythagorean*, 1986);

3. $x \perp y \ (U)$ if and only if $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp \|y\|^{-1}y \ (C)$ (*Unitary Carlsson*);

4. $x \perp y \ (aI)$ if and only if $\|x - ay\| = \|x + ay\|$ for some fixed $a \neq 0$ (*a-Isosceles*, 1988);

5. $x \perp y$ (aP) if and only if $\|x - ay\|^2 = \|x\|^2 + a\|y\|^2$ for some fixed $a \neq 0$ (*a-Pythagorean*, 1988);

6. $x \perp y$ (ab) if and only if $\|ax + by\|^2 + \|x + y\|^2 = \|ax + y\|^2 + \|x + by\|^2$, for some fixed $a, b \in (0, 1)$ (1978);

7. $x \perp y$ (a) if and only if $(1 + a)^2\|x + y\|^2 = \|ax + y\|^2 + \|x + ay\|^2$, for some fixed $a \neq 1$ (1983).

In any normed space, C-orthogonality satisfies non-degeneracy, simplification and continuity [3, p. 4]. The following example shows that C-orthogonality is not symmetric in general.

Example 6.1.18. Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. Define x to be orthogonal to y if and only if $\|x + 2y\| = \|x - 2y\|$ [3, p. 9]. This is a particular case of C-orthogonality and is not symmetric.

However, C-orthogonality is symmetric in some cases [3, p. 9]. In particular, P-orthogonality and I-orthogonality are symmetric [61].

With regards to existence, P-orthogonality and I-orthogonality are existent [61]. Since C-orthogonality is nonsymmetric, the existence to the right and to the left must be distinguished.

Proposition 6.1.19 (Carlsson [19])**.** *C-orthogonality is existent to the right and to the left.*

With regards to uniqueness, Alonso and Benitez [3] noted that in general, C-orthogonality, is not unique, when the space is not strictly convex. In particular, we have the following proposition.

Proposition 6.1.20 (Kapoor and Prasad [68])**.** *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then,*

1. *P-orthogonality is unique;*

2. *I-orthogonality is unique, if and only if $\mathbf{X}$ is strictly convex.*

James [61] remarked that if I-orthogonality is homogeneous or additive, then it is unique.

The following example shows that C-orthogonality is nonadditive and nonhomogeneous.

Example 6.1.21. In the space ℓ_2^1, choose $x = (3,6)$ and $y = (-8,4)$, then $x \perp y\ (P)$, but $x \not\perp 2y\ (P)$. This shows that P-orthogonality is neither additive nor homogeneous [55]. In the space ℓ_2^1, choose $x = (2,1)$ and $y = (1,-2)$, then $x \perp y\ (I)$, but $x \not\perp 2y\ (I)$. This shows that I-orthogonality is neither additive nor homogeneous [55].

In the next few results, it is shown that some characterizations of inner product spaces follow by the homogeneity (and the additivity) of P-, I- and C-orthogonality.

Proposition 6.1.22 (James [61])**.** *Let* $\mathbf{X}$ *be a normed space. Then,*

1. *if P-orthogonality is homogeneous (additive) in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space;*
2. *if I-orthogonality is homogeneous (additive) in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space.*

Carlsson [19] introduced the following definition.

Definition 6.1.23 (Carlsson [19])**.** Let m be a positive integer. Then, C-orthogonality is said to have *property (H)* in a normed linear space $(\mathbf{X}, \|\cdot\|)$ if $x \perp y\ (C)$ implies that

$$\lim_{n\to\infty} n^{-1} \sum_{\nu=1}^{m} \alpha_\nu \|n\beta_\nu x + \gamma_\nu y\|^2 = 0 \quad (n \text{ positive integer}).$$

It is important to note that if C-orthogonality is homogeneous or additive to the left in a normed space $\mathbf{X}$, then it has property (H) in $\mathbf{X}$ [19, p. 302].

Proposition 6.1.24 (Carlsson [19])**.** *Let* $\mathbf{X}$ *be a normed space. Then, the following statements are true.*

1. *If C-orthogonality has property (H) in* $\mathbf{X}$*, then it is symmetric and equivalent to B-orthogonality in* $\mathbf{X}$*.*
2. *If C-orthogonality is homogeneous or additive to the left in* $\mathbf{X}$*, then it has property (H).*
3. *If C-orthogonality has property (H) in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space; hence, C-orthogonality is homogeneous (additive to the left) if and only if* $\mathbf{X}$ *is an inner product space.*

The following result is a generalization of the parallelogram law.

Theorem 6.1.25 (Carlsson [19])**.** *Let m be a positive integer. Suppose that $\alpha_\nu \neq 0$, β_ν, γ_ν, $\nu = 1, 2, \ldots, m$ are real numbers such that (β_ν, γ_ν) and (β_μ, γ_μ) are linearly* *independent for* $\nu \neq \mu$. *If* $(\mathbf{X}, \|\cdot\|)$ *is a normed linear space satisfying the condition*

$$\sum_{\nu=1}^{m} \alpha_\nu \|\beta_\nu x + \gamma_\nu y\|^2 = 0, \text{ for } x, y \in \mathbf{X},$$

then $\mathbf{X}$ *is an inner product space.*

6.1.4 Relations between main orthogonalities

In this subsection, we gather some results concerning relations between main orthogonalities. These results are mostly taken from the survey paper by Alonso and Benitez [4].

In Subsection 6.1.1, it is noted that every orthogonality is existent, except R-orthogonality. However, R-orthogonality is existent only in inner product spaces, giving the following propositions.

Proposition 6.1.26 (Alonso and Benitez [4]). *Let* $\mathbf{X}$ *be a normed space. Then,* R*-orthogonality is equivalent to any other orthogonality if and only if* $\mathbf{X}$ *is an inner product space.*

Proposition 6.1.27 (Alonso and Benitez [4]). *Let* $\mathbf{X}$ *be a normed space. If any of the existing orthogonality implies* R*-orthogonality, then* $\mathbf{X}$ *is an inner product space.*

As shown in Example 6.1.2, there are spaces in which R-orthogonality is not existent. For these spaces, R-orthogonality implies any other orthogonality [4, p. 125]. Furthermore, in every case, R-orthogonality implies B-, I- and Singer's (Unitary-Isosceles) orthogonalities [4, p. 125].

It has been pointed out that B-orthogonality is homogeneous. Alonso and Benitez [4] stated that Unitary-Carlsson orthogonality is positively homogeneous. However, C-orthogonality is positively homogeneous only in an inner product space [19].

Proposition 6.1.28 (Alonso and Benitez [4]). *Let* $\mathbf{X}$ *be a normed space. Then,* C*-orthogonality is equivalent to* B*-orthogonality, as well as Unitary-Carlsson orthogonality, if and only if* $\mathbf{X}$ *is an inner product space.*

In general, C-orthogonality is not unique. However, Unitary-Carlsson orthogonality is unique [4]. It has also been pointed out that B-orthogonality is unique to the left (right) only in strictly convex (smooth, respectively) spaces. These facts give the following proposition.

Proposition 6.1.29 (Alonso and Benitez [4]). *Let* $\mathbf{X}$ *be a normed space. Then,* C*-orthogonality implies* B*-orthogonality in a strictly convex or smooth space* $\mathbf{X}$*, if and only if* $\mathbf{X}$ *is an inner product space.* C*-orthogonality implies Singer's (Unitary-Isosceles) orthogonality if and only if* $\mathbf{X}$ *is an inner product space.*

6.2 Hermite-Hadamard type orthogonality

Let $(\mathbf{X}, \langle \cdot, \cdot \rangle)$ be an inner product space. For any $x, y \in \mathbf{X}$, we have

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2.$$

When x is orthogonal to y, the above equality becomes

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2, \tag{6.1}$$

that is, the Pythagorean law, which motivates the notion of P-orthogonality.

If $\mathbf{X}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$, then the 2-HH-norm of the pair (x, y) in $\mathbf{X}^2$ is

$$\|(x,y)\|_{2-HH}^2 = \int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \langle x, y\rangle + \|y\|^2).$$

If $x \perp y$, that is, $\langle x, y\rangle = 0$, then

$$\int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2). \tag{6.2}$$

Conversely, when x and y satisfy (6.2), then $x \perp y$, so the two statements are equivalent. Therefore, we consider a notion of orthogonality as follows:

Definition 6.2.1 (Kikianty and Dragomir [70]). In any normed space $(\mathbf{X}, \|\cdot\|)$, a vector $x \in \mathbf{X}$ is said to be *HH-P-orthogonal* to $y \in \mathbf{X}$ if and only if they satisfy (6.2); and we denote it by $x \perp_{HH-P} y$.

Remark 6.2.2. We note that P-orthogonality is not equivalent to HH-P-orthogonality, as pointed out by the following:

1. In $\mathbb{R}^2$ equipped with the ℓ^1-norm, $x = (-3, 6)$ is P-orthogonal to $y = (8, 4)$, but $x \not\perp_{HH-P} y$.

2. In $\mathbb{R}^2$ equipped with the ℓ^1-norm, $x = (2,1)$ is HH-P-orthogonal to $y = (\frac{11}{2} - \frac{\sqrt{145}}{2}, 1)$, but $x \not\perp y$ (P).

Furthermore, the following statements are true.

1. If $x, y \in \mathbf{X}$ such that $(1-t)x \perp ty$ (P) for almost every $t \in [0,1]$, then by the continuity of P-orthogonality, $(1-t)x \perp ty$ (P) for every $t \in [0,1]$. Moreover, it implies that $\alpha x \perp \beta y$ (P) for any $\alpha, \beta \in \mathbb{R}$, which gives us the homogeneity of P-orthogonality.

2. If $x \perp y$ (P) implies that $(1-t)x \perp ty$ (P), then the P-orthogonality is homogeneous. Therefore the underlying space is an inner product space. Thus, $x \perp_{HH-P} y$.

The HH-P-orthogonality is connected to P-orthogonality, as presented in the following proposition (we omit the proof).

Proposition 6.2.3 (Kikianty and Dragomir [70])**.** *Let* $\mathbf{X}$ *be a normed space. If* $x, y \in \mathbf{X}$ *such that* $(1-t)x \perp ty$ (P) *for almost every* $t \in [0,1]$*, then,* $x \perp_{HH-P} y$*.*

In the same manner, suppose that $x, y \in \mathbf{X}$ such that $(1-t)x \perp ty$ (I) for almost every $t \in [0,1]$, that is,

$$\|(1-t)x + ty\| = \|(1-t)x - ty\|,$$

almost everywhere on $[0,1]$. By integrating the last equality over $[0,1]$,

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt. \tag{6.3}$$

Note that (6.3) is equivalent to $x \perp y$ in an inner product space (the proof is omitted).

Definition 6.2.4 (Kikianty and Dragomir [70])**.** In any normed space $(\mathbf{X}, \|\cdot\|)$, a vector $x \in \mathbf{X}$ is said to be *HH-I-orthogonal* to $y \in \mathbf{X}$ if and only if they satisfy (6.3); and we denote it by $x \perp_{HH-I} y$.

Remark 6.2.5. We also remark that I-orthogonality is not equivalent to HH-I-orthogonality, as pointed out in the following:

1. In $\mathbb{R}^2$ equipped with the ℓ^1-norm, $x = (2,-1)$ is I-orthogonal to $y = (1,1)$, but $x \not\perp_{HH-I} y$.

2. In $\mathbb{R}^2$ equipped with the ℓ^1-norm, $x = (-\frac{1}{8} + \frac{\sqrt{129}}{8}, 1)$ is HH-I-orthogonal to $y = (-\frac{1}{8} + \frac{\sqrt{129}}{8}, -2)$, but $x \not\perp y$ (P).

Furthermore, if $x \perp y\ (I)$ implies that $(1-t)x \perp ty\ (I)$, then the I-orthogonality is homogeneous. Therefore the underlying space is an inner product space. Thus, $x \perp_{HH-I} y$.

Recall C-orthogonality, which is a generalization of P-orthogonality and I-orthogonality. As the definition of HH-I-orthogonality arises from Proposition 6.2.3, we have the following definition, in the same manner.

Definition 6.2.6 (Kikianty and Dragomir [69])**.** Let m be a positive integer. In a normed space $(\mathbf{X}, \|\cdot\|)$, $x \in \mathbf{X}$ is said to be *HH-C-orthogonal* to $y \in \mathbf{X}$ (we denote it by $x \perp_{HH-C} y$) if and only if

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2\ dt = 0, \tag{6.4}$$

where α_i, β_i, γ_i are real numbers such that

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1. \tag{6.5}$$

The main properties of this orthogonality are discussed in the subsequent sections.

Proposition 6.2.7 (Kikianty and Dragomir [69])**.** *Let $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ be an inner product space and $x, y \in \mathbf{X}$. Then, $x \perp y$ if and only if $x \perp_{HH-C} y$.*

Proof. Let m be a positive integer. Since $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ is an inner product space, we have

$$\begin{aligned} & \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2\ dt \\ = \ & \sum_{i=1}^{m} \alpha_i \int_0^1 \left((1-t)^2\beta_i^2\|x\|^2 + 2t(1-t)\beta_i\gamma_i\langle x, y\rangle + t^2\gamma_i^2\|y\|^2\right)\ dt \\ = \ & \frac{1}{3}\left(\sum_{i=1}^{m} \alpha_i\beta_i^2\|x\|^2 + \sum_{i=1}^{m} \alpha_i\beta_i\gamma_i\langle x, y\rangle + \sum_{i=1}^{m} \alpha_i\gamma_i\|y\|^2\right) \\ = \ & \frac{1}{3}\langle x, y\rangle, \end{aligned}$$

by (6.5). If $x \perp y$, then $\langle x, y\rangle = 0$. Therefore,

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2\ dt = 0,$$

that is, $x \perp_{HH-C} y$.

If $x \perp_{HH-C} y$, then

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt = \frac{1}{3}\langle x, y\rangle = 0,$$

which implies that $\langle x, y\rangle = 0$, that is, $x \perp y$. $\square$

Remark 6.2.8. Note that HH-P-orthogonality is a particular case of HH-C-orthogonality, which is obtained by choosing $m = 3$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = -1$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $\gamma_1 = \gamma_3 = 1$ and $\gamma_2 = 0$. Similarly, HH-I-orthogonality is also a particular case of HH-C-orthogonality, which is obtained by choosing $m = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = 1$ and $\gamma_2 = -1$.

Remark 6.2.9. Similarly to Remark 6.1.17, we consider some particular cases of HH-C-orthogonality.

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. Then,

1. $x \perp_{HH-S} y$ if and only if $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp_{HH-I} \|y\|^{-1}y$;

2. $x \perp_{HH-UP} y$ if and only if $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp_{HH-P} \|y\|^{-1}y$;

3. $x \perp_{HH-U} y$ if and only if $\|x\|\|y\| = 0$ or $\|x\|^{-1}x \perp_{HH-C} \|y\|^{-1}y$;

4. $x \perp_{HH-aI} y$ if and only if $x \perp_{HH-I} ay$ for some fixed $a \neq 0$;

5. $x \perp_{HH-aP} y$ if and only if $x \perp_{HH-P} ay$ for some fixed $a \neq 0$;

6. $x \perp_{HH-ab} y$ if and only if

$$\begin{aligned} \int_0^1 \|a(1-t)x + bty\|^2 \quad &+ \quad \|(1-t)x + ty\|^2 \, dt \\ &= \quad \int_0^1 \|a(1-t)x + ty\|^2 + \|(1-t)x + bty\|^2 \, dt \end{aligned}$$

   for some fixed $a, b \in (0, 1)$;

7. $x \perp_{HH-a} y$ if and only if $(ax + y) \perp_{HH-P} (x + ay)$, for some fixed $a \neq 1$.

The properties of HH-C-orthogonality are considered in the following section. In particular, these properties hold for some special cases, namely, HH-P- and HH-I-orthogonalities and those of Remark 6.2.9.

6.3 Existence and main properties

The properties of HH-C-orthogonality are discussed in this section. The properties of HH-P-orthogonality and HH-I-orthogonality follow by those of HH-C-orthogonality, unless stated in special cases.

The following lemma covers the main properties of HH-C-orthogonality.

Lemma 6.3.1 (Kikianty and Dragomir [69]). *HH-C-orthogonality satisfies the non-degeneracy, simplification and continuity.*

Proof. Let m be a positive integer. Suppose that $x \perp_{HH-C} x$, then

$$\begin{aligned}\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i x\|^2 \, dt &= \sum_{i=1}^{m} \alpha_i \|x\|^2 \int_0^1 |(1-t)\beta_i + t\gamma_i|^2 \, dt \\ &= \frac{1}{3}\|x\|^2 \sum_{i=1}^{m} \alpha_i(\beta_i^2 + \beta_i\gamma_i + \gamma_i^2) = \frac{1}{3}\|x\|^2 = 0,\end{aligned}$$

which implies that $x = 0$ (note the use of (6.5)). This shows that HH-C-orthogonality is non-degenerate.

Suppose that $x \perp_{HH-C} y$ and $\lambda \in \mathbb{R}$. Then,

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i \lambda x + t\gamma_i \lambda y\|^2 \, dt = |\lambda|^2 \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt = 0,$$

that is, $\lambda x \perp_{HH-C} \lambda y$, which proves the simplification property of HH-C-orthogonality.

If $x_n \to x$, $y_n \to y$ and $x_n \perp_{HH-C} y_n$ for any $n \in \mathbb{N}$, then by continuity of the norm,

$$\begin{aligned}&\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt \\ =\ &\sum_{i=1}^{m} \alpha_i \int_0^1 \left\|(1-t)\beta_i \lim_{n\to\infty} x_n + t\gamma_i \lim_{n\to\infty} y_n\right\|^2 dt \\ =\ &\lim_{n\to\infty} \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x_n + t\gamma_i y_n\|^2 \, dt = 0,\end{aligned}$$

that is, $x \perp_{HH-C} y$; and the continuity of HH-C-orthogonality is proven. $\square$

With regards to symmetry, HH-C-orthogonality is symmetric in some cases, for example, HH-P- and HH-I-orthogonalities are symmetric, as shown in the next proposition.

Proposition 6.3.2 (Kikianty and Dragomir [70]). *Let* $\mathbf{X}$ *be a normed space. Then, HH-P-orthogonality and HH-I-orthogonality are symmetric in* $\mathbf{X}$.

Proof. Let $(\mathbf{X}, \|\cdot\|)$ be a normed space and $x, y \in \mathbf{X}$. If $x \perp_{HH-P} y$, then

$$\begin{aligned} \int_0^1 \|(1-t)y + tx\|^2 dt &= \int_0^1 \|(1-t)x + ty\|^2 dt \\ &= \frac{1}{3}(\|x\|^2 + \|y\|^2) = \frac{1}{3}(\|y\|^2 + \|x\|^2), \end{aligned}$$

since the 2-HH-norm is symmetric. Hence, HH-P-orthogonality is symmetric. If $x \perp_{HH-I} y$, then

$$\begin{aligned} \int_0^1 \|(1-t)y + tx\|^2 dt &= \int_0^1 \|(1-t)x + ty\|^2 dt \\ &= \int_0^1 \|(1-t)x - ty\|^2 dt = \int_0^1 \|(1-t)y - tx\|^2 dt, \end{aligned}$$

which proves that HH-I-orthogonality is symmetric. □

The following provides an example of a nonsymmetric HH-C-orthogonality.

Example 6.3.3. Define $x \perp_{HH-C'} y$ to be

$$\int_0^1 \|(1-t)x + 2ty\|^2 \, dt = \int_0^1 \|(1-t)x - 2ty\|^2 \, dt.$$

In the plane $\mathbb{R}^2$ with the ℓ^1-norm, $x = (2,1)$ is HH-C′-orthogonal to $y = (\frac{1}{2}, -1)$ but $y \not\perp_{HH-C'} x$.

Therefore, it is important to distinguish the existence (as well as the additivity) to the left and to the right.

The following lemma is due to Carlsson [19, p. 299], which will be used in proving the existence of HH-C-orthogonality.

Lemma 6.3.4 (Carlsson [19]). *Let* $x, y \in \mathbf{X}$. *Then,*

$$\lim_{\lambda \to \pm\infty} \lambda^{-1} \left[\|(\lambda + a)x + y\|^2 - \|\lambda x + y\|^2 \right] = 2a\|x\|^2.$$

In the following theorem, it is shown that HH-C-orthogonality is existent to the left and to the right. Hence, HH-P-orthogonality and HH-I-orthogonality are also existent. In Section 6.6, alternative proofs for these cases are provided.

Theorem 6.3.5 (Kikianty and Dragomir [69])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. Then, HH-C-orthogonality is existent to the right and to the left.*

Proof. The proof uses a similar idea to that of Carlsson [19, p. 301]. The proof is provided for the existence to the right, as the other case follows analogously. Let m be a positive integer and g be a function on $\mathbb{R}$ defined by

$$g(\lambda) := \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i(\lambda x + y)\|^2 \, dt,$$

where α_i, β_i and γ_i are real numbers that satisfy (6.5). Note that our domain of integration is on $(0,1)$ (excluding the extremities) to ensure that we can employ Lemma 6.3.4. Therefore, for any $\lambda \neq 0$,

$$\begin{aligned} \lambda^{-1} g(\lambda) &= \lambda^{-1} \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i(\lambda x + y)\|^2 \, dt \qquad (6.6) \\ &= \lambda^{-1} \sum_{i=1}^{m} \alpha_i \int_0^1 \left[\|(1-t)\beta_i x + t\gamma_i(\lambda x + y)\|^2 \; - \|t\gamma_i(\lambda x + y)\|^2\right] \, dt. \end{aligned}$$

Note the use of $\sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0$. Hence, (6.6) becomes

$$\begin{aligned} &\lambda^{-1} \Bigg[\sum_{\gamma_i \neq 0} \alpha_i \int_0^1 \|[t\lambda + (1-t)\beta_i\gamma_i^{-1}]\gamma_i x + t\gamma_i y\|^2 \; - \|t\lambda\gamma_i x + t\gamma_i y\|^2 \, dt \\ &+ \sum_{\gamma_i = 0} \alpha_i \int_0^1 \|(1-t)\beta_i x\|^2 \, dt\Bigg] \\ = \; &\lambda^{-1} \Bigg[\sum_{\gamma_i \neq 0} \alpha_i \int_0^1 \|[t\lambda + (1-t)\beta_i\gamma_i^{-1}]\gamma_i x + t\gamma_i y\|^2 \; - \|t\lambda\gamma_i x + t\gamma_i y\|^2 \, dt \\ &+ \frac{1}{3} \sum_{\gamma_i = 0} \alpha_i \beta_i^2 \|x\|^2\Bigg]. \end{aligned}$$

Note that

$$\lim_{\lambda \to \pm\infty} \frac{1}{3} \lambda^{-1} \sum_{\gamma_i = 0} \alpha_i \beta_i^2 \|x\|^2 = 0.$$

By using Lemma 6.3.4, we obtain

$$\begin{aligned}\lim_{\lambda\to\pm\infty}\lambda^{-1}g(\lambda) &= \sum_{\gamma_i\neq 0}2\alpha_i\int_0^1 t(1-t)\beta_i\gamma_i^{-1}\ \|\gamma_i x\|^2\ dt\\ &= \frac{1}{3}\sum_{\gamma_i\neq 0}\alpha_i\beta_i\gamma_i\|x\|^2=\frac{1}{3}\|x\|^2,\end{aligned}$$

since $\sum_{i=1}^m \alpha_i\beta_i\gamma_i = 1$. It follows that $g(\lambda)$ is positive for sufficiently large positive λ and negative for sufficiently large negative λ. By the continuity of g, we conclude that there exists a λ_0 such that $g(\lambda_0)=0$, as required. $\square$

The following example shows that both HH-P-orthogonality and HH-I-orthogonality are neither additive nor homogeneous.

Example 6.3.6. In $\mathbb{R}^2$ equipped with the ℓ^1-norm, set $x=(0,-1)$ and $y=(1,\sqrt[3]{2}-1)$. Then, $x\perp_{HH-P} y$, but $x\not\perp_{HH-P} 2y$. Again, in $\mathbb{R}^2$, with the ℓ^1-norm, $x=(2,1)$ is HH-I-orthogonal to $y=(1,-2)$, but $x\not\perp_{HH-I} 2y$.

This example implies that HH-C-orthogonality is neither additive nor homogeneous. We will discuss these properties further in Section 6.5 with regards to some characterizations of an inner product space.

6.4 Uniqueness

In this section, we consider the uniqueness of HH-P-orthogonality and HH-I-orthogonality. The uniqueness of HH-C-orthogonality in general will be discussed in Section 6.5.

The following lemma will be used to prove the uniqueness property of HH-P-orthogonality.

Lemma 6.4.1 (Kikianty and Dragomir [70]). *Let $(\mathbf{X},\|\cdot\|)$ be a normed space and $x,y\in\mathbf{X}$. Let g be a function on $\mathbb{R}$ defined by*

$$g(k):=\int_0^1\|(1-t)y+k(tx)\|^2\ dt.$$

Then, g is a convex function on $\mathbb{R}$ and furthermore, for any $s\in(0,1)$ and $k_1,k_2\in\mathbb{R}$ where $g(k_1)\neq g(k_2)$, we have

$$g[sk_1+(1-s)k_2]<sg(k_1)+(1-s)g(k_2).$$

Proof. The proof has a similar idea to that of Kapoor and Prasad in [68, p. 406]. Let $s \in (0,1)$ and $k_1, k_2 \in \mathbb{R}$, where $k_1 \neq k_2$. Then,

$$
\begin{aligned}
& g[sk_1 + (1-s)k_2] \\
= & \int_0^1 \|(1-t)y + [sk_1 + (1-s)k_2](tx)\|^2 \, dt \qquad (6.7) \\
= & \int_0^1 \|s[(1-t)y + k_1 tx] + (1-s)[(1-t)y + k_2 tx]\|^2 \, dt \\
\leq & \; s^2 \int_0^1 \|(1-t)y + k_1 tx\|^2 \, dt + (1-s)^2 \int_0^1 \|(1-t)y + k_2 tx\|^2 \, dt \\
& + 2s(1-s) \int_0^1 \|(1-t)y + k_1 tx\| \|(1-t)y + k_2 tx\| \, dt \\
= & \; s \int_0^1 \|(1-t)y + k_1 tx\|^2 \, dt + (1-s) \int_0^1 \|(1-t)y + k_2 tx\|^2 \, dt \\
& + (s^2 - s) \left(\int_0^1 \|(1-t)y + k_1 tx\|^2 \, dt + \int_0^1 \|(1-t)y + k_2 tx\|^2 \, dt \right) \\
& + 2s(1-s) \int_0^1 \|(1-t)y + k_1 tx\| \|(1-t)y + k_2 tx\| \, dt \\
= & \; s \int_0^1 \|(1-t)y + k_1 tx\|^2 \, dt + (1-s) \int_0^1 \|(1-t)y + k_2 tx\|^2 \, dt \qquad (6.8) \\
& - s(1-s) \left(\int_0^1 |\|(1-t)y + k_1 tx\| - \|(1-t)y + k_2 tx\||^2 \, dt \right) \\
\leq & \; s \int_0^1 \|(1-t)y + k_1 tx\|^2 \, dt + (1-s) \int_0^1 \|(1-t)y + k_2 tx\|^2 \, dt \\
= & \; sg(k_1) + (1-s)g(k_2).
\end{aligned}
$$

Note that when equality holds, we conclude from (6.8) that

$$\int_0^1 |\|(1-t)y + k_1 tx\| - \|(1-t)y + k_2 tx\||^2 \, dt = 0,$$

that is, $\|(1-t)y + k_1 tx\| = \|(1-t)y + k_2 tx\|$ almost everywhere on $(0,1)$, which implies that

$$g(k_1) = \int_0^1 |\|(1-t)y + k_1 tx\|^2 dt = \int_0^1 \|(1-t)y + k_2 tx\|^2 dt = g(k_2).$$

Therefore, if $g(k_1) \neq g(k_2)$, then the inequality is strict. □

With this lemma in hand, we prove the uniqueness of HH-P-orthogonality, as presented in the next theorem.

Theorem 6.4.2 (Kikianty and Dragomir [70]). *Let* $\mathbf{X}$ *be a normed space. Then, HH-P-orthogonality is unique on* $\mathbf{X}$.

Proof. The proof has a similar idea to that of Kapoor and Prasad in [68, p. 406]. Suppose that HH-P-orthogonality is not unique. Then there exist $x, y \in \mathbf{X}$, $x \neq 0$ and $\alpha > 0$, such that

$$\begin{aligned} y &\perp_{HH-P} x, &(6.9)\\ \text{and}\quad \alpha x + y &\perp_{HH-P} x. &(6.10)\end{aligned}$$

Recall the convex function g as defined in Lemma 6.4.1. Observe that (6.9) implies that

$$g(0) = \frac{1}{3}\|y\|^2,$$

and

$$g(1) = \int_0^1 \|(1-t)y + tx\|^2\, dt = \frac{\|y\|^2}{3} + \frac{\|x\|^2}{3} = g(0) + \frac{\|x\|^2}{3}. \tag{6.11}$$

Set $\alpha'(t) = \frac{(1-t)\alpha}{t}$ and observe that $g(\alpha'(t)) = \frac{1}{3}\|\alpha x + y\|^2$, and

$$\begin{aligned} g(\alpha'(t)+1) &= \int_0^1 \|(1-t)y + (\alpha'(t)+1)tx\|^2\, dt\\ &= \int_0^1 \|(1-t)(\alpha x + y) + tx\|^2\, dt\\ &= \frac{\|\alpha x + y\|^2}{3} + \frac{\|x\|^2}{3} = g(\alpha'(t)) + \frac{\|x\|^2}{3}, \end{aligned} \tag{6.12}$$

by (6.10). Now, suppose that $0 < \alpha'(t) < 1$, and note that $g(1) \neq g(0)$ (since $x \neq 0$). Lemma 6.4.1 then gives us

$$g(\alpha'(t)) < \alpha'(t)\; g(1) + (1-\alpha'(t))\; g(0). \tag{6.13}$$

Also, $g(\alpha'(t)+1) \neq g(\alpha'(t))$ (since $x \neq 0$). By Lemma 6.4.1, we have

$$\begin{aligned} g(1) &< \alpha'(t)\; g(\alpha'(t)) + (1-\alpha'(t))\; g(\alpha'(t)+1)\\ &= \alpha'(t)\; g(\alpha'(t)) + (1-\alpha'(t))\; \left[g(\alpha'(t)) + \frac{\|x\|^2}{3}\right]\\ &= \alpha'(t)\; g(\alpha'(t)) + (1-\alpha'(t))\; \left[g(\alpha'(t)) + g(1) - g(0)\right], \end{aligned}$$

by (6.11) and (6.12). Therefore (by rearranging the last inequality), we have

$$\alpha'(t)g(1) + (1 - \alpha'(t))g(0) < g(\alpha'(t)),$$

which contradicts (6.13).

Now, consider the case that $\alpha'(t) > 1$. We have,

$$g(1) \leq \frac{\alpha'(t) - 1}{\alpha'(t)} g(0) + \frac{1}{\alpha'(t)} g(\alpha'(t)),$$

that is,

$$\frac{\|x\|^2}{3} = g(1) - g(0) \leq \frac{1}{\alpha'(t)}[g(\alpha'(t)) - g(0)]. \tag{6.14}$$

Since $x \neq 0$, (6.14) implies that $g(\alpha'(t)) \neq g(0)$. Thus, Lemma 6.4.1 gives us

$$g(1) < \frac{\alpha'(t) - 1}{\alpha'(t)} g(0) + \frac{1}{\alpha'(t)} g(\alpha'(t)). \tag{6.15}$$

Also, $g(1) \neq g(\alpha'(t) + 1)$ (since $g(\alpha'(t)) \neq g(0)$) by (6.11) and (6.12). By employing Lemma 6.4.1, we have

$$\begin{aligned} g(\alpha'(t)) &< \frac{1}{\alpha'(t)}\, g(1) + \frac{\alpha'(t) - 1}{\alpha'(t)}\, g(\alpha'(t) + 1) \\ &= \frac{1}{\alpha'(t)} g(1) + \frac{\alpha'(t) - 1}{\alpha'(t)} \left[g(\alpha'(t)) + \frac{\|x\|^2}{3} \right] \\ &= \frac{1}{\alpha'(t)} g(1) + \frac{\alpha'(t) - 1}{\alpha'(t)} [g(\alpha'(t)) + g(1) - g(0)], \end{aligned}$$

by (6.11) and (6.12). Therefore (by rearranging the last inequality), we have

$$\frac{1}{\alpha'(t)} g(\alpha'(t)) + \frac{\alpha'(t) - 1}{\alpha'(t)}\, g(0) < g(1),$$

which contradicts (6.15). For the case where $\alpha'(t) = 1$, we have

$$g(\alpha'(t) + 1) = g(2) = \frac{\|x\|^2}{3} + g(1) = g(0) + \frac{2\|x\|^2}{3}.$$

Again, note that $g(0) \neq g(2)$ since $x \neq 0$; and Lemma 6.4.1 gives us

$$g(1) < \frac{1}{2}\, g(0) + \frac{1}{2} g(2) = g(0) + \frac{1}{3}\|x\|^2$$

which contradicts (6.11). Therefore, HH-P-orthogonality must be unique. □

The following lemma will be used to prove the uniqueness property of HH-I-orthogonality.

Lemma 6.4.3 (Kikianty and Dragomir [70]). *Let $(\mathbf{X}, \|\cdot\|)$ be a strictly convex normed space, $x, y \in \mathbf{X}$ and $t \in (0,1)$. Let g be a function on $\mathbb{R}$ defined by*

$$g(k) := \int_0^1 \|(1-t)y + k(tx)\|^2 \, dt.$$

Then, g is a strictly convex function on $\mathbb{R}$.

The proof follows readily from the fact that $\mathbf{X}$ is strictly convex. The details are omitted.

The uniqueness of HH-I-orthogonality is a characterization of strictly convex space, as pointed out in the following theorem.

Theorem 6.4.4 (Kikianty and Dragomir [70]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, HH-I-orthogonality is unique if and only if $\mathbf{X}$ is strictly convex.*

Proof. The proof has a similar idea to that of Kapoor and Prasad in [68, p. 405]. Suppose that $\mathbf{X}$ is strictly convex and HH-I-orthogonality is not unique. Then, there exist $x, y \in \mathbf{X}$, where $x \neq 0$ and $\alpha > 0$ such that

$$\begin{aligned} y &\perp_{HH-I} x, \quad \text{and} && (6.16)\\ \alpha x + y &\perp_{HH-I} x. && (6.17) \end{aligned}$$

Recall the strictly convex function g as defined in Lemma 6.4.3, for the given x, y and $t \in (0,1)$ as follows

$$g(k) := \int_0^1 \|(1-t)y + k(tx)\|^2 \, dt.$$

Note that (6.16) gives us

$$\begin{aligned} g(1) &= \int_0^1 \|(1-t)y + tx\|^2 \, dt \\ &= \int_0^1 \|(1-t)y - tx\|^2 \, dt = g(-1). \end{aligned}$$

Set $\alpha'(t) = \frac{(1-t)\alpha}{t}$, $t \in (0,1)$, then

$$\begin{aligned}
g(\alpha'(t)-1) &= \int_0^1 \|(1-t)y + (\alpha'(t)-1)tx\|^2 \, dt \\
&= \int_0^1 \|(1-t)(\alpha x + y) - tx\|^2 \, dt \\
&= \int_0^1 \|(1-t)(\alpha x + y) + tx\|^2 \, dt \\
&= \int_0^1 \|(1-t)y + (\alpha'(t)+1)(tx)\|^2 \, dt = g(\alpha'(t)+1),
\end{aligned}$$

from (6.17). Consider the case where $0 < \alpha'(t) \leq 2$. We have

$$\begin{aligned}
g(\alpha'(t)-1) &= g\left[\left(1 - \frac{\alpha'(t)}{2}\right)(-1) + \frac{\alpha'(t)}{2}(1)\right] \\
&< \left(1 - \frac{\alpha'(t)}{2}\right) g(-1) + \frac{\alpha'(t)}{2}\, g(1) \\
&= g(1) \\
&= g\left[\frac{\alpha'(t)}{2}(\alpha'(t)-1) + \left(1 - \frac{\alpha'(t)}{2}\right)(\alpha'(t)+1)\right] \\
&< \frac{\alpha'(t)}{2}\, g\left(\alpha'(t)-1\right) + \left(1 - \frac{\alpha'(t)}{2}\right) g\left(\alpha'(t)+1\right) \\
&= g(\alpha'(t)-1),
\end{aligned}$$

which leads us to a contradiction. Now consider the case where $\alpha'(t) > 2$. The intervals $[-1,1]$ and $[\alpha'(t)-1, \alpha'(t)+1]$ are disjoint. Therefore, we have two distinct local minimum, one on each of these intervals. But, g is strictly convex and thus can only have one (global) minimum, which yields a contradiction.

Conversely, let us assume that $\mathbf{X}$ is not strictly convex. Let $x, y \in \mathbf{X}$, $x \neq y$, such that $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$. Then, the quantities

$$\int_0^1 \left\|(1-t)\frac{x+y}{1-t}\right\|^2 \, dt = \|x+y\|^2,$$

$$\int_0^1 \left\|(1-t)\frac{x+y}{1-t} + t\left(\frac{x-y}{t}\right)\right\|^2 \, dt = 4\|x\|^2,$$

and

$$\int_0^1 \left\|(1-t)\frac{x+y}{1-t} - t\left(\frac{x-y}{t}\right)\right\|^2 \, dt = 4\|y\|^2,$$

are all equal by our assumption.

Set $x' = \frac{x+y}{1-t}$ and $y' = \frac{x-y}{t}$, so we have

$$\int_0^1 \|(1-t)x'\|^2 \; dt = \int_0^1 \|(1-t)x' + ty'\|^2 \; dt \tag{6.18}$$

$$= \int_0^1 \|(1-t)x' - ty'\|^2 \; dt. \tag{6.19}$$

Note that by (6.18), we have

$$\begin{aligned}
& \int_0^1 \left\|(1-t)\left(x' + \frac{ty'}{2(1-t)}\right) + t\left(\frac{y'}{2}\right)\right\|^2 \; dt \\
= & \int_0^1 \|(1-t)x' + ty'\|^2 \; dt \\
= & \int_0^1 \|(1-t)x'\|^2 \; dt \\
= & \int_0^1 \left\|(1-t)\left(x' + \frac{ty'}{2(1-t)}\right) - t\left(\frac{y'}{2}\right)\right\|^2 \; dt,
\end{aligned}$$

that is,

$$\left[x' + \frac{t}{1-t}\left(\frac{y'}{2}\right)\right] \perp_{HH-I} \left(\frac{y'}{2}\right). \tag{6.20}$$

Also, by (6.19) we have

$$\begin{aligned}
& \int_0^1 \left\|(1-t)\left(x' - \frac{ty'}{2(1-t)}\right) - t\left(\frac{y'}{2}\right)\right\|^2 \; dt \\
= & \int_0^1 \|(1-t)x' - ty'\|^2 \; dt \\
= & \int_0^1 \|(1-t)x'\|^2 \; dt \\
= & \int_0^1 \left\|(1-t)\left(x' - \frac{ty'}{2(1-t)}\right) + t\left(\frac{y'}{2}\right)\right\|^2 \; dt,
\end{aligned}$$

that is,

$$\left[x' - \frac{t}{1-t}\left(\frac{y'}{2}\right)\right] \perp_{HH-I} \left(\frac{y'}{2}\right). \tag{6.21}$$

By (6.20) and (6.21), we conclude that HH-I-orthogonality is not unique. □

6.5 Characterizations of inner product spaces

The main problem in the study of Banach space geometry is that most geometric properties fail to hold in a general normed space, unless it is an inner product space. The characterization of inner product space has become an important area of research due to this fact. Dan Amir has gathered numerous results concerning this field in his book "*Characterizations of inner product spaces*" [5].

The main result of this section is a characterization of inner product space via the homogeneity (and the additivity to the left) of HH-C-orthogonality (hence, HH-P-orthogonality and HH-I-orthogonality).

Theorem 6.5.1 (Kikianty and Dragomir [69])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space in which HH-C-orthogonality is homogeneous (or additive to the left). Then,* $\mathbf{X}$ *is an inner product space.*

It is important to note that since both HH-P-orthogonality and HH-I-orthogonality are symmetric, Theorem 6.5.1 also holds for the additivity property to the right, in these two cases. In Section 6.6, the alternative proofs for these cases are given.

The proof of this theorem is described in this section in two separate cases: the case for normed space of dimension 3 and higher, and the 2-dimensional case. In both cases, we consider a property introduced by Carlsson [19, p. 301], which is weaker than the homogeneity and the additivity of the orthogonality (cf. Definition 6.1.23).

The following is a 'modified' definition of the property.

Definition 6.5.2 (Kikianty and Dragomir [69])**.** Let m be a positive integer. Then, HH-C-orthogonality is said to have property (H) in a normed space $\mathbf{X}$ if $x \perp_{HH-C} y$ implies that

$$\lim_{n\to\infty} n^{-1} \int_0^1 \sum_{i=1}^{m} \alpha_i \|n\beta_i(1-t)x + \gamma_i t y\|^2 \, dt = 0. \tag{6.22}$$

Note that

1. if HH-C-orthogonality is homogeneous (or additive to the left) in $\mathbf{X}$, then it has property (H);
2. if $\mathbf{X}$ is an inner product space, then HH-C-orthogonality is homogeneous (or additive), and therefore has property (H).

Thus, in order to prove Theorem 6.5.1, it is sufficient to show that if the HH-C-orthogonality has property (H) in $\mathbf{X}$, then $\mathbf{X}$ is an inner product space.

The case of dimension 3 and higher

For the case of normed spaces with dimension 3 and higher, the proof of Theorem 6.5.1 follows by the fact that property (H) of HH-C-orthogonality implies that this orthogonality is symmetric and equivalent to B-orthogonality, whose proof will be described in this subsection. Recall that the symmetry of B-orthogonality characterizes inner product space of dimension 3 and higher (cf. Proposition 6.1.6).

The following propositions will be used to prove the theorem. We refer to Lemma 2.6. and Lemma 2.7. of Carlsson [19] for the proofs. Before stating the propositions, we recall the following notation:

$$(\nabla_+\|\cdot\|(x))(y) := \lim_{t\to 0^+}\frac{\|x+ty\|-\|x\|}{t}$$

$$\text{and}\quad (\nabla_-\|\cdot\|(x))(y) := \lim_{t\to 0^-}\frac{\|x+ty\|-\|x\|}{t}\qquad x,y\in(\mathbf{X},\|\cdot\|)$$

that is, the right- and left-Gâteaux derivatives at $x\in\mathbf{X}\setminus\{0\}$.

Proposition 6.5.3 (Carlsson [19]). *Let $(\mathbf{X},\|\cdot\|)$ be a normed space. Then, for $\lambda\mu>0$ we have*

$$\begin{aligned}(\nabla_+\|\cdot\|(\lambda x))(\mu y) &= |\mu|(\nabla_+\|\cdot\|(x))(y),\\ \textit{and}\quad (\nabla_-\|\cdot\|(\lambda x))(\mu y) &= |\mu|(\nabla_-\|\cdot\|(x))(y);\end{aligned}$$

and for $\lambda\mu<0$

$$\begin{aligned}(\nabla_+\|\cdot\|(\lambda x))(\mu y) &= -|\mu|(\nabla_-\|\cdot\|(x))(y),\\ \textit{and}\quad (\nabla_-\|\cdot\|(\lambda x))(\mu y) &= -|\mu|(\nabla_+\|\cdot\|(x))(y).\end{aligned}$$

Proposition 6.5.4 (Carlsson [19]). *Let $(\mathbf{X},\|\cdot\|)$ be a normed linear space. If there exist two real numbers λ and μ with $\lambda+\mu\neq 0$, such that*

$$\lambda(\nabla_+\|\cdot\|(x))(y)+\mu(\nabla_-\|\cdot\|(x))(y)$$

is a continuous function of $x,y\in\mathbf{X}$, then the norm $\|\cdot\|$ is Gâteaux differentiable.

The following lemma will also be employed to prove Theorem 6.5.1 for the case of normed spaces of dimension 3 and higher. This lemma also gives us the uniqueness of HH-C-orthogonality.

Lemma 6.5.5 (Kikianty and Dragomir [69]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space where HH-C-orthogonality has property (H). Suppose that for any* $x, y \in \mathbf{X}$*, there exists* $\lambda \in \mathbb{R}$ *such that* $x \perp_{HH-C} (\lambda x + y)$*. Then,*

$$\lambda = -\|x\|^{-1}\left[\sum_{\beta_i\gamma_i>0} \alpha_i\beta_i\gamma_i(\nabla_+\|\cdot\|(x))(y) + \sum_{\beta_i\gamma_i<0} \alpha_i\beta_i\gamma_i(\nabla_-\|\cdot\|(x))(y)\right].$$

Proof. Suppose that m is a positive integer. By assumption, we have

$$\lim_{n\to\infty} n^{-1}\int_0^1 \sum_{i=1}^m \alpha_i\|n\beta_i(1-t)x + \gamma_i t(\lambda x + y)\|^2\, dt = 0. \tag{6.23}$$

Note that by Lemma 6.3.4, we have the following for any i and $t \in (0,1)$ (again, note that we exclude the extremities to ensure that we can employ Lemma 6.3.4)

$$\begin{aligned} & n^{-1}\|[n\beta_i(1-t) + \gamma_i t\lambda]x + \gamma_i t y\|^2 \\ = \;& n^{-1}\|n\beta_i(1-t)x + \gamma_i t y\|^2 + 2\beta_i(1-t)\gamma_i t\lambda\|x\|^2 + \varepsilon_i(n), \end{aligned} \tag{6.24}$$

where $\varepsilon_i(n) \to 0$ when $n \to 0$. Now, we multiply (6.24) by α_i and integrate it over $(0,1)$, to get

$$\begin{aligned} & n^{-1}\alpha_i\int_0^1 \|[n\beta_i(1-t) + \gamma_i t\lambda]x + \gamma_i t y\|^2\, dt \\ = \;& n^{-1}\alpha_i\int_0^1 \|n\beta_i(1-t)x + \gamma_i t y\|^2\, dt + 2\alpha_i\beta_i\gamma_i\lambda\|x\|^2\int_0^1 (1-t)t\, dt + \varepsilon_i(n). \end{aligned} \tag{6.25}$$

Take the sum over $i \in \{1, \ldots, m\}$ and let $n \to \infty$ to get

$$0 = \lim_{n\to\infty} n^{-1}\sum_{i=1}^m \alpha_i\int_0^1 \|n\beta_i(1-t)x + \gamma_i t y\|^2 dt + \frac{1}{3}\lambda\|x\|^2 \tag{6.26}$$

(note the use of (6.25) and $\sum_{i=1}^m \alpha_i\beta_i\gamma_i = 1$).

Now, since $\sum_{i=1}^m \alpha_i\beta_i^2 = 0$, then

$$n^{-1}\sum_{i=1}^m \alpha_i\int_0^1 \|n\beta_i(1-t)x + \gamma_i t y\|^2 dt$$

$$\begin{aligned}
&= n^{-1}\sum_{i=1}^{m}\alpha_i\int_0^1[\|n\beta_i(1-t)x+\gamma_i ty\|^2-\|n\beta_i(1-t)x\|^2]dt \\
&= \sum_{i=1}^{m}\alpha_i\int_0^1[\|n\beta_i(1-t)x+\gamma_i ty\|-\|n\beta_i(1-t)x\|] \\
&\qquad\times n^{-1}[\|n\beta_i(1-t)x+\gamma_i ty\|+\|n\beta_i(1-t)x\|]dt.
\end{aligned}$$

Rewrite

$$\|n\beta_i(1-t)x+\gamma_i ty\|-\|n\beta_i(1-t)x\|$$

as

$$n\left(\|\beta_i(1-t)x+\tfrac{1}{n}\gamma_i ty\|-\|\beta_i(1-t)x\|\right),$$

to obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\left(\|\beta_i(1-t)x+\tfrac{1}{n}\gamma_i ty\|-\|\beta_i(1-t)x\|\right) \\
&= \lim_{s\to 0^+}\frac{\|\beta_i(1-t)x+s\gamma_i ty\|-\|\beta_i(1-t)x\|}{s} \\
&= (\nabla_+\|\cdot\|(\beta_i(1-t)x))(\gamma_i ty).
\end{aligned}$$

Note also that

$$\begin{aligned}
&\lim_{n\to\infty} n^{-1}[\|n\beta_i(1-t)x+\gamma_i ty\|+\|n\beta_i(1-t)x\|] \\
&= \lim_{n\to\infty}[\|\beta_i(1-t)x+n^{-1}\gamma_i ty\|+\|\beta_i(1-t)x\|] \\
&= 2\|\beta_i(1-t)x\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} n^{-1}\sum_{i=1}^{m}\alpha_i\int_0^1\|n\beta_i(1-t)x+\gamma_i ty\|^2dt \\
&= 2\sum_{i=1}^{m}\alpha_i\int_0^1(\nabla_+\|\cdot\|(\beta_i(1-t)x))(\gamma_i ty)\|\beta_i(1-t)x\|dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda &= -3\|x\|^{-2}\sum_{i=1}^{m}\alpha_i\int_0^1(\nabla_+\|\cdot\|(\beta_i(1-t)x))(\gamma_i ty)2\|\beta_i(1-t)x\|\ dt \\
&= -6\|x\|^{-1}\sum_{i=1}^{m}\alpha_i|\beta_i|\int_0^1(1-t)\tau_+(\beta_i(1-t)x,\gamma_i ty)\ dt.
\end{aligned}$$

By Proposition 6.5.3, (6.26) gives us

$$\begin{aligned}
\lambda &= -6\|x\|^{-1}\int_0^1 (1-t)t\,dt \\
&\quad\times\left[\sum_{\beta_i\gamma_i>0}\alpha_i\beta_i\gamma_i(\nabla_+\|\cdot\|(x))(y)+\sum_{\beta_i\gamma_i<0}\alpha_i\beta_i\gamma_i(\nabla_-\|\cdot\|(x))(y)\right] \\
&= -\|x\|^{-1}\left[\sum_{\beta_i\gamma_i>0}\alpha_i\beta_i\gamma_i(\nabla_+\|\cdot\|(x))(y)+\sum_{\beta_i\gamma_i<0}\alpha_i\beta_i\gamma_i(\nabla_-\|\cdot\|(x))(y)\right].
\end{aligned}$$

This completes the proof. □

Now, we have a unique λ for any $x, y \in \mathbf{X}$ such that $x \perp_{HH-C} \lambda x + y$. As a function of x and y, $\lambda = \lambda(x, y)$ is a continuous function [19, p. 303]. Thus,

$$\sum_{\beta_i\gamma_i>0}\alpha_i\beta_i\gamma_i(\nabla_+\|\cdot\|(x))(y)+\sum_{\beta_i\gamma_i<0}\alpha_i\beta_i\gamma_i(\nabla_-\|\cdot\|(x))(y)$$

is also a continuous function in $x, y \in \mathbf{X}$. By Proposition 6.5.4, the norm $\|\cdot\|$ is Gâteaux differentiable. Hence, we have the following consequence.

Corollary 6.5.6 (Kikianty and Dragomir [69])**.** *If HH-C-orthogonality has property (H), then the norm of* $\mathbf{X}$ *is Gâteaux differentiable and* $x \perp_{HH-C} y$ *holds if and only if* $(\nabla\|\cdot\|(x))(y) = 0$, *that is,* $x \perp y$ (B).

Remark 6.5.7. We note that the function $(\nabla\|\cdot\|(x))(y)$ is also continuous as a function of x and y.

Let us assume that x is HH-C-anti-orthogonal to y if and only if $y \perp_{HH-C} x$. We have shown that when HH-C-orthogonality has property (H), then it is equivalent to B-orthogonality and therefore is homogeneous (since B-orthogonality is homogeneous). This fact implies that HH-C-anti-orthogonality has property (H), as the homogeneity property implies property (H). Therefore, the above results also hold for HH-C-anti-orthogonality. In particular, $(\nabla\|\cdot\|(x))(y) = 0$ implies that $x \perp_{HH-C} y$, that is, y is HH-C-anti-orthogonal to x. Hence, $(\nabla\|\cdot\|(y))(x) = 0$. Thus, B-orthogonality is symmetric; and we obtain the following consequence.

Corollary 6.5.8 (Kikianty and Dragomir [69])**.** *If HH-C-orthogonality has property (H), then it is symmetric and equivalent to* B*-orthogonality.*

Proof of Theorem 6.5.1 for three-dimensional case (and higher). Assume HH-C-orthogonality has property (H). By Proposition 6.1.6 and Corollary 6.5.8, it follows that for 3-dimensional normed spaces (and higher), the symmetrical B-orthogonality implies that the norm is induced by an inner product. This completes the proof. $\square$

The 2-dimensional case

Previously, we have defined that $x \perp_{HH-C} y$, when x and y satisfy

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt = 0,$$

where

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1,$$

and m a positive integer. In this subsection, we use a slightly different notation, in order to resolve the 2-dimensional problem. Observe that

$$\begin{aligned} &\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt \qquad (6.27) \\ = &\sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \int_0^1 \left\|(1-t)x + t\frac{\gamma_i}{\beta_1} y\right\|^2 dt + \frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i = 0} \alpha_i \beta_i^2 \|x\|^2 + \frac{1}{3} \sum_{\beta_i = 0, \gamma_i \neq 0} \alpha_i \gamma_i^2 \|y\|^2. \end{aligned}$$

Since $\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0$,

$$\frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i = 0} \alpha_i \beta_i^2 \|x\|^2 = -\frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \|x\|^2,$$

and similarly,

$$\frac{1}{3} \sum_{\beta_i = 0, \gamma_i \neq 0} \alpha_i \gamma_i^2 \|y\|^2 = -\frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \gamma_i^2 \|y\|^2.$$

Therefore, (6.27) becomes

$$\begin{aligned} &\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt \\ = &\sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \int_0^1 \left\|(1-t)x + t\frac{\gamma_i}{\beta_1} y\right\|^2 dt - \frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \|x\|^2 - \frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \gamma_i^2 \|y\|^2. \end{aligned}$$

We set $p_i = \alpha_i \beta_i^2$ and $q_i = \gamma_i/\beta_i$ and rearrange the indices to obtain

$$\begin{aligned} &\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \; dt \\ = &\sum_{k=1}^{r} p_k \int_0^1 \|(1-t)x + tq_k y\|^2 \; dt - \frac{1}{3}\sum_{k=1}^{r} p_k \|x\|^2 - \frac{1}{3}\sum_{k=1}^{r} p_k q_k^2 \|y\|^2. \end{aligned} \tag{6.28}$$

Assume that HH-C-orthogonality has property (H). Then, it is equivalent to B-orthogonality, and therefore is homogeneous. Denote $S_{\mathbf{X}}$ to be the unit circle in $\mathbf{X}$ with respect to the norm $\|\cdot\|$ and let $x, y \in S_{\mathbf{X}}$ such that $x \perp_{HH-C} y$. Then, (6.28) gives us

$$3\sum_{k=1}^{r} p_k \int_0^1 \|(1-t)x + tq_k \alpha y\|^2 \; dt = C_1 + C_2\alpha^2,$$

where $C_1 = \sum_{k=1}^{r} p_k$ and $C_2 = \sum_{k=1}^{r} p_k q_k^2$. We may conclude that the function

$$\phi(\alpha) = 3\int_0^1 \|(1-t)x + t\alpha y\|^2 \; dt$$

is the solution of the functional equation

$$\sum_{k=1}^{r} p_k F(q_k \alpha) = C_1 + C_2\alpha^2, \quad -\infty < \alpha < \infty, \tag{6.29}$$

where

$$\sum_{k=1}^{r} p_k = C_1, \quad \sum_{k=1}^{r} p_k q_k^2 = C_2, \quad \sum_{k=1}^{r} p_k q_k = 1, \quad q_k \neq 0, \; k = 1, \ldots, r. \tag{6.30}$$

We note that the function ϕ is continuously differentiable from Corollary 6.5.6 and Remark 6.5.7.

In the following results by Carlsson [19], it is shown that the behaviour of ϕ for large and small values of $|\alpha|$ gives us an explicit formula for ϕ.

Definition 6.5.9 (Carlsson [19]). Let r and s be two positive integers. Given a functional equation

$$\sum_{k=1}^{r} p_k F(q_k \alpha) = C_1 + C_2\alpha^2, \quad -\infty < \alpha < \infty,$$

for some real numbers C_1, C_2, p_k and q_k. We say that the equation is *symmetric* if it can be written in the form

$$\sum_{k=1}^{s} m_k F(n_k\alpha) - \sum_{k=1}^{s} m_k F(-n_k\alpha) = C_1 + C_2\alpha^2$$

for some real numbers C_1, C_2, m_k and n_k; otherwise, it is *nonsymmetric.*

Lemma 6.5.10 (Carlsson [19])**.** *Let $\phi(\alpha)$ be a continuously differentiable solution of the functional equation (6.29) satisfying (6.30) and*

$$\begin{aligned} \phi(\alpha) &= 1 + O(\alpha^2) \quad \textit{when } \alpha \to 0 \\ \phi(\alpha) &= \alpha^2 + O(\alpha) \quad \textit{when } \alpha \to \pm\infty. \end{aligned}$$

If (6.29) is nonsymmetric, then $\phi(\alpha) = 1 + \alpha^2$ for $-\infty < \alpha < \infty$. If (6.29) is (nontrivially) symmetric, then $\phi(\alpha) = \phi(-\alpha)$ for $-\infty < \alpha < \infty$.

A 2-dimensional normed space has certain properties that enable us to work on a smaller subset. One of the useful properties is stated in Lemma 6.5.11. Before stating the lemma, recall that the norm $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ is said to be *Fréchet differentiable* (or, differentiable) at $x \in \mathbf{X}$ if and only if there exists a continuous linear functional φ'_x on $\mathbf{X}$ such that

$$\lim_{\|z\|\to 0} \frac{|\|x+z\| - \|x\| - \varphi'_x(z)|}{\|z\|} = 0.$$

It is said to be *twice (Fréchet) differentiable* at $x \in \mathbf{X}$ if and only if there exists a continuous bilinear functional φ''_x on $\mathbf{X}^2$ such that

$$\lim_{\|z\|\to 0} \frac{|\|x+z\| - \|x\| - \varphi'_x(z) - \varphi''_x(z,z)|}{\|z\|^2} = 0.$$

Lemma 6.5.11 (Amir [5])**.** *If $(\mathbf{X}, \|\cdot\|)$ is a 2-dimensional normed space, then the norm is twice (Fréchet) differentiable almost everywhere on the unit circle $S_{\mathbf{X}} = \{u \in \mathbf{X} : \|u\| = 1\}$.*

This result follows by the fact that the direction of the left-side tangent is a monotone function and therefore, by Lebesgue's theorem, is differentiable almost everywhere [5, p. 22].

By assuming that HH-C-orthogonality is homogeneous, we may restrict ourself to work on the unit circle. Furthermore, the previous proposition enables us to work on a dense subset of the unit circle.

Proof of Theorem 6.5.1 for 2-dimensional case. Assume HH-C-orthogonality has property (H). Then, it is equivalent to B-orthogonality, and therefore is homogeneous. Since $\dim(\mathbf{X}) = 2$, then the norm $\|\cdot\|$ is twice differentiable for almost every $u \in S_{\mathbf{X}}$.

Let D be the subset of $S_{\mathbf{X}}$ consists of all points where the norm $\|\cdot\|$ is twice differentiable. Let $x \in D$ and $x \perp_{HH-C} y$ (or, equivalently $x \perp y$ (B)) with $\|y\| = 1$. Then, the function

$$\phi(\alpha) = 3\int_0^1 \|(1-t)x + t\alpha y\|^2 \ dt$$

is a continuously differentiable solution of the functional equation (6.29) satisfying (6.30).

Claim 6.5.12. The function ϕ satisfies

$$\begin{aligned} \phi(\alpha) &= 1 + O(\alpha^2) \quad \text{when } \alpha \to 0 \\ \phi(\alpha) &= \alpha^2 + O(\alpha) \quad \text{when } \alpha \to \pm\infty. \end{aligned}$$

The proof of this claim will be stated at the end of this section as Lemmas 6.5.13 and 6.5.14.

Case 1: Equation (6.30) is nonsymmetric. It follows from Lemma 6.5.10 that $\phi(\alpha) = 1 + \alpha^2$. If we choose x and y as the unit vectors of a coordinate system in the plane $\mathbf{X}$ and write $w = \nu x + \eta y$, we see that $\|w\| = 1$ if and only if $\nu^2 + \eta^2 = 1$. This means that the unit circle has the equation $\nu^2 + \eta^2 = 1$, that is, an Euclidean circle. Therefore, $\mathbf{X}$ is an inner product space.

Case 2: Equation (6.30) is symmetric. It follows from Lemma 6.5.10 that $\phi(\alpha) = \phi(-\alpha)$ for all α, that is,

$$\int_0^1 \|(1-t)x + t\alpha y\|^2 \ dt = \int_0^1 \|(1-t)x - t\alpha y\|^2 \ dt \tag{6.31}$$

holds for any $\alpha \in \mathbb{R}$, $x, y \in \mathbf{X}$ where $x \in D$ and $x \perp_{HH-C} y$.

Since D is a dense subset of C and HH-C-orthogonality is homogeneous, we conclude that (6.31) also holds for any $x \in \mathbf{X}$ where $x \perp_{HH-C} y$. Let $t \in (0,1)$ and for any $\beta \in \mathbb{R}$, choose $\alpha = \frac{(1-t)}{t}\beta$. Then, (6.31) gives us

$$\int_0^1 \|(1-t)x + (1-t)\beta y\|^2 \ dt = \int_0^1 \|(1-t)x - (1-t)\beta y\|^2 \ dt,$$

or equivalently

$$\|x + \beta y\| = \|x - \beta y\|,$$

that is, $x \perp y$ (R). Thus, HH-C-orthogonality implies R-orthogonality. Since HH-C-orthogonality is existent, $\mathbf{X}$ is an inner product space by Proposition 6.1.27. $\square$

The proof of claim is stated as the following lemmas:

Lemma 6.5.13 (Kikianty and Dragomir [69])**.** *Suppose that* $(\mathbf{X}, \|\cdot\|)$ *be a 2-dimensional normed space and denote its unit circle by* $S_{\mathbf{X}}$. *Let* $u, v \in S_{\mathbf{X}}$. *Then, the function*

$$\phi(\alpha) = 3\int_0^1 \|(1-t)u + t\alpha v\|^2 \ dt$$

satisfies the condition

$$\phi(\alpha) = \alpha^2 + O(\alpha) \quad \textit{when } \alpha \to \pm\infty.$$

Proof. For any $u, v \in S_{\mathbf{X}}$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}
& |\phi(\alpha) - \alpha^2| \\
= & \left|3\left(\int_0^1 \|(1-t)u + t\alpha v\|^2 \ dt - \frac{1}{3}\alpha^2\right)\right| \\
= & \left|3\int_0^1 \|(1-t)u + t\alpha v\|^2 - \|t\alpha v\|^2 \ dt\right| \\
\leq & \ 3\int_0^1 \left|\|(1-t)u + t\alpha v\|^2 - \|t\alpha v\|^2\right| \ dt \\
= & \ 3\int_0^1 \left|\|(1-t)u + t\alpha v\| - \|t\alpha v\|\right|\Big(\|(1-t)u + t\alpha v\| + \|t\alpha v\|\Big) \ dt \\
\leq & \ 3\int_0^1 (1-t)\,\|u\|\Big((1-t)\|u\| + 2t\|\alpha v\|\Big) \ dt \\
= & \ 3\int_0^1 ((1-t)^2 + 2t(1-t)|\alpha|) \ dt = 1 + |\alpha|.
\end{aligned}$$

Thus, $\phi(\alpha) - \alpha^2 = O(\alpha)$, when $\alpha \to \pm\infty$. $\square$

Lemma 6.5.14 (Kikianty and Dragomir [69])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a 2-dimensional normed space and denote its unit circle by* $S_{\mathbf{X}}$*. Then, there is a dense subset* D *of* $S_{\mathbf{X}}$ *such that if* $u \in D$ *and* $u \perp v$ (B)*, the function*

$$\phi(\alpha) = 3 \int_0^1 \|(1-t)u + t\alpha v\|^2 \ dt$$

satisfies

$$\phi(\alpha) = 1 + O(\alpha^2) \quad \textit{when } \alpha \to 0. \tag{6.32}$$

Proof. Since $\dim(\mathbf{X}) = 2$, then the norm $\|\cdot\|$ is twice differentiable for almost every $u \in S_{\mathbf{X}}$ by Lemma 6.5.11 [5, p. 22]. Let D be the subset of $S_{\mathbf{X}}$ consists of all points where the norm $\|\cdot\|$ is twice differentiable. We conclude that D is a dense subset of $S_{\mathbf{X}}$. Denote $\varphi(x) = \|x\|$, then for any $u \in D$, the derivative φ'_u is a linear functional and the second derivative φ''_u is a bilinear functional. Furthermore, we have

$$\lim_{\|z\|\to 0} \left| \frac{\|u+z\| - \|u\| - \varphi'_u(z) - \varphi''_u(z,z)}{\|z\|^2} \right| = 0. \tag{6.33}$$

Let $u \in D$ and $u \perp v$ (B), where $\|v\| = 1$. Set $z = \frac{t}{1-t}\alpha v$ $(t \in (0,1))$. Therefore, when $\alpha \to 0$, $\|z\| \to 0$. Since $u \perp v$ (B), $\varphi'_u(v) = 0$; and (6.33) gives us

$$\lim_{\alpha \to 0} \left| \frac{\left\|u + \left(\frac{t}{1-t}\alpha\right) v\right\| - \|u\| - \left(\frac{t}{1-t}\alpha\right)^2 \varphi''_u(v,v)}{\left(\frac{t}{1-t}\alpha\right)^2 \|v\|^2} \right| = 0,$$

that is, for any $\epsilon > 0$, there exists $\delta_0 > 0$, such that for any $|\alpha| < \delta_0$, we have

$$\left| \frac{\left\|u + \left(\frac{t}{1-t}\alpha\right) v\right\| - 1}{\left(\frac{t}{1-t}\alpha\right)^2} - \varphi''_u(v,v) \right| < \epsilon.$$

Furthermore, by triangle inequality, we obtain

$$\left| \frac{\left\|u + \frac{t}{(1-t)}\alpha v\right\| - 1}{\frac{t^2}{(1-t)^2}\alpha^2} \right| < \epsilon + |\varphi''_u(v,v)| = M.$$

Equivalently, we have,

$$\left| \left\|u + \frac{t}{1-t}\alpha v\right\| - 1 \right| < M \frac{t^2}{(1-t)^2}\alpha^2.$$

Note that for any $t \in (0,1)$, $\left\|u + \frac{1-t}{t}\alpha v\right\| + 1 \to 2$ when $\alpha \to 0$. It can be shown that there exists δ_1 such that for any $|\alpha| < \delta_1$, we have

$$\left|\left\|u + \frac{t}{1-t}\alpha v\right\| + 1 - 2\right| < 1,$$

that is,

$$\left\|u + \frac{t}{1-t}\alpha v\right\| + 1 < 1 + 2 = 3.$$

Now, for any $|\alpha| < \min\{\delta_0, \delta_1\}$, we have

$$\begin{aligned}
|\phi(\alpha) - 1| &= \left|3\left(\int_0^1 \|(1-t)u + t\alpha v\|^2 \ dt - \frac{1}{3}\right)\right| \\
&= \left|3\int_0^1 \|(1-t)u + t\alpha v\|^2 - \|(1-t)u\|^2 \ dt\right| \\
&\leq 3\int_0^1 \left|\|(1-t)u + t\alpha v\|^2 - \|(1-t)u\|^2\right| \ dt \\
&= 3\int_0^1 (1-t)^2 \left|\left\|u + \frac{t}{1-t}\alpha v\right\|^2 - \|u\|^2\right| \ dt \\
&= 3\int_0^1 (1-t)^2 \left(\left\|u + \frac{t}{1-t}\alpha v\right\| + 1\right)\left|\left\|u + \frac{t}{1-t}\alpha v\right\| - 1\right| \ dt \\
&< 9\int_0^1 (1-t)^2 M \frac{t^2}{(1-t)^2}\alpha^2 \ dt = 3M\alpha^2,
\end{aligned}$$

that is, $\phi(\alpha) - 1 = O(\alpha^2)$, when $\alpha \to 0$. $\square$

The last results conclude that the homogeneity (also, the right-additivity) of HH-C-orthogonality is a necessary and sufficient condition for the normed space to be an inner product space.

6.6 Alternative proofs for special cases

In this section, alternative proofs for the existence and characterization of inner product spaces via the homogeneity and the additivity properties, are provided in particular settings of HH-P-orthogonality and HH-I-orthogonality.

6.6.1 Existence

The following theorem gives us the existence of HH-P-orthogonality. The proof for the existence property employs the similar continuity argument and the intermediate value theorem, which was used by James [61, p. 299–300].

Theorem 6.6.1 (Kikianty and Dragomir [70]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. Then, HH-P-orthogonality is existent.*

Proof. Fix $x, y \in \mathbf{X}$. The proof is trivial when $x = 0$, thus, we assume that $x \neq 0$.

Let $f : \mathbb{R} \times (0,1) \to \mathbb{R}$ be a function defined by

$$f(\alpha, t) := t^2\|x\|^2 + (1-t)^2\|(\alpha x + y)\|^2 - \|(1-t)(\alpha x + y) + tx\|^2, \tag{6.34}$$

and F be a function on $\mathbb{R}$ defined by $F(\alpha) := \int_0^1 f(\alpha, t)\, dt$.

Note that when $F(\alpha) = 0$, we have

$$\|x\|^2 \int_0^1 t^2 dt + \|(\alpha x + y)\|^2 \int_0^1 (1-t)^2 dt - \int_0^1 \|(1-t)(\alpha x + y) + tx\|^2 dt = 0,$$

which is equivalent to

$$\int_0^1 \|(1-t)(\alpha x + y) + tx\|^2 dt = \frac{1}{3}\left(\|(\alpha x + y)\|^2 + \|x\|^2\right),$$

that is, $(\alpha x + y) \perp_{HH-C} x$. To show that there exists an α such that the continuous function F is zero, we apply the intermediate value theorem to show that there exist two distinct α_1 and α_2 such that $F(\alpha_1) < 0$ and $F(\alpha_2) > 0$.

Let $\alpha > 0$. Since $t \neq 1$, we have the following identity

$$1 = -\frac{2t(1-t)\alpha + t^2}{(1-t)^2\alpha^2} + \left(\frac{(1-t)\alpha + t}{(1-t)\alpha}\right)^2.$$

We have

$$\begin{aligned} f(\alpha, t) &= t^2\|x\|^2 - \frac{2t(1-t)\alpha + t^2}{(1-t)^2\alpha^2}(1-t)^2\|\alpha x + y\|^2 \\ &\quad + \left(\frac{(1-t)\alpha + t}{(1-t)\alpha}\right)^2 (1-t)^2\|\alpha x + y\|^2 - \|[(1-t)\alpha + t]x + (1-t)y\|^2 \end{aligned}$$

$$
\begin{aligned}
&= t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left\|x + \frac{y}{\alpha}\right\|^2 \\
&\quad + \left\|[(1-t)\alpha + t]x + \left[(1-t) + \frac{t}{\alpha}\right]y\right\|^2 - \|[(1-t)\alpha + t]x + (1-t)y\|^2 \\
&= t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left\|x + \frac{y}{\alpha}\right\|^2 \\
&\quad + \left[\left\|[(1-t)\alpha + t]x + \left[(1-t) + \frac{t}{\alpha}\right]y\right\| - \|[(1-t)\alpha + t]x + (1-t)y\|\right] \\
&\quad \times \left[\left\|[(1-t)\alpha + t]x + \left[(1-t) + \frac{t}{\alpha}\right]y\right\| + \|[(1-t)\alpha + t]x + (1-t)y\|\right].
\end{aligned}
$$

Note that by the triangle inequality, we have

$$
\begin{aligned}
&\left[\left\|[(1-t)\alpha + t]x + \left[(1-t) + \frac{t}{\alpha}\right]y\right\| - \|[(1-t)\alpha + t]x + (1-t)y\|\right] \\
\leq\ & \left|\frac{t}{\alpha}\right|\|y\| = \frac{t}{\alpha}\|y\|,
\end{aligned}
$$

since $\alpha > 0$. Again, by the triangle inequality,

$$
\begin{aligned}
&\left[\left\|[(1-t)\alpha + t]x + \left[(1-t) + \frac{t}{\alpha}\right]y\right\| + \|[(1-t)\alpha + t]x + (1-t)y\|\right] \\
\leq\ & 2|(1-t)\alpha + t|\|x\| + \left[\left|(1-t) + \frac{t}{\alpha}\right| + (1-t)\right]\|y\| \\
=\ & 2[(1-t)\alpha + t]\|x\| + \left[2(1-t) + \frac{t}{\alpha}\right]\|y\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&f(\alpha, t) \\
\leq\ & t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left\|x + \frac{y}{\alpha}\right\|^2 \\
&+ \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
\leq\ & t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\left(\|x\| - \left\|\frac{y}{\alpha}\right\|\right)^2 \\
&+ \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
=\ & t^2\|x\|^2 - \left[2t(1-t)\alpha + t^2\right]\|x\|^2 \\
&+2\left[2t(1-t)\alpha + t^2\right]\|x\|\left\|\frac{y}{\alpha}\right\| - \left[2t(1-t)\alpha + t^2\right]\left\|\frac{y}{\alpha}\right\|^2 \\
&+ \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2
\end{aligned}
$$

$$\begin{aligned}
&= -2t(1-t)\alpha\|x\|^2 + \left[4t(1-t) + \frac{2t^2}{\alpha}\right]\|x\|\|y\| - \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&\quad + \left(2t(1-t) + \frac{2t^2}{\alpha}\right)\|x\|\|y\| + \left[\frac{2t(1-t)}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&= -2t(1-t)\alpha\|x\|^2 + \left[6t(1-t) + \frac{4t^2}{\alpha}\right]\|x\|\|y\|.
\end{aligned}$$

Integrate the last inequality with respect to t on $(0,1)$ to get

$$\begin{aligned}
F(\alpha) &= \int_0^1 f(\alpha,t)dt \\
&\leq \int_0^1 \left(-2t(1-t)\alpha\|x\|^2 + \left[6t(1-t) + \frac{4t^2}{\alpha}\right]\|x\|\|y\|\right) dt \\
&= -\frac{1}{3}\alpha\|x\|^2 + \left[1 + \frac{4}{3\alpha}\right]\|x\|\|y\|.
\end{aligned}$$

By taking α sufficiently large (denote this value by α_1) and since x is nonzero, we have

$$F(\alpha_1) = \int_0^1 f(\alpha_1,t)dt < 0.$$

Again, let $\alpha > 0$, we have

$$f(-\alpha,t) = t^2\|x\|^2 + (1-t)^2\|\alpha x - y\|^2 - \|(1-t)(\alpha x - y) - tx\|^2.$$

Since $t \neq 1$, we have the following identity

$$1 = \frac{2t(1-t)\alpha - t^2}{(1-t)^2\alpha^2} + \left(\frac{(1-t)\alpha - t}{(1-t)\alpha}\right)^2.$$

Thus, we have the following

$$\begin{aligned}
f(-\alpha,t) &= t^2\|x\|^2 + \frac{2t(1-t)\alpha - t^2}{(1-t)^2\alpha^2}(1-t)^2\|\alpha x - y\|^2 \\
&\quad + \left(\frac{(1-t)\alpha - t}{(1-t)\alpha}\right)^2 (1-t)^2\|\alpha x - y\|^2 - \|[(1-t)\alpha - t]x - (1-t)y\|^2 \\
&= t^2\|x\|^2 + [2t(1-t)\alpha - t^2]\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad + \left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\|^2 - \|[(1-t)\alpha - t]x - (1-t)y\|^2
\end{aligned}$$

$$\begin{aligned}
&= t^2\|x\|^2 + [2t(1-t)\alpha - t^2]\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad + \left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| - \|[(1-t)\alpha - t]x - (1-t)y\|\right] \\
&\quad \times \left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| + \|[(1-t)\alpha - t]x - (1-t)y\|\right].
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}
&\left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| - \|[(1-t)\alpha - t]x - (1-t)y\|\right] \\
\geq\ & -\left|\frac{t}{\alpha}\right| \|y\| = -\frac{t}{\alpha}\|y\|.
\end{aligned}$$

Again, by the triangle inequality, we also have

$$\begin{aligned}
&\left[\left\|[(1-t)\alpha - t]x - \left(1 - t - \frac{t}{\alpha}\right)y\right\| + \|[(1-t)\alpha - t]x - (1-t)y\|\right] \\
\leq\ & \left[2|(1-t)\alpha - t|\ \ \|x\| + \left[\left|1 - t - \frac{t}{\alpha}\right| + 1 - t\right]\|y\|\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(-\alpha, t) &\geq t^2\|x\|^2 + 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - t^2\left\|x - \frac{y}{\alpha}\right\|^2 \\
&\quad -2\left|(1-t)t - \frac{t^2}{\alpha}\right|\ \|x\|\,\|y\| - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \frac{(1-t)t}{\alpha}\right]\|y\|^2 \\
&\geq t^2\|x\|^2 + 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - t^2\left(\|x\| + \left\|\frac{y}{\alpha}\right\|\right)^2 \\
&\quad -2\left|(1-t)t - \frac{t^2}{\alpha}\right|\ \|x\|\,\|y\| - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \frac{(1-t)t}{\alpha}\right]\|y\|^2 \\
&= t^2\|x\|^2 + 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - t^2\|x\|^2 - 2t^2\|x\|\left\|\frac{y}{\alpha}\right\| - t^2\left\|\frac{y}{\alpha}\right\|^2 \\
&\quad -2\left|(1-t)t - \frac{t^2}{\alpha}\right|\ \|x\|\,\|y\| - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \left|\frac{(1-t)t}{\alpha}\right|\right]\|y\|^2 \\
&= 2t(1-t)\alpha\left\|x - \frac{y}{\alpha}\right\|^2 - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{\alpha}\right]\ \|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \frac{(1-t)t}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2
\end{aligned}$$

$$\begin{aligned}
&\geq 2t(1-t)\alpha\|x\|^2 - 4t(1-t)\alpha\|x\|\left\|\frac{y}{\alpha}\right\| + 2t(1-t)\alpha\left\|\frac{y}{\alpha}\right\|^2 \\
&\quad - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{\alpha}\right]\|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| + \frac{(1-t)t}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2 \\
&= 2t(1-t)\alpha\|x\|^2 - \left[2\left|(1-t)t - \frac{t^2}{\alpha}\right| + \frac{2t^2}{\alpha} + 4t(1-t)\right]\|x\|\,\|y\| \\
&\quad - \left[\left|\frac{t(1-t)}{\alpha} - \frac{t^2}{\alpha^2}\right| - \frac{(1-t)t}{\alpha} + \frac{t^2}{\alpha^2}\right]\|y\|^2.
\end{aligned}$$

Integrate on $(0,1)$, to get

$$\begin{aligned}
F(-\alpha) &= \int_0^1 f(-\alpha,t)dt \\
&\geq \frac{1}{3}\alpha\|x\|^2 - \left[\frac{\alpha^3+3\alpha+2}{3\alpha\,(\alpha+1)^2} + \frac{2}{3\alpha} + \frac{2}{3}\right]\|x\|\|y\| \\
&\quad - \left[\frac{\alpha^3+3\alpha+2}{6\alpha^2\,(\alpha+1)^2} - \frac{1}{6\alpha} + \frac{1}{3\alpha^2}\right]\|y\|^2.
\end{aligned}$$

For α sufficiently large (denote this $-\alpha$ by α_2) and since x is nonzero, we have

$$F(\alpha_2) = \int_0^1 f(\alpha_2,t)dt > 0.$$

We conclude that there exists an α strictly between α_1 and α_2 such that $F(\alpha)=0$. This completes the proof. □

The following lemma is due to James [61] and is used to prove the existence of HH-I-orthogonality.

Lemma 6.6.2 (James [61]). *Let $x,y \in \mathbf{X}$. Then, for any $a \in \mathbb{R}$,*

$$\lim_{\alpha\to\infty} \|(\alpha+a)x+y\| - \|\alpha x+y\| = a\|x\|.$$

We prove the following theorem by using a similar continuity argument and the intermediate value theorem, which was used by James [61, p. 296–297].

Theorem 6.6.3 (Kikianty and Dragomir [70]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, HH-I-orthogonality is existent.*

Proof. Let $x, y \in \mathbf{X}$. The proof is trivial for $x = 0$. Therefore, we assume that $x \neq 0$. Suppose that $h : \mathbb{R} \times (0,1) \to \mathbb{R}$ be a function defined by

$$\begin{aligned} h(\alpha, t) &:= \|(1-t)(\alpha x + y) + tx\| - \|(1-t)(\alpha x + y) - tx\| \\ &= \|[(1-t)\alpha + t]x + (1-t)y\| - \|[(1-t)\alpha - t]x + (1-t)y\|, \end{aligned}$$

and associated to h, a function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(\alpha) := \int_0^1 h(\alpha, t)dt.$$

Note that, for any $t \in (0,1)$,

$$\begin{aligned} \lim_{\alpha\to\infty} h(\alpha, t) &= \lim_{\alpha\to\infty} \left[\|[(1-t)\alpha + t]x + (1-t)y\| - \|[(1-t)\alpha - t]x + (1-t)y\|\right] \\ &= (1-t) \lim_{\alpha\to\infty} \left[\left\|\left(\alpha + \frac{t}{1-t}\right) x + y\right\| - \left\|\left(\alpha - \frac{t}{1-t}\right) x + y\right\|\right] \\ &= (1-t) \lim_{\alpha\to\infty} \left[\left\|\left(\alpha + \frac{2t}{1-t}\right) x + y\right\| - \|\alpha x + y\|\right] \\ &= (1-t)\frac{2t}{(1-t)}\|x\| = 2t\|x\|, \end{aligned}$$

by Lemma 6.6.2, and that

$$\lim_{\alpha\to\infty} H(\alpha) = \lim_{\alpha\to\infty} \int_0^1 h(\alpha, t)\ dt = \int_0^1 \lim_{\alpha\to\infty} h(\alpha, t)\ dt,$$

by the continuity of h. Therefore,

$$\lim_{\alpha\to\infty} H(\alpha) = \int_0^1 2t\|x\|\ dt = \|x\| > 0.$$

We also note that for any $t \in (0,1)$

$$\begin{aligned} &\lim_{\alpha\to\infty} h(-\alpha, t) \\ =\ &\lim_{\alpha\to\infty} \left[\|[(1-t)(-\alpha) + t]x + (1-t)y\| - \|[(1-t)(-\alpha) - t]x + (1-t)y\|\right] \\ =\ &\lim_{\alpha\to\infty} \left[\|[(1-t)\alpha - t]x - (1-t)y\| - \|[(1-t)\alpha + t]x - (1-t)y\|\right] \\ =\ &(1-t) \lim_{\alpha\to\infty} \left[\left\|\left(\alpha - \frac{t}{1-t}\right) x - y\right\| - \left\|\left(\alpha + \frac{t}{1-t}\right) x - y\right\|\right] \end{aligned}$$

$$\begin{aligned} &= (1-t) \lim_{\alpha\to\infty} \left[\left\| \left(\alpha - \frac{2t}{1-t} \right) x - y \right\| - \|\alpha x - y\| \right] \\ &= (1-t) \left(-\frac{2t}{1-t} \right) \|x\| = -2t\|x\|, \end{aligned}$$

again by Lemma 6.6.2, and by the continuity of h,

$$\lim_{\alpha\to\infty} H(-\alpha) = \lim_{\alpha\to\infty} \int_0^1 h(-\alpha,t)\ dt = \int_0^1 \lim_{\alpha\to\infty} h(-\alpha,t)\ dt.$$

Therefore,

$$\lim_{\alpha\to\infty} H(-\alpha) = \int_0^1 (-2t)\|x\|\ dt = -\|x\| < 0.$$

Now, we have shown that there exist $\alpha_1 > 0$ such that $H(\alpha_1) > 0$ and $\alpha_2 < 0$ such that $H(\alpha_2) < 0$. By continuity of H, we conclude that there exists an α_0 such that $H(\alpha_0) = 0$, and therefore

$$\int_0^1 \|(1-t)(\alpha_0 x + y) + tx\|^2 dt = \int_0^1 \|(1-t)(\alpha_0 x + y) - tx\|^2 dt,$$

as required. □

6.6.2 Characterizations of inner product spaces

In this subsection, we provide the alternative proofs for the characterization of inner product spaces via HH-P-orthogonality and HH-I-orthogonality. The following theorem gives us a characterization of inner product spaces via the homogeneity of HH-P-orthogonality.

Theorem 6.6.4 (Kikianty and Dragomir [70])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. Then,* $\mathbf{X}$ *is an inner product space if and only if HH-P-orthogonality is homogeneous.*

Proof. We use a similar argument to that of James [61, p. 301]. If $\mathbf{X}$ is an inner product space, then HH-P-orthogonality is equivalent to the usual orthogonality, and therefore is homogeneous. Conversely, assume that the homogeneity property of HH-P-orthogonality holds and let $x, y \in \mathbf{X}$. By existence, there exists an $\alpha \in \mathbf{X}$, such that

$$\int_0^1 \|(1-t)(\alpha x + y) + tx\|^2 dt = \frac{1}{3}\left(\|\alpha x + y\|^2 + \|x\|^2\right).$$

Since the homogeneity property holds, we have that for any $k \in \mathbb{R}$

$$\int_0^1 \|(1-t)(\alpha x + y) + tkx\|^2 dt = \frac{1}{3}\left(\|\alpha x + y\|^2 + k^2\|x\|^2\right). \tag{6.35}$$

Assuming $t \in (0,1)$, we set $k = \frac{(1-t)(1-\alpha)}{t}$. The left-hand side of (6.35) becomes

$$\int_0^1 \|(1-t)(\alpha x + y) + (1-t)(1-\alpha)x\|^2 dt = \|x+y\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3}\|x+y\|^2.$$

The right-hand side of (6.35) becomes

$$\frac{1}{3}\left(\|\alpha x + y\|^2 + \frac{(1-t)^2(1-\alpha)^2}{t^2}\|x\|^2\right).$$

Thus, by (6.35), we have

$$\|x+y\|^2 = \|\alpha x + y\|^2 + \frac{(1-t)^2(1-\alpha)^2}{t^2}\|x\|^2.$$

Since $t \neq 0$, we have

$$t^2\|x+y\|^2 = t^2\|\alpha x + y\|^2 + (1-t)^2(1-\alpha)^2\|x\|^2.$$

Integrating with respect to t over $(0,1)$ produces

$$\|x+y\|^2 = \|\alpha x + y\|^2 + (1-\alpha)^2\|x\|^2. \tag{6.36}$$

Analogously, we set $k = \frac{-(1-t)(1+\alpha)}{t}$ for any $t \in (0,1)$. The left-hand side of (6.35) becomes

$$\int_0^1 \|(1-t)(\alpha x + y) - (1-t)(1+\alpha)x\|^2 dt = \|x-y\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3}\|x-y\|^2.$$

The right-hand side of (6.35) becomes

$$\frac{1}{3}\left(\|\alpha x + y\|^2 + \frac{(1-t)^2(1+\alpha)^2}{t^2}\|x\|^2\right),$$

and thus

$$\|x-y\|^2 = \|\alpha x + y\|^2 + \frac{(1-t)^2(1+\alpha)^2}{t^2}\|x\|^2.$$

Since $t \neq 0$, we have

$$t^2\|x-y\|^2 = t^2\|\alpha x + y\|^2 + (1-t)^2(1+\alpha)^2\|x\|^2.$$

Integrating with respect to t over $(0,1)$ gives

$$\|x-y\|^2 = \|\alpha x + y\|^2 + (1+\alpha)^2\|x\|^2. \tag{6.37}$$

Adding (6.36) and (6.37), we get

$$\begin{aligned} & \|x+y\|^2 + \|x-y\|^2 \\ = \;& 2\|\alpha x + y\|^2 + [(1-\alpha)^2 + (1+\alpha)^2]\|x\|^2 \qquad (6.38) \\ = \;& 2\|\alpha x + y\|^2 + (2+2\alpha^2)\|x\|^2. \end{aligned}$$

Now, we note that by homogeneity, we also have $\alpha x + y \perp_{HH-P} \frac{(1-t)}{t}\alpha x$ for all $t \in (0,1)$. That is,

$$\begin{aligned} \int_0^1 \|(1-t)(\alpha x + y) + (1-t)\alpha x\|^2 dt &= \|y\|^2 \int_0^1 (1-t)^2 dt \\ &= \frac{1}{3}\|y\|^2 \\ &= \frac{1}{3}\left(\|\alpha x + y\|^2 + \frac{(1-t)^2\alpha^2}{t^2}\|x\|^2\right). \end{aligned}$$

Since $t \neq 0$, we have

$$t^2\|y\|^2 = t^2\|\alpha x + y\|^2 + (1-t)^2\alpha^2\|x\|^2,$$

from which integrating over $(0,1)$ produces

$$\|y\|^2 = \|\alpha x + y\|^2 + \alpha^2\|x\|^2,$$

or equivalently,

$$\|\alpha x + y\|^2 = \|y\|^2 - \alpha^2\|x\|^2.$$

Therefore, (6.38) gives us

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|y\|^2 - \alpha^2\|x\|^2) + (2+2\alpha^2)\|x\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

thus completes the proof. □

The following theorem gives us a characterization of inner product spaces via the additivity of HH-P-orthogonality.

Theorem 6.6.5 (Kikianty and Dragomir [70]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. Then,* $\mathbf{X}$ *is an inner product space if and only if HH-P-orthogonality is additive.*

Proof. The proof has a similar idea to that of James [61, p. 301–302]. Theorem 6.6.4 implies that it is sufficient to prove that the additivity and the homogeneity properties are equivalent in $\mathbf{X}$. If HH-P-orthogonality is homogeneous, then the underlying space is an inner product space and therefore is additive. Assume that the additivity property holds, and that $x \perp_{HH-P} y$. Consider x and $-y$, the existence property gives us an $\alpha \in \mathbb{R}$ such that $x \perp_{HH-P} \alpha x - y$. By additivity, we conclude that $x \perp_{HH-P} \alpha x$. Therefore, $\alpha = 0$ when $x \neq 0$. Thus, $x \perp_{HH-P} -y$. By symmetry and additivity, we conclude that $nx \perp_{HH-P} my$ for all integers n and m. In particular, when $n \neq 0$,

$$\int_0^1 \left\|(1-t)x + t\left(\frac{m}{n}\right)y\right\|^2 dt = \frac{1}{3}\left(\|x\|^2 + \frac{m^2}{n^2}\|y\|^2\right),$$

which implies that $x \perp_{HH-P} ky$ for any $k \in \mathbb{Q}$. By continuity of the norm, $x \perp_{HH-P} ky$ for any $k \in \mathbb{R}$. The proof is completed by the symmetry of HH-P-orthogonality. □

Before presenting the proof for the case of HH-I-orthogonality, we recall the following lemma.

Lemma 6.6.6 (Ficken [51]). *A normed space* $(\mathbf{X}, \|\cdot\|)$ *is an inner product space, if and only if*

$$\|kx + y\| = \|x + ky\|,$$

for any $k \in \mathbb{R}$ *and* $x, y \in \mathbf{X}$*, with* $\|x\| = \|y\|$.

The following theorem gives us a characterization of inner product spaces via the homogeneity of HH-I-orthogonality.

Theorem 6.6.7 (Kikianty and Dragomir [70]). *If HH-I-orthogonality is homogeneous in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space.*

Proof. The proof has a similar idea to that of James [61, p. 298]. Assume that the homogeneity property of HH-orthogonality holds and let $x, y \in \mathbf{X}$, where $\|x\| = \|y\|$.

For any $t \in (0,1)$, set

$$A(t) = \frac{x+y}{(1-t)}, \quad \text{and} \quad B(t) = \frac{x-y}{t}.$$

Note that

$$\int_0^1 \|(1-t)A(t) + tB(t)\|^2 dt = \int_0^1 \|x+y+x-y\|^2 dt = 4\|x\|^2,$$

and

$$\int_0^1 \|(1-t)A(t) - tB(t)\|^2 dt = \int_0^1 \|x+y-(x-y)\|^2 dt = 4\|y\|^2.$$

Since $\|x\| = \|y\|$,

$$\int_0^1 \|(1-t)A(t) + tB(t)\|^2 dt = \int_0^1 \|(1-t)A(t) - tB(t)\|^2 dt,$$

that is, $A(t) \perp_{HH-I} B(t)$, for all $t \in (0,1)$. Since we are assuming the homogeneity of HH-I-orthogonality, for any $k \in \mathbb{R}$, we have $\frac{k+1}{2}A(t) \perp_{HH-I} \frac{k-1}{2}B(t)$,

$$\int_0^1 \left\|(1-t)\left(\frac{k+1}{2}\right)A(t) + t\left(\frac{k-1}{2}\right)B(t)\right\|^2 dt = \|kx+y\|^2,$$

and

$$\int_0^1 \left\|(1-t)\left(\frac{k+1}{2}\right)A(t) - t\left(\frac{k-1}{2}\right)B(t)\right\|^2 dt = \|x+ky\|^2.$$

Thus,

$$\|kx+y\| = \|x+ky\|,$$

for all $k \in \mathbb{R}$. By Lemma 6.6.6, we conclude that $\mathbf{X}$ is an inner product space. $\square$

The following theorem gives us a characterization of inner product spaces via the additivity of HH-P-orthogonality.

Theorem 6.6.8 (Kikianty and Dragomir [70]). *If HH-I-orthogonality is homogeneous in* $\mathbf{X}$*, then* $\mathbf{X}$ *is an inner product space.*

The proof is similar to that of Theorem 6.6.5. The details are omitted.

Chapter 7

The p-HH-norms on Cartesian powers and sequence spaces

The content of this chapter is largely due to the joint work with Professor Gord Sinnamon [75], from the University of Western Ontario. Our main goal is to extend the p-HH-norms to the nth Cartesian power of a normed space $\mathbf{X}$.

The classical means, exemplified by ℓ^p, extend from means on $[0, \infty)$ to means in a normed vector space $\mathbf{X}$ in an unfortunately simple fashion; one evaluates the norms of n vectors in $\mathbf{X}$ and then calculates the mean of the resulting n real numbers. Consequently, these means depend on the original vectors only through their norms. This process does give a norm on $\mathbf{X}^n$, but one that is relatively insensitive to the geometry of $\mathbf{X}^n$. The weighted arithmetic means (as distinct from weighted ℓ^1 norms) are exceptional in this regard because one first computes, *within* $\mathbf{X}$, a fixed linear combination of the original vectors and then evaluates the $\mathbf{X}$-norm of the result. This preserves more of the structure of $\mathbf{X}^n$. However, a weighted arithmetic mean of non-zero vectors can be zero so it does not give us a norm on $\mathbf{X}^n$.

To calculate the hypergeometric mean of n vectors in $\mathbf{X}$ one evaluates a number of different weighted arithmetic means, indexed by the points of an $(n-1)$-simplex, and then finds the L^p norm of this collection of means by integrating over the simplex (cf. Section 1.3). Theorem 7.1.2 shows that for each $p \geq 1$ this procedure does give a norm on $\mathbf{X}^n$, called the $p - HH$ norm. The $p - HH$ norms retain the sensitivity of the arithmetic means to the geometry of $\mathbf{X}^n$; they depend on the relative positions of the n original vectors in the space $\mathbf{X}$, not just on the size of each vector. Example 7.1.9 shows one concrete way that a change in the "shape" of the space $\mathbf{X}$ affects the $p - HH$ norms.

Spaces of sequences with entries in a normed space $\mathbf{X}$ can be normed using classical means in much the same way as the space $\mathbf{X}^n$ can be, provided one is willing to restrict the sequence space to ensure finiteness of the norm. Here again, the norm of the sequence depends only on the norms of the entries. Extending the $p-HH$ norms, and hence the hypergeometric means, to sequence spaces $h^p[\mathbf{X}]$ is done in Section 7.3. The sensitivity of these norms to the geometry of $\mathbf{X}$ is markedly different than, for instance, the spaces $\ell^p(\mathbf{X})$. A simple example of this is provided by Remark 7.3.8 and Example 7.3.9. These prove that although $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots)$ and $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ are both in ℓ^2, the first is in $h^2[\mathbb{R}]$ but the second is not. The reason for this is that, even though the entries of the two sequences are the same size, the first sequence is spread out around zero and so has significantly smaller weighted arithmetic means than the second, which is concentrated on one side of zero. A more persuasive example comes from Harmonic Analysis. Consider the sequence of terms of the trigonometric polynomial

$$f(x) = \sum_{n=-N}^{N} a_n e^{inx}.$$

Its ℓ^2 norm does not depend on x. Indeed, for any x,

$$\|(a_n e^{inx})_{n=-N}^{N}\|_{\ell^2(\mathbb{C})} = \|f\|_{L^2(-\pi,\pi)}.$$

However, Theorem 7.1.12 shows that its $2-HH$ norm does depend on x. The formula is quite straightforward;

$$\|(a_n e^{inx})_{n=-N}^{N}\|_{2-HH} = \left(\frac{\|f\|_{L^2(-\pi,\pi)} + |f(x)|^2}{(2N+1)(2N+2)}\right)^{1/2}.$$

Letting $N \to \infty$ we can, at least formally, apply Theorem 7.3.7 (for two-sided sequences) to get

$$\|(a_n e^{inx})_{n=-\infty}^{\infty}\|_{h^2[\mathbb{C}]} = \tfrac{1}{\sqrt{2}} \left(\|f\|_{L^2(-\pi,\pi)} + |f(x)|^2\right)^{1/2}.$$

This norm may be different, may be finite or infinite, for different x depending on the pointwise convergence of the trigonometric polynomials as $N \to \infty$. This is not the case with the $\ell^2(\mathbb{C})$ norm.

It would be interesting to investigate in what precise sense the series for $f(x)$ must converge for the above formula to hold, and to explore the differences between the spaces $\ell^2(\mathbb{C})$ and $h^2[\mathbb{C}]$ but our task is to introduce the $p-HH$ norms and the spaces $h^p[\mathbf{X}]$, and to establish some basic properties.

The equivalence of the p-norms and the p-HH-norms on $\mathbf{X}^2$ follows from the Hermite-Hadamard inequality and Theorem 3.3.2. For the case of $n > 2$, we utilize the n-dimensional Hermite-Hadamard inequality to get the upper bound for the p-HH-norms in terms of the p-norms. For the lower bound, the strict analogue of Theorem 3.3.2 fails, but a substitute is given, which is also sharp.

A brief examination of the smoothness and convexity properties of the p-HH-norms on $\mathbf{X}^n$ follows. In keeping with the methods of Chapter 3, an isometric embedding of $\mathbf{X}^n$ into a Lebesgue-Bochner space is given. This embedding facilitates the proofs of the geometrical results. A formula for the semi-inner products is also presented and is used to prove that (Gâteaux) smoothness of the space $\mathbf{X}^n$ is inherited from $\mathbf{X}$.

Extending the p-HH-norm from $\mathbf{X}^n$ to a suitable space of sequences reveals fundamental differences between the p-HH-norms and the p-norms. Although the resulting sequence spaces all lie between $\ell^1(\mathbf{X})$ and $\ell^\infty(\mathbf{X})$, it seems that the resemblance to $\ell^p(\mathbf{X})$ ends there. Examples are given, in the case $\mathbf{X} = \mathbb{R}$, to show that the 2-HH-norm extends to a sequence space that strictly contains ℓ^1, that these sequence spaces need not be lattices; they need not be complete spaces; and they need not even be closed under a permutation of the terms of the sequence.

7.1 The p-HH-norm on $\mathbf{X}^n$

In Chapter 3, we introduced the p-HH-norms on the Cartesian square of a normed space $(\mathbf{X}, \|\cdot\|)$. The extension of these norms to $\mathbf{X}^n$ is presented in this section, as well as their equivalence to the p-norms. Similarly to the case of $n = 2$, the consequences of this equivalence include the completeness and the reflexivity of the p-HH-norms in $\mathbf{X}^n$, provided that $\mathbf{X}$ is complete and reflexive, respectively. We also note that the 2-HH-norm in $\mathbf{X}^n$ is induced by an inner product, when $\mathbf{X}$ is an inner product space.

7.1.1 Extending the p-HH-norms

In Chapter 3, we introduced the p-HH-norms on the Cartesian square of a normed space $(\mathbf{X}, \|\cdot\|)$ as follows

$$\|\mathbf{x}\|_{p-HH} = \left(\int_0^1 \|(1-t)x_1 + tx_2\|^p \, dt \right)^{1/p}$$

for all $\mathbf{x} = (x_1, x_2) \in \mathbf{X}^2$ and $1 \le p < \infty$. In this section we extend the definition of the p-HH-norm to $\mathbf{X}^n$ for $n > 2$.

In the case of $n = 2$, we consider the integral mean on a line segment generated by two vectors. The line segment is essentially the convex combination of the given pair of vectors. Therefore, in the case of $n > 2$, the domain of integration is the convex combination of n vectors, that is, a simplex generated by n vectors.

Definition 7.1.1 (Kikianty and Sinnamon [75])**.** Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \ge 2$ be an integer and $1 \le p < \infty$. Set

$$E_n = \{(u_1, \dots, u_{n-1}) \in (0,1)^{n-1} : u_1 + \cdots + u_{n-1} < 1\}.$$

When $(u_1, \dots, u_{n-1}) \in E_n$ set $u_n = 1 - u_1 - \cdots - u_{n-1}$ and $du' = du_{n-1} \dots du_1$. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}^n$,

$$\|\mathbf{x}\|_{p-HH} = \left(\frac{1}{|E_n|} \int_{E_n} \|u_1 x_1 + \cdots + u_n x_n\|^p \, du' \right)^{1/p}.$$

Here $|E_n| = \int_{E_n} du'$ is the measure of the set E_n.

Note that when $n = 2$ this definition agrees with the one given in Chapter 3. When $n = 1$ it is convenient to set $\|\mathbf{x}\|_{p-HH} = \|x_1\|$ for $\mathbf{x} = (x_1) \in \mathbf{X}^1$.

Theorem 7.1.2 (Kikianty and Sinnamon [75])**.** *Suppose* $(\mathbf{X}, \|\cdot\|)$ *is a normed space,* n *is a positive integer and* $1 \le p < \infty$*. Then* $\|\cdot\|_{p-HH}$ *is a norm on* $\mathbf{X}^n$*.*

Proof. The triangle inequality in $\mathbf{X}$ shows that

$$(u_1, \dots, u_{n-1}) \mapsto \|u_1 x_1 + \cdots + u_n x_n\|^p$$

defines a continuous function on the closure of E_n, a compact set of finite measure. It follows that the integral defining the p-HH-norm is finite. The norm is clearly non-negative and homogeneous. The triangle inequality follows readily from the triangle inequality in $\mathbf{X}$ and the Minkowski inequality. Now suppose that $\|\mathbf{x}\|_{p-HH} = 0$. Then, $\|u_1 x_1 + \cdots + u_n x_n\|^p = 0$ for almost every $(u_1, \cdots, u_{n-1}) \in E_n$. By continuity it is identically zero on E_n; furthermore on the closure of E_n. In particular, it vanishes at the points $(1, 0, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 0, 1)$ and $(0, 0, \cdots, 0)$. This shows that $x_1 = x_2 = \cdots = x_n = 0$ and completes the proof. $\square$

There is a natural definition of the p-HH-norm when $p = \infty$ but it does not give a new norm. Indeed, for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$,

$$\begin{aligned} \|\mathbf{x}\|_{\infty-HH} &= \sup_{(u_1,\ldots,u_{n-1})\in E_n} \|u_1x_1 + \cdots + u_nx_n\| \\ &= \max\{\|x_1\|, \ldots, \|x_n\|\} = \|\mathbf{x}\|_\infty. \end{aligned}$$

When $\mathbf{X} = \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector of positive real numbers, the p-HH-norm of $\mathbf{x}$ is the pth-hypergeometric mean of $(x_1, \ldots, x_n)$, which is constructed from the unweighted hypergeometric R-function evaluated at $(x_1, \ldots, x_n)$ (cf. Chapter 1).

The p-HH-norms enjoy a simple relationship with each other and with the p-norms on $\mathbf{X}^n$. Since the integral defining the p-HH-norm is an average, Hölder's inequality shows that the p-HH-norm is increasing as a function of p on $[1, \infty)$. So for $1 < p \leq q < \infty$ we have for all $x \in \mathbf{X}^n$

$$\|\mathbf{x}\|_{1-HH} \leq \|\mathbf{x}\|_{p-HH} \leq \|\mathbf{x}\|_{q-HH} \leq \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_1$$

(cf. Remark 3.2.7). It is interesting to compare this observation with Theorem 7.3.2 in Section 7.3.

7.1.2 Equivalency of the p-norms and the p-HH-norms

We have proven that the p-norms and the p-HH-norms are equivalent in the Cartesian square, in Chapter 3. In this subsection, we prove that they are equivalent in the nth Cartesian power, for any positive integer n. We investigate upper and lower bounds for this new norm, in terms of the p-norms.

For the upper bound, we apply the unweighted case of the n-dimensional Hermite-Hadamard inequality. The general case is Theorem 1.2.1 of Chapter 1 (Theorem 5.20 of [100]), but we provide an elementary proof of the special case that we use.

In Theorem 3.3.2, we provide a sharp lower bound for the p-HH-norm in terms of the p-norm on $\mathbf{X}^2$. Moreover, the best constant in this lower bound is the same for every normed space $\mathbf{X}$. This is not the case when $n > 2$. Example 7.1.9 shows that when $n > 2$ the sharp lower bound for the p-HH-norm in terms of the p-norm on $\mathbf{X}^n$ may genuinely depend on the norm of the underlying space $\mathbf{X}$. As a substitute for the sharp lower bound obtained when $n = 2$, we provide a sharp lower bound for the p-HH-norm in terms of the ∞-norm,

$$\|\mathbf{x}\|_\infty = \max\{\|x_1\|, \ldots, \|x_n\|\}.$$

In this result the best constant does not depend on the space $\mathbf{X}$.

To work effectively with the p-HH-norms we often need to make calculations involving integration over the simplex E_n. To assist with such calculations we offer the following useful changes of variable.

Lemma 7.1.3 (Kikianty and Sinnamon [75])**.** *Let n be a positive integer and $f : (0,1)^n \to \mathbb{R}$ be integrable. For $(u_1, \ldots, u_{n-1}) \in E_n$, set $u_n = 1 - u_1 - \cdots - u_{n-1}$ and $du' = du_{n-1} \ldots du_1$. If σ is a permutation of $\{1, 2, \ldots, n\}$, then*

$$\int_{E_n} f(u_1, \ldots, u_n)\, du' = \int_{E_n} f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})\, du'. \tag{7.1}$$

Proof. Let σ be a permutation of $\{1, 2, \ldots, n\}$ and consider the change of variables

$$w_i = u_{\sigma(i)}, \quad \text{for all } i \in \{1, 2, \ldots, n\}.$$

Then, the Jacobian determinant J of this transformation is 1, when σ is an even permutation; and it is -1 when σ is an odd permutation. Therefore,

$$\begin{aligned}
\int_{E_n} f(u_1, \ldots, u_n)\, du' &= \int_{E_n} f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})|J|\, du' \\
&= \int_{E_n} f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})\, du'
\end{aligned}$$

which completes the proof. $\square$

Lemma 7.1.4 (Kikianty and Sinnamon [75])**.** *Let $m \geq 2$ and $n \geq 2$ be integers and $f : (0,1)^{m+n} \to \mathbb{R}$ be integrable. For $(u_1, \ldots, u_{n-1}) \in E_n$, $(v_1, \ldots, v_{m-1}) \in E_m$ and $(w_1, \ldots, w_{m+n-1}) \in E_{m+n}$, set*

$$\begin{aligned}
u_n = 1 - u_1 - \cdots - u_{n-1}, &\quad du' = du_{n-1} \ldots du_1, \\
v_m = 1 - v_1 - \cdots - v_{m-1}, &\quad dv' = dv_{m-1} \ldots dv_1, \\
w_{m+n} = 1 - w_1 - \cdots - w_{m+n-1}, &\quad dw' = dw_{m+n-1} \ldots dw_1.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_{E_{m+n}} f(w_1,\ldots,w_{m+n})\,dw' \qquad (7.2)\\
&= \int_0^1\int_{E_m}\int_{E_n} f(tv_1,\ldots,tv_m,(1-t)u_1,\ldots,(1-t)u_n)\,du'\,dv' t^{m-1}(1-t)^{n-1}\,dt,
\end{aligned}$$

$$\begin{aligned}
&\int_{E_{n+1}} f(w_1,\ldots,w_{n+1})\,dw' \qquad (7.3)\\
&= \int_0^1\int_{E_n} f(t,(1-t)u_1,\ldots,(1-t)u_n)\,du'(1-t)^{n-1}\,dt
\end{aligned}$$

and

$$\int_{E_{m+1}} f(w_1,\ldots,w_{m+1})\,dw' = \int_0^1\int_{E_m} f(tv_1,\ldots,tv_m,1-t)\,dv' t^{m-1}\,dt. \qquad (7.4)$$

Proof. We only prove (7.2), as the proofs for (7.3) and (7.4) follows as its particular cases. We consider the following change of variables

$$w_i = \begin{cases} tv_i, & \text{when } i=1,\ldots,m;\\ (1-t)u_{i-m}, & \text{when } i=m+1,\ldots,m+n-1.\end{cases}$$

The Jacobian determinant J of this transformation is

$$\begin{aligned}
J &= \left|\frac{\partial(w_1,\ldots,w_{m+n-1})}{\partial(v_1,\ldots,v_{m-1},t,u_1,\ldots,u_{n-1})}\right|\\
&= \begin{vmatrix}
t & \cdots & 0 & v_1 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & t & v_{m-1} & 0 & \cdots & 0\\
-t & \cdots & -t & v_m & 0 & \cdots & 0\\
0 & \cdots & 0 & -u_1 & (1-t) & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots\\
0 & \cdots & 0 & -u_{n-1} & 0 & \cdots & (1-t)
\end{vmatrix}\\
&= \sum_{i=1}^{m}(-1)^{m+i}v_i(-t)t^{m-2}(-1)^{m-i-1}(1-t)^{n-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} v_i t^{m-1}(1-t)^{n-1} \\
&= t^{m-1}(1-t)^{n-1},
\end{aligned}
$$

since $\sum_{i=1}^{m} v_i = 1$. Therefore,

$$
\begin{aligned}
&\int_{E_{m+n}} f(w_1, \ldots, w_{m+n})\, dw' \\
= &\int_0^1 \int_{E_m} \int_{E_n} f(tv_1, \ldots, tv_m, (1-t)u_1, \ldots, (1-t)u_n)\, du'\, dv' |J|\, dt \\
= &\int_0^1 \int_{E_m} \int_{E_n} f(tv_1, \ldots, tv_m, (1-t)u_1, \ldots, (1-t)u_n)\, du'\, dv' t^{m-1}(1-t)^{n-1}\, dt,
\end{aligned}
$$

as required. □

Remark 7.1.5. With $f \equiv 1$ equation (7.3) becomes

$$
|E_{n+1}| = \int_{E_{n+1}} dw' = \int_0^1 \int_{E_n} du'(1-t)^{n-1}\, dt = \frac{1}{n}|E_n|;
$$

and by induction we find that $|E_n| = 1/(n-1)!$.

With these in hand we can easily prove the following (unweighted) n-dimensional Hermite-Hadamard inequality.

Theorem 7.1.6 (Kikianty and Sinnamon [75])**.** *Suppose* $\mathbf{X}$ *is a vector space,* $n \geq 2$ *is an integer and* $f : \mathbf{X} \to \mathbb{R}$ *is convex. If* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$*, then*

$$
\begin{aligned}
&f\left(\frac{x_1 + \cdots + x_n}{n}\right) \\
\leq\ &\frac{1}{|E_n|} \int_{E_n} f(u_1 x_1 + \cdots + u_n x_n)\, du' \\
\leq\ &\frac{f(x_1) + \cdots + f(x_n)}{n}.
\end{aligned}
$$

Proof. Let S_n denote the collection of all permutations of $\{1, \ldots, n\}$ and note that S_n has $n!$ elements. Let $(u_1, \ldots, u_{n-1}) \in E_n$ and set $u_n = 1 - u_1 - \cdots - u_{n-1}$. For each i,

$$
\sum_{\sigma \in S_n} u_{\sigma(i)} = (n-1)!
$$

because each of $u_1, \ldots, u_n$ occurs exactly $(n-1)!$ times in the sum and $u_1 + \cdots + u_n = 1$.

By Lemma 7.1.3,

$$
\begin{aligned}
& \frac{1}{|E_n|}\int_{E_n} f(u_1x_1+\cdots+u_nx_n)\,du' \\
= \; & \frac{1}{n!}\sum_{\sigma\in S_n}\frac{1}{|E_n|}\int_{E_n} f(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n)\,du' \\
= \; & \frac{1}{|E_n|}\int_{E_n}\frac{1}{n!}\sum_{\sigma\in S_n} f(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n)\,du'. \qquad (7.5)
\end{aligned}
$$

Since f is convex and $u_{\sigma(1)}+\cdots+u_{\sigma(n)}=1$ for all $\sigma\in S_n$, the Jensen inequality yields

$$
\begin{aligned}
& \frac{1}{n!}\sum_{\sigma\in S_n} f(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n) \\
\le \; & \frac{1}{n!}\sum_{\sigma\in S_n}\big(u_{\sigma(1)}f(x_1)+\cdots+u_{\sigma(n)}f(x_n)\big) \\
= \; & \left(\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(1)}\right)f(x_1)+\cdots+\left(\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(n)}\right)f(x_n) \\
= \; & \frac{f(x_1)+\cdots+f(x_n)}{n}.
\end{aligned}
$$

On the other hand, the convexity of f also yields

$$
\begin{aligned}
& \frac{1}{n!}\sum_{\sigma\in S_n} f(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n) \\
\ge \; & f\left(\frac{1}{n!}\sum_{\sigma\in S_n}(u_{\sigma(1)}x_1+\cdots+u_{\sigma(n)}x_n)\right) \\
= \; & f\left(\left(\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(1)}\right)x_1+\cdots+\left(\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(n)}\right)x_n\right) \\
= \; & f\left(\frac{x_1+\cdots+x_n}{n}\right).
\end{aligned}
$$

Using these upper and lower bounds for the integrand in (7.5) completes the proof. □

Remark 7.1.7. The results involving integration over simplex in this subsection are special cases of the results in Chapter 5 of Carlson [18]. In particular, the double inequality in Theorem 7.1.6 is a special case of the inequality in Exercise 5.2-1 of Carlson [18, p. 118].

The following corollary gives a sharp upper bound for the p-HH-norm in terms of the p-norm on $\mathbf{X}^n$.

Corollary 7.1.8 (Kikianty and Sinnamon [75]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, n a positive integer and $1 \le p < \infty$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$,*

$$\left\| \frac{x_1 + \cdots + x_n}{n} \right\| \le \|\mathbf{x}\|_{p-HH} \le n^{-1/p} \|\mathbf{x}\|_p. \tag{7.6}$$

The inequalities reduce to equality when $x_1 = \cdots = x_n$.

Proof. If $n = 1$ the statement holds trivially. If $n \ge 2$, note that $f(x) = \|x\|^p$ is a convex function on $\mathbf{X}$. With this f, the conclusion of the previous theorem easily implies (7.6); just take the pth roots.

When $\mathbf{x} = (x, \ldots, x)$ for some $x \in \mathbf{X}$,

$$\begin{aligned} \left\| \frac{x_1 + \cdots + x_n}{n} \right\| &= \|x\|, \\ \|\mathbf{x}\|_{p-HH} &= \left(\frac{1}{|E_n|} \int_{E_n} \|x\|^p \, du' \right)^{1/p} = \|x\|, \\ \text{and} \quad n^{-1/p} \|\mathbf{x}\|_p &= n^{-1/p} \left(\|x\|^p + \cdots + \|x\|^p \right)^{1/p} = \|x\| \end{aligned}$$

as required. □

Obtaining a lower bound for the p-HH-norm in terms of the p-norm is more delicate. Recall that for any $\mathbf{x} \in \mathbf{X}^2$,

$$(2p+2)^{-1/p} \|\mathbf{x}\|_p \le \|\mathbf{x}\|_{p-HH}.$$

In view of this result it is natural to ask for the best (greatest) constant c in the inequality

$$c\|\mathbf{x}\|_p \le \|\mathbf{x}\|_{p-HH} \tag{7.7}$$

for $\mathbf{x} \in \mathbf{X}^n$. However, as the next example shows, the constant c may be different for different spaces $\mathbf{X}$.

Example 7.1.9 (Kikianty and Sinnamon [75]). Let $n = 3$ and $p = 2$. If $\mathbf{X} = \mathbb{R}$, the best constant for which (7.7) holds is $c = 1/\sqrt{12}$. However, if $\mathbf{X} = \mathbb{R}^2_\infty$ then (7.7) fails with $c = 1/\sqrt{12}$.

Proof. First take $\mathbf{X} = \mathbb{R}$. For $\mathbf{x} = (x_1, x_2, x_3)$, a straightforward calculation shows that

$$\|\mathbf{x}\|_{2-HH}^2 = \tfrac{1}{6}(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1).$$

Since $\|\mathbf{x}\|_2^2 = x_1^2 + x_2^2 + x_3^2$, we see that

$$0 \leq (x_1 + x_2 + x_3)^2 = 12\|\mathbf{x}\|_{2-HH}^2 - \|\mathbf{x}\|_2^2,$$

which proves (7.7) with $c = 1/\sqrt{12}$. Take $\mathbf{x} = (1, -1, 0)$ to see that no larger value of c will do.

Now let $\mathbf{X} = \mathbb{R}_\infty^2$, that is, $\mathbf{X} = \mathbb{R}^2$ with norm $\|(t_1, t_2)\| = \max\{|t_1|, |t_2|\}$. Set $\mathbf{x} = (x_1, x_2, x_3)$, where $x_1 = (-1, 2)$, $x_2 = (-1, -2)$ and $x_3 = (2, 0)$. Calculations show that

$$\|\mathbf{x}\|_2^2 = 12 \quad \text{and} \quad \|\mathbf{x}\|_{2-HH}^2 = 437/450.$$

For (7.7) to hold for this vector $\mathbf{x}$ we must have $12c^2 \leq 437/450$ so (7.7) fails with $c = 1/\sqrt{12}$. $\square$

Rather than continuing to pursue a lower bound involving the p-norm directly, we turn our attention to the ∞-norm and get a lower bound for the p-HH-norm in which the same constant is sharp for each normed space $\mathbf{X}$. Since the p-norm and the ∞-norm are equivalent, this approach gives, indirectly, a lower bound for the p-HH-norm in terms of the p-norm.

Theorem 7.1.10 (Kikianty and Sinnamon [75])**.** *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ and integer and $1 \leq p < \infty$. The inequality*

$$\|\mathbf{x}\|_{p-HH} \geq c\|\mathbf{x}\|_\infty \tag{7.8}$$

holds for all $\mathbf{x} \in \mathbf{X}^n$, where

$$c^p = \inf_{1 \leq s \leq 2} (n-1) \int_0^1 |1 - ts|^p t^{n-2}\, dt.$$

The constant c is strictly positive and best possible.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$. The identity (7.1) implies that the p-HH-norm is invariant under permutations of $x_1, \ldots, x_n$ so we may permute $x_1, \ldots, x_n$ without changing either side of the inequality above. Therefore we may suppose without loss of generality that $\|x_1\| = \max\{\|x_1\|, \ldots, \|x_n\|\}$. Set $\bar{x} = (x_2 + \cdots + x_n)/(n-1)$ and note that $\|\bar{x}\| \leq \|x_1\|$.

Let σ be the $(n-1)$-cycle $(2 \ldots n)$ and apply σ to $x_1, \ldots, x_n$ repeatedly to get

$$\begin{aligned}
\|(x_1, x_2, \ldots, x_n)\|_{p-HH} &= \frac{1}{n-1}\left(\|(x_1, x_2, \ldots, x_n)\|_{p-HH}\right. \\
&+ \|(x_1, x_3, \ldots, x_n, x_2)\|_{p-HH} \\
&+ \left.\cdots + \|(x_1, x_n, x_2 \ldots, x_{n-1})\|_{p-HH}\right) \\
&\geq \|(x_1, \bar{x}, \ldots, \bar{x})\|_{p-HH}^p.
\end{aligned}$$

The last inequality above is the triangle inequality in the p-HH-norm.

If $n \geq 3$, (7.3) implies that

$$\begin{aligned}
\|(x_1, \bar{x}, \ldots, \bar{x})\|_{p-HH}^p &= \frac{1}{|E_n|}\int_{E_n} \|w_1 x_1 + (1-w_1)\bar{x}\|^p \, dw' \\
&= \frac{1}{|E_n|}\int_0^1 \int_{E_{n-1}} \|t x_1 + (1-t)\bar{x}\|^p \, du'(1-t)^{n-2} \, dt \\
&= (n-1)\int_0^1 \|(1-t)x_1 + t\bar{x}\|^p t^{n-2} \, dt.
\end{aligned}$$

It is straightforward to check that this equation also holds when $n = 2$. Putting these together and applying the triangle inequality in $\mathbf{X}$ shows that

$$\begin{aligned}
\|(x_1, x_2, \ldots, x_n)\|_{p-HH}^p &\geq (n-1)\int_0^1 |(1-t)\|x_1\| - t\|\bar{x}\||^p t^{n-2} \, dt \\
&= (n-1)\int_0^1 |1 - t(1 + \|\bar{x}\|/\|x_1\|)|^p t^{n-2} \, dt \|x_1\|^p \\
&\geq c^p \|x_1\|^p.
\end{aligned}$$

Observe that $\int_0^1 |1 - ts|^p t^{n-2} \, dt$ is a strictly positive, continuous function of s on $[1, 2]$. The infimum of such a function is strictly positive so c is strictly positive.

To complete the proof we show that c is the best possible constant in (7.8). If $1 \leq s \leq 2$ and $x \neq 0$, set

$$\mathbf{x} = (x, (1-s)x, \ldots, (1-s)x) \in \mathbf{X}^n$$

and note that $\|\mathbf{x}\|_\infty = \|x\|$. On the other hand, if $n \geq 3$ then (7.3) implies

$$\begin{aligned}
\|\mathbf{x}\|_{p-HH}^p &= \frac{1}{|E_n|}\int_{E_n} \|w_1 x + (1-w_1)(1-s)x\|^p \, dw' \\
&= \frac{\|x\|^p}{|E_n|}\int_0^1 \int_{E_{n-1}} |t + (1-t)(1-s)|^p \, du'(1-t)^{n-2} \, dt \\
&= (n-1)\|x\|^p \int_0^1 |1-ts|^p t^{n-2} \, dt.
\end{aligned}$$

It is straightforward to check that this equation also holds when $n = 2$. It follows that (7.8) fails for any constant larger than c, so c is best possible. □

Corollary 7.1.11 (Kikianty and Sinnamon [75]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, n a positive integer and $1 \leq p < \infty$. Then, the p-HH-norm is equivalent to the p-norm on $\mathbf{X}^n$. If $\mathbf{X}$ is a Banach space, then $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is a Banach space. If $\mathbf{X}$ is reflexive, then $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is also reflexive.*

Proof. The Hermite-Hadamard inequality gives an upper bound for the p-HH-norm in terms of the the p-norm and the previous theorem gives a lower bound for the p-HH-norm in terms of the ∞-norm. Since the ∞-norm is equivalent to the p-norm there is a lower bound for the p-HH-norm in terms of the p-norm and so the two norms are equivalent.

As stated in Proposition 2.2.2, it is known that if $\mathbf{X}$ is complete then $(\mathbf{X}^n, \|\cdot\|_p)$ is also complete. Since the p-HH-norm is equivalent to the p-norm, $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is complete as well.

If $\mathbf{X}$ is reflexive then $(\mathbf{X}^n, \|\cdot\|_p)$ is also reflexive (cf. Chapter 2). The equivalence of the p-norm and the p-HH-norm implies that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is reflexive whenever $\mathbf{X}$ is. □

7.1.3 The 2-HH-norm

In this subsection, we point out that if the norm in $\mathbf{X}$ is induced by a (real) inner product then the 2-HH-norm in $\mathbf{X}^n$ is also induced by an inner product. For convenience in expressing the formula for the inner product that induces the 2-HH-norm we define $s(\mathbf{x}) = x_1 + \cdots + x_n$ for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$.

Theorem 7.1.12 (Kikianty and Sinnamon [75])**.** *Suppose* $(\mathbf{X}, \langle\cdot,\cdot\rangle)$ *is an inner product space and* $n \geq 2$ *is an integer. Then,* $\mathbf{X}^n$ *is an inner product space with respect to the inner product*

$$\langle \mathbf{x}, \mathbf{y}\rangle_{2-HH} = \frac{1}{n(n+1)}\Big(\langle \mathbf{x}, \mathbf{y}\rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y})\rangle\Big)$$

and for all $\mathbf{x} \in \mathbf{X}^n$ *we have*

$$\|\mathbf{x}\|_{2-HH}^2 = \langle \mathbf{x}, \mathbf{x}\rangle_{2-HH}.$$

Proof. It is a simple matter to check that the formula for $\langle\cdot,\cdot\rangle_{2-HH}$ given above does define an inner product.

To verify that it induces the 2-HH-norm suppose $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$. Then,

$$\begin{aligned}
\|\mathbf{x}\|_{2-HH}^2 &= \frac{1}{|E_n|}\int_{E_n} |u_1x_1 + \cdots + u_nx_n|^2\, du' \\
&= (n-1)!\int_{E_n} \langle u_1x_1 + \cdots + u_nx_n, u_1x_1 + \cdots + u_nx_n\rangle\, du' \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}(n-1)!\int_{E_n} u_ju_k\, du' \langle x_j, x_k\rangle \quad \text{and} \\
\langle \mathbf{x}, \mathbf{x}\rangle_{2-HH} &= \frac{1}{n(n+1)}\Big(\langle \mathbf{x}, \mathbf{x}\rangle_2 + \langle s(\mathbf{x}), s(\mathbf{x})\rangle\Big) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\delta_{j\,k}+1}{n(n+1)}\langle x_j, x_k\rangle
\end{aligned}$$

where $\delta_{j\,k}$ is 1 when $j = k$ and 0 otherwise.

It remains to show that

$$(n+1)!\int_{E_n} u_ju_k\, du' = \delta_{j\,k} + 1 \tag{7.9}$$

for all j, k. By (7.1) it is enough to show that

$$(n+1)!\int_{E_n} u_1u_2\, du' = 1 \quad \text{and} \quad (n+1)!\int_{E_n} u_1^2\, du' = 2.$$

When $n = 3$, we have

$$4!\int_0^1\int_0^{1-s} ts\; dtds = 1 \quad \text{and} \quad 4!\int_0^1\int_0^{1-s} t^2\; dtds = 2.$$

When $n > 3$, we employ (7.2), as follows

$$\begin{aligned} & (n+1)! \int_{E_n} u_1 u_2 \, du' \\ = \; & (n+1)! \int_0^1 \int_{E_3} \int_{E_{n-3}} t^2 s_1 s_2 t^2 (1-t)^{n-4} \; dv' ds_1 ds_2 dt \\ = \; & \frac{(n+1)!}{(n-4)!} \int_0^1 t^4 (1-t)^{n-4} dt \int_0^1 \int_0^{1-s_2} s_1 s_2 \; ds_1 ds_2 = 1 \end{aligned}$$

and

$$\begin{aligned} (n+1)! \int_{E_n} u_1^2 \, du' &= (n+1)! \int_0^1 \int_{E_2} \int_{E_{n-2}} t^2 s^2 t (1-t)^{n-3} \; dv' ds dt \\ &= \frac{(n+1)!}{(n-3)!} \int_0^1 t^3 (1-t)^{n-3} dt \int_0^1 s^2 ds = 2. \end{aligned}$$

This completes the proof. □

Remark 7.1.13. The identity (7.9) is a special case of identity (4.4-8) of Carlson [18]. Carlson [18, p. 66] stated that for any n-tuples $b = (b_1, \ldots, b_n)$ and $m = (m_1, \ldots, m_n)$, we have

$$\int_{E_n} u_1^{m_1} \ldots u_k^{m_k} d\mu_b(u) = \frac{B(b+m)}{B(b)},$$

where $d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1 - 1} \ldots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \cdots - u_{k-1})^{b_k - 1} du'$ and B is the usual Beta function. By choosing $b = (1, \ldots, 1)$ and m to be the vector with all elements of zero values except for the the jth and kth elements having the value 1, we obtain identity (7.9).

7.2 Convexity and smoothness

Although the p-HH-norm on $\mathbf{X}^n$ is equivalent to the p-norm, it is not identical. Geometrical properties such as convexity and smoothness are not preserved under equivalence of norms.

In this section we investigate the extent to which geometrical properties of $\mathbf{X}$ are inherited by $\mathbf{X}^n$ when it is given the p-HH-norm. In addition, we give simple formulas for the semi-inner products on $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ in terms of the semi-inner products on $\mathbf{X}$.

7.2.1 Strict convexity and uniform convexity

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ an integer and $1 \leq p < \infty$. Recall that

$$E_n = \{(u_1, \dots, x_{n-1}) \in (0,1)^{n-1} : u_1 + \dots + u_{n-1} < 1\},$$

$du' = du_{n-1} \dots du_1$ and $u_n = 1 - u_1 - \dots - u_{n-1}$.

The Lebesgue-Bochner space $L^p(E_n, \mathbf{X})$ is the vector space of all $f : E_n \to \mathbf{X}$ such that the function

$$(u_1, \dots, u_{n-1}) \mapsto \|f(u_1, \dots, u_{n-1})\|^p$$

is integrable on E_n. The norm is given by

$$\|f\|_{L^p(E_n,\mathbf{X})} = \left(\frac{1}{|E_n|} \int_{E_n} \|f(u_1, \dots, u_{n-1})\|^p \, du' \right)^{1/p}$$

and, as usual, functions that agree almost everywhere are taken to be equal. This is a special case of the Bochner function spaces as stated in Chapter 2. For properties of the Lebesgue-Bochner spaces, we refer to III.3 of Dunford and Schwartz [48] and for applications to the geometry of Banach spaces, we refer to Lemma 2.3.4 and the paper by Randrianantoanina and Saab [103].

For each $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{X}^n$ we define the function $f : E_n \to \mathbf{X}$ by $f_{\mathbf{x}}(u_1, \dots, u_{n-1}) = u_1 x_1 + \dots + u_n x_n$. Evidently, the map $\mathbf{x} \mapsto f_{\mathbf{x}}$ is an isometry from $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ into $L^p(E_n, \mathbf{X})$.

Using this embedding we show that both types of convexity are preserved as we pass from $\mathbf{X}$ to $(\mathbf{X}^n, \|\cdot\|_{p-HH})$, although we must exclude the case $p = 1$.

Theorem 7.2.1 (Kikianty and Sinnamon [75]). *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space, $n \geq 2$ an integer and $1 < p < \infty$. If $\mathbf{X}$ is uniformly convex, then so is $(\mathbf{X}^n, \|\cdot\|_{p-HH})$. If $\mathbf{X}$ is strictly convex, then so is $(\mathbf{X}^n, \|\cdot\|_{p-HH})$.*

Proof. Suppose first that $\mathbf{X}$ is uniformly convex. Then, $L^p(E_n, \mathbf{X})$ is also uniformly convex. It is clear from the definition that any subspace of a uniformly convex space is also uniformly convex. The above embedding shows that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is isometrically isomorphic to a subspace of $L^p(E_n, \mathbf{X})$. Therefore $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is uniformly convex.

The uniform convexity of $\mathbb{R}$ is trivial; and it follows that $L^p(E_n, \mathbb{R})$ is uniformly convex and hence strictly convex (cf. Theorem 5.2.6 of Megginson [85]).

Now suppose that $\mathbf{X}$ is strictly convex. The strict convexity of $L^p(E_n, \mathbb{R})$ and Theorem 6 of Day [29] together imply that $L^p(E_n, \mathbf{X})$ is strictly convex. The definition of strict convexity shows that any subspace of a strictly convex space is strictly convex. Since $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is isometrically isomorphic to a subspace of $L^p(E_n, \mathbf{X})$, it is also strictly convex. □

7.2.2 Smoothness

For Fréchet smoothness we exclude the case $p = 1$ and also require that $\mathbf{X}$ be complete.

Theorem 7.2.2 (Kikianty and Sinnamon [75])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a Banach space,* $n \geq 2$ *an integer and* $1 < p < \infty$*. If* $\mathbf{X}$ *is Fréchet smooth, then so is* $(\mathbf{X}^n, \|\cdot\|_{p-HH})$*.*

Proof. The norm in the Banach space $\mathbf{X}$ is Fréchet differentiable away from zero so, according to Theorem 2.5 of Leonard and Sundaresan [79], the norm in $L^p(E_n, \mathbf{X})$ is also Fréchet differentiable away from zero. In particular, the norm in $L^p(E_n, \mathbf{X})$ is Fréchet differentiable at each nonzero point of the isometric image of $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ in $L^p(E_n, \mathbf{X})$. It follows that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is Fréchet smooth. □

The next result gives formulas for the one-sided derivatives for the p-HH-norm. In a slight abuse of notation we let

$$\mathbf{u} \cdot \mathbf{x} = u_1 x_1 + \cdots + u_n x_n$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$ and $(u_1, \ldots, u_{n-1}) \in E_n$, with $u_n = 1 - u_1 - \cdots - u_{n-1}$ as usual.

Theorem 7.2.3 (Kikianty and Sinnamon [75])**.** *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space,* $n \geq 2$ *an integer and* $1 \leq p < \infty$*. For any* $\mathbf{x}, \mathbf{y} \in \mathbf{X}^n$ *with* $\mathbf{y} \neq 0$*,*

$$(\nabla_+ \|\cdot\|_{p-HH}(\mathbf{y}))(\mathbf{x}) = \|\mathbf{y}\|_{p-HH}^{1-p} \frac{1}{|E_n|} \int_{E_n} \|\mathbf{u} \cdot \mathbf{y}\|^{p-1} (\nabla_+ \|\cdot\|(\mathbf{u} \cdot \mathbf{y}))(\mathbf{u} \cdot \mathbf{x})\, du'.$$

and

$$\langle \mathbf{x}, \mathbf{y} \rangle_{p-HH,s} = \|\mathbf{y}\|_{p-HH}^{2-p} \frac{1}{|E_n|} \int_{E_n} \|\mathbf{u} \cdot \mathbf{y}\|^{p-2} \langle \mathbf{u} \cdot \mathbf{x}, \mathbf{u} \cdot \mathbf{y} \rangle_s \, du'.$$

Corresponding formulas hold for the left-hand derivative and the inferior semi-inner product.

Proof. First, observe that if $\mathbf{y} \neq 0$ then the set

$$\{(u_1, \ldots, u_{n-1}) \in E_n : \mathbf{u} \cdot \mathbf{y} = 0\}$$

is a section of an affine set of dimension $n-2$ and is therefore of measure zero in the $(n-1)$-dimensional set E_n. This ensures that the expressions $\|\mathbf{u} \cdot \mathbf{y}\|^{p-1}$ and $\|\mathbf{u} \cdot \mathbf{y}\|^{p-2}$ appearing above are well-defined and finite almost everywhere.

Fix $\mathbf{x}, \mathbf{y} \in \mathbf{X}^n$ with $\mathbf{y} \neq 0$ and define

$$f_t = f_t(u_1, \ldots, u_{n-1}) = \|\mathbf{u} \cdot (\mathbf{y} + t\mathbf{x})\|$$

for all $t \in (0, 1)$ and for all $(u_1, \ldots, u_{n-1}) \in E_n$ satisfying $\mathbf{u} \cdot \mathbf{y} \neq 0$. The triangle inequality shows that $|f_t| \leq \|\mathbf{y}\|_1 + \|\mathbf{x}\|_1$ for all t and that

$$\tfrac{1}{t}(f_t - f_0) \leq \|\mathbf{u} \cdot \mathbf{x}\| \leq \|\mathbf{x}\|_1 \leq \|\mathbf{y}\|_1 + \|\mathbf{x}\|_1.$$

By the mean value theorem,

$$|\tfrac{1}{t}(f_t^p - f_0^p)| \leq p(\|\mathbf{y}\|_1 + \|\mathbf{x}\|_1)^{p-1} |\tfrac{1}{t}(f_t - f_0)| \leq p(\|\mathbf{y}\|_1 + \|\mathbf{x}\|_1)^p.$$

Thus, $\frac{1}{t}(f_t^p - f_0^p)$ is dominated by a constant independent of t and $(u_1, \ldots, u_{n-1})$.

For almost every $(u_1, \ldots, u_{n-1}) \in E_n$, $f_0 = \|\mathbf{u} \cdot \mathbf{y}\| \neq 0$ so the chain rule implies

$$\lim_{t \to 0^+} \tfrac{1}{t}(f_t^p - f_0^p) = p f_0^{p-1} (\nabla_+ \| \cdot \| (\mathbf{u} \cdot \mathbf{y}))(\mathbf{u} \cdot \mathbf{x})$$

and by Lebesgue's dominated convergence theorem,

$$\lim_{t \to 0^+} \frac{1}{t} \left(\int_{E_n} f_t^p \, du' - \int_{E_n} f_0^p \, du' \right) = \int_{E_n} p f_0^{p-1} (\nabla_+ \| \cdot \| (\mathbf{u} \cdot \mathbf{y}))(\mathbf{u} \cdot \mathbf{x}) \, du'.$$

Applying the chain rule again gives

$$\lim_{t \to 0^+} \tfrac{1}{t}(\|\mathbf{y} + t\mathbf{x}\|_{p-HH} - \|\mathbf{y}\|_{p-HH}) = \|\mathbf{y}\|_{p-HH}^{1-p} \int_{E_n} \|\mathbf{u} \cdot \mathbf{y}\|^{p-1} (\nabla_+ \| \cdot \| (\mathbf{u} \cdot \mathbf{y}))(\mathbf{u} \cdot \mathbf{x}) \, du',$$

the first formula of the theorem.

The second formula follows from the first by applying (2.2). With obvious minor modifications the proof will apply to the left-hand derivative and the inferior semi-inner product. □

These formulas imply that if the superior and inferior semi-inner products of $\mathbf{X}$ agree, then the superior and inferior semi-inner products of $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ agree, giving the following corollary.

Corollary 7.2.4 (Kikianty and Sinnamon [75]). *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space,* $n \geq 2$ *an integer and* $1 \leq p < \infty$. *If* $\mathbf{X}$ *is smooth, then so is* $(\mathbf{X}^n, \|\cdot\|_{p-HH})$.

Proof. Since $\mathbf{X}$ is smooth, $\langle x, y\rangle_s = \langle x, y\rangle_i$ for all $x, y \in \mathbf{X}$. It follows that for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}^n$ with $\mathbf{y} \neq 0$ and for almost all $(u_1, \dots, u_{n-1}) \in E_n$,

$$\|\mathbf{u}\cdot\mathbf{y}\|^{2-p}\langle \mathbf{u}\cdot\mathbf{x}, \mathbf{u}\cdot\mathbf{y}\rangle_s = \|\mathbf{u}\cdot\mathbf{y}\|^{2-p}\langle \mathbf{u}\cdot\mathbf{x}, \mathbf{u}\cdot\mathbf{y}\rangle_i.$$

Theorem 7.2.3 implies that $\langle \mathbf{x}, \mathbf{y}\rangle_{p-HH,s} = \langle \mathbf{x}, \mathbf{y}\rangle_{p-HH,i}$ for all $\mathbf{y} \neq 0$. It also holds when $\mathbf{y} = 0$, from the definition of the semi-inner products. Equality of these two semi-inner products for the $p - HH$ norm implies that $(\mathbf{X}^n, \|\cdot\|_{p-HH})$ is smooth. □

7.3 The h^p spaces

In this section (cf. Kikianty and Sinnamon [75]), we introduce a space of sequences of elements of the normed space $\mathbf{X}$. The norm in this sequence space will be based on the p-HH-norm in $\mathbf{X}^n$. To do this we first renormalize the p-HH-norms so that the embedding $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$ of $\mathbf{X}^n$ into $\mathbf{X}^{n+1}$ is an isometry. For $1 \leq p < \infty$ and $n \geq 2$ we define the space $h_n^p = h_n^p[\mathbf{X}]$ to be $\mathbf{X}^n$ with norm

$$\|(x_1, \dots, x_n)\|_{h_n^p} = \left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)}\right)^{1/p} \|(x_1, \dots, x_n)\|_{p-HH}.$$

For convenience we let $h_1^p[\mathbf{X}] = \mathbf{X}$, with identical norms.

Define

$$h^p = h^p[\mathbf{X}] = \left\{(x_1, x_2, \dots) : \lim_{N\to\infty} \sup_{n>m\geq N} \|(x_{m+1}, \dots, x_n)\|_{h_{n-m}^p} = 0\right\}$$

and, for $(x_1, x_2, \dots) \in h^p$, define

$$\|(x_1, x_2, \dots)\|_{h^p} = \lim_{n\to\infty} \|(x_1, x_2, \dots, x_n)\|_{h_n^p}. \tag{7.10}$$

Some work is required before we can show that h^p is a normed space.

Theorem 7.3.1 (Kikianty and Sinnamon [75]). *The embedding $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$ of h_n^p into h_{n+1}^p is an isometry for $n \geq 1$.*

Proof. If $n = 1$ and $x_1 \in \mathbf{X}$ we have

$$\begin{aligned} \|(x_1, 0)\|_{h_2^p}^p &= \frac{\Gamma(p+2)}{\Gamma(p+1)} \int_{E_2} \|w_1 x_1 + (1-w_1)0\|^p \, dw' \\ &= \|x_1\|^p (p+1) \int_0^1 w_1^p \, dw_1 = \|x_1\|^p = \|x_1\|_{h_1^p}^p. \end{aligned}$$

Suppose $n > 1$ and $x_1, \dots x_n \in \mathbf{X}$. Applying (7.4) with m replaced by n yields

$$\begin{aligned} \|(x_1, \dots, x_n, 0)\|_{h_{n+1}^p}^p &= \frac{\Gamma(p+n+1)}{\Gamma(p+1)} \int_{E_{n+1}} \|w_1 x_1 + \dots + w_n x_n + w_{n+1} 0\|^p \, dw' \\ &= \frac{\Gamma(p+n+1)}{\Gamma(p+1)} \int_0^1 \int_{E_n} \|t v_1 x_1 + \dots + t v_n x_n\|^p \, dv' t^{n-1} \, dt \\ &= \frac{\Gamma(p+n+1)}{\Gamma(p+1)} \int_0^1 t^{p+n-1} \, dt \int_{E_n} \|v_1 x_1 + \dots + v_n x_n\|^p \, dv' \\ &= \frac{\Gamma(p+n+1)}{\Gamma(p+n)} \frac{1}{p+n} \|(x_1, \dots, x_n)\|_{h_n^p}^p \\ &= \|(x_1, \dots, x_n)\|_{h_n^p}^p. \end{aligned}$$

This completes the proof. $\square$

The change of variable (7.1) shows that the norm in h_n^p is invariant under permutations of $x_1, \dots, x_n$. This observation, together with the embedding lemma just given, enables us to show that the limit in (7.10) exists for every $(x_1, x_2, \dots) \in h^p$: It is enough to show that the sequence $\|(x_1, \dots, x_n)\|_{h_n^p}$ is Cauchy. If $m < n$, then

$$\begin{aligned} \|(x_1, \dots, x_n)\|_{h_n^p} &\leq \|(x_1, \dots, x_m, 0 \dots, 0)\|_{h_n^p} + \|(0, \dots, 0, x_{m+1}, \dots, x_n)\|_{h_n^p} \\ &= \|(x_1, \dots, x_m)\|_{h_m^p} + \|(x_{m+1}, \dots, x_n)\|_{h_{n-m}^p} \end{aligned}$$

so

$$\|(x_1, \dots, x_n)\|_{h_n^p} - \|(x_1, \dots, x_m)\|_{h_m^p} \leq \|(x_{m+1}, \dots, x_n)\|_{h_{n-m}^p}.$$

The definition of h^p shows that the last term goes to zero as m and n go to infinity.

The next theorem shows that the h^p spaces are normed spaces.

Theorem 7.3.2 (Kikianty and Sinnamon [75]). *If $\mathbf{X}$ is a normed space, then $h^p = h^p[\mathbf{X}]$ is a normed space. Moreover, if $1 \le p \le q < \infty$ then*

$$\ell^1(\mathbf{X}) \subset h^q[\mathbf{X}] \subset h^p[\mathbf{X}] \subset \ell^\infty(\mathbf{X})$$

with continuous inclusions.

Proof. It is easy to verify that h^p is a vector space of sequences of elements of $\mathbf{X}$ and that (7.10) defines a non-negative function that is positive homogeneous and satisfies the triangle inequality. Theorem 7.1.10 may be used to show that the limit in (7.10) is zero only when $(x_1, x_2, \dots) = (0, 0, \dots)$ but first we need an estimate of the constant c for $n \ge 2$. Set

$$\varphi(s) = (n-1)\int_0^1 |1 - ts| t^{n-2}\, dt$$

and split the integral at $t = 1/s$ to calculate

$$\varphi(s) = s - 1 - \frac{s}{n}\left(1 - \frac{2}{s^n}\right) \quad \text{and} \quad \varphi'(s) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{s^n}\right).$$

Since φ is decreasing on $[1, 2^{1/n}]$ and increasing on $[2^{1/n}, 2]$ its infimum is $\varphi(2^{1/n}) = 2^{1/n} - 1$.

By Hölder's inequality,

$$\begin{aligned} c &= \inf_{1\le s\le 2}\left((n-1)\int_0^1 |1 - ts|^p t^{n-2}\, dt\right)^{1/p} \\ &\ge \inf_{1\le s\le 2}(n-1)\int_0^1 |1 - ts| t^{n-2}\, dt = 2^{1/n} - 1. \end{aligned}$$

By Theorem 7.1.10 and the definition of the norm in h_n^p

$$\|(x_1, \dots, x_n)\|_{h_n^p} \ge (2^{1/n} - 1)\left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)}\right)^{1/p} \max\{\|x_1\|, \dots, \|x_n\|\}.$$

The limit as $n \to \infty$ of $\max\{\|x_1\|, \dots, \|x_n\|\}$ is $\|(x_1, x_2, \dots)\|_{\ell^\infty(\mathbf{X})}$; and Stirling's formula shows that

$$\lim_{n\to\infty}(2^{1/n} - 1)\left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)}\right)^{1/p} = \frac{\log 2}{\Gamma(p+1)^{1/p}}.$$

Thus,

$$\|(x_1, x_2 \dots)\|_{h^p} \geq (\log 2)\Gamma(p+1)^{-1/p} \|(x_1, x_2, \dots)\|_{\ell^\infty(\mathbf{X})}.$$

This finishes the proof that (7.10) defines a norm by showing that only the zero vector in h^p can have zero norm. It also proves that h^p is contained in $\ell^\infty(\mathbf{X})$ with continuous inclusion.

Next we show that h^p contains $\ell^1(\mathbf{X})$. If $0 \leq m < n$ then the permutation invariance of the h^p_n norm, together with the isometry of the embeddings $h^p_n \hookrightarrow h^p_{n+1}$ yields

$$\begin{aligned} \|(x_{m+1}, \dots, x_n)\|_{h^p_{n-m}} &\leq \|(x_{m+1}, 0, \dots, 0)\|_{h^p_{n-m}} + \dots + \|(0, \dots, 0, x_n)\|_{h^p_{n-m}} \\ &= \|x_{m+1}\| + \dots + \|x_n\|. \end{aligned}$$

If $(x_1, x_2, \dots) \in \ell^1(\mathbf{X})$ then this sum tends to zero as $m, n \to \infty$ so, by definition, $(x_1, x_2, \dots) \in h^p$. Moreover, taking $m = 0$ above gives,

$$\begin{aligned} \|(x_1, x_2, \dots)\|_{h^p} &= \lim_{n\to\infty} \|(x_1, \dots, x_n)\|_{h^p_n} \\ &\leq \lim_{n\to\infty} (\|x_1\| + \dots + \|x_n\|) = \|(x_1, x_2, \dots)\|_{\ell^1(\mathbf{X})}. \end{aligned}$$

This shows that the inclusion is continuous.

As mentioned previously, the p-HH-norm is defined as an integral average so Hölder's inequality shows that for any $\mathbf{x} \in \mathbf{X}^n$,

$$\|\mathbf{x}\|_{p-HH} \leq \|\mathbf{x}\|_{q-HH}$$

when $p \leq q$. In terms of the h^p and h^q norms this is,

$$\|\mathbf{x}\|_{h^p_n} \leq \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \left(\frac{\Gamma(p+n)}{\Gamma(n)}\right)^{1/p} \left(\frac{\Gamma(n)}{\Gamma(q+n)}\right)^{1/p} \|\mathbf{x}\|_{h^q_n}. \tag{7.11}$$

By Stirling's formula,

$$\lim_{n\to\infty} \left(\frac{\Gamma(p+n)}{\Gamma(n)}\right)^{1/p} \left(\frac{\Gamma(n)}{\Gamma(q+n)}\right)^{1/p} = 1.$$

Therefore, the constant

$$C_{p,q} = \sup_n \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \left(\frac{\Gamma(p+n)}{\Gamma(n)}\right)^{1/p} \left(\frac{\Gamma(n)}{\Gamma(q+n)}\right)^{1/p},$$

is finite, independent of n and satisfies

$$\|\mathbf{x}\|_{h_n^p} \leq C_{p,q}\|\mathbf{x}\|_{h_n^q}.$$

This implies that $h^q \subset h^p$. In addition, taking the limit in (7.11) yields

$$\|\mathbf{x}\|_{h^p} \leq \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}}\|\mathbf{x}\|_{h^q}$$

for all $\mathbf{x} \in h^q$, showing that the inclusion is continuous. □

Remark 7.3.3. Since h^p contains ℓ^1 it contains all sequences that are eventually zero. Theorem 7.3.1 shows that for these sequences the norm in h^p reduces to the norm in h_n^p for some n. That is,

$$\|(x_1, x_2, \dots, x_n, 0, 0, \dots)\|_{h^p} = \|(x_1, x_2, \dots, x_n)\|_{h_n^p}.$$

Remark 7.3.4. It is important to distinguish between the spaces $h^p[\mathbf{X}]$ and $h^p[\mathbb{R}](\mathbf{X})$. The latter provides a norm on the space

$$h^p[\mathbb{R}](\mathbf{X}) = \{(x_1, x_2, \dots) : (\|x_1\|, \|x_2\|, \dots) \in h^p[\mathbb{R}]\}$$

given by

$$\|(x_1, x_2, \dots)\|_{h^p[\mathbb{R}](\mathbf{X})} = \|(\|x_1\|, \|x_2\|, \dots)\|_{h^p[\mathbb{R}]}.$$

Even in the case $\mathbf{X} = \mathbb{R}$ the spaces $h^p[\mathbf{X}]$ and $h^p[\mathbb{R}](\mathbf{X})$ are not the same, although in this special case the two norms do coincide on vectors with non-negative entries.

The next example shows that the spaces $h^p[\mathbf{X}]$ need not be complete, even if the underlying space $\mathbf{X}$ is complete. In the example, $\mathbf{X} = \mathbb{R}$ but, since every non-trivial normed space contains an isometric copy of $\mathbb{R}$, the example is easily adapted to any $\mathbf{X}$.

Example 7.3.5 (Kikianty and Sinnamon [75])**.** The normed space $h^2[\mathbb{R}]$ is not complete.

Proof. Consider the sequence $(a, \dots, a, b, \dots, b, 0, 0, \dots)$ in which the first m entries equal $a \in \mathbb{R}$, the next n entries equal $b \in \mathbb{R}$ and the rest of the entries are zero. If $m, n \geq 2$ we use (7.2) to get

$$\begin{aligned} &\|(a, \dots, a, b, \dots, b, 0, 0, \dots)\|_{h^2}^2 \\ = \ &\|(a, \dots, a, b, \dots, b)\|_{h_{m+n}^2}^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{(m+n+1)!}{2}\int_{E_{m+n}} |(w_1+\cdots+w_m)a+(w_{m+1}+\cdots+w_{m+n})b|^2\,dw' \\
&= \frac{(m+n+1)!}{2}\int_0^1\int_{E_m}\int_{E_n}(ta+(1-t)b)^2\,dv'\,du'\,t^{m-1}(1-t)^{n-1}\,dt \\
&= \frac{(m+n+1)!}{2(m-1)!(n-1)!}\int_0^1 (ta+(1-t)b)^2\,t^{m-1}(1-t)^{n-1}\,dt \\
&= \tfrac{1}{2}m(m+1)a^2+mnab+\tfrac{1}{2}n(n+1)b^2.
\end{aligned}$$

Similar arguments using (7.3), (7.4) show that the conclusion remains valid when $m,n \geq 0$.

In particular, if $\xi_n = (\frac{1}{n},\dots\frac{1}{n},0,0,\dots)$ is chosen to have exactly n non-zero entries then $\|\xi_n\|_{h^2}^2 = (n+1)/(2n)$. Since $\|\xi_n\|_{h^2} \to 1/\sqrt{2}$ as $n\to\infty$ the sequence $\{\xi_n\}$ does not converge to 0 in h^2. However, ξ_n does converge to 0 in ℓ^∞ so $\{\xi_n\}$ cannot have a limit at all in the smaller space h^2.

On the other hand, the above calculation shows that

$$\begin{aligned}
\|\xi_{m+n}-\xi_m\|_{h^2}^2 &= \tfrac{1}{2}m(m+1)\left(\frac{1}{m+n}-\frac{1}{m}\right)^2 \\
&+ mn\left(\frac{1}{m+n}-\frac{1}{m}\right)\left(\frac{1}{m+n}\right) \\
&+ \tfrac{1}{2}n(n+1)\left(\frac{1}{m+n}\right)^2 \\
&= \frac{n}{2m(m+n)} \leq \frac{1}{2m}.
\end{aligned}$$

Since $\|\xi_{m+n}-\xi_m\|_{h^2} \to 0$ uniformly in n as $m\to\infty$ the sequence $\{\xi_n\}$ is a Cauchy sequence in h^2. As we have seen, $\{\xi_n\}$ does not converge in h^2. Thus h^2 is not complete. □

Remark 7.3.6. The formula,

$$\|(a,\dots,a,b,\dots,b,0,0,\dots)\|_{h^2}^2 = \tfrac{1}{2}m(m+1)a^2+mnab+\tfrac{1}{2}n(n+1)b^2$$

given above, shows that $h^2[\mathbb{R}]$ does not have the lattice property since replacing a by $-a$ may affect the norm in $h^2[\mathbb{R}]$.

When $\mathbf{X}$ is an inner product space, $h^2[\mathbf{X}]$ is too. Also, there is a simple formula relating their inner products. Recall that $s:\mathbf{X}^n\to\mathbf{X}$ was defined earlier by

$s(x_1,\dots,x_n) = x_1+\dots+x_n$. By identifying $(x_1,\dots,x_n) \in \mathbf{X}^n$ with the sequence $(x_1,\dots,x_n,0,0,\dots)$ we can extend this definition to

$$s(x_1,x_2,\dots) = x_1+x_2+\dots$$

for all sequences $(x_1,x_2,\dots)$ that are eventually zero.

Theorem 7.3.7 (Kikianty and Sinnamon [75]). *If $\mathbf{X}$ is a (real) inner product space, then $h^2 = h^2[\mathbf{X}] \subset \ell^2(\mathbf{X})$, the operator s extends uniquely to a bounded linear operator on h^2; and h^2 is an inner product space satisfying*

$$\langle \mathbf{x},\mathbf{y}\rangle_{h^2} = \frac{1}{2}\Big(\langle \mathbf{x},\mathbf{y}\rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y})\rangle \Big) \tag{7.12}$$

for all $\mathbf{x},\mathbf{y} \in h^2$.

Proof. By Theorem 7.1.12, $(\mathbf{X}^n, \|\cdot\|_{2-HH})$ is an inner product space and consequently so is h_n^2. Moreover, for all $\mathbf{x},\mathbf{y} \in \mathbf{X}^n$,

$$\langle \mathbf{x},\mathbf{y}\rangle_{h_n^2} = \tfrac{1}{2}n(n+1)\langle \mathbf{x},\mathbf{y}\rangle_{2-HH} = \frac{1}{2}\Big(\langle \mathbf{x},\mathbf{y}\rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y})\rangle \Big).$$

Taking $\mathbf{y} = \mathbf{x}$ in this equation implies

$$\|\mathbf{x}\|_2 \le \sqrt{2}\|\mathbf{x}\|_{h_n^2} \tag{7.13}$$

and

$$\|s(\mathbf{x})\| \le \sqrt{2}\|\mathbf{x}\|_{h_n^2}. \tag{7.14}$$

For $\mathbf{x} = (x_1,x_2,\dots) \in h^2$ set $\mathbf{x}^{(n)} = (x_1,\dots,x_n) \in \mathbf{X}^n$. Inequality (7.13) shows that

$$\|\mathbf{x}\|_2 = \lim_{n\to\infty} \|\mathbf{x}^{(n)}\|_2 \le \sqrt{2}\lim_{n\to\infty} \|\mathbf{x}^{(n)}\|_{h^2} = \sqrt{2}\|\mathbf{x}\|_{h^2}.$$

Thus, $\mathbf{x} \in \ell^2(\mathbf{X})$ and we have $h^2 \subset \ell^2(\mathbf{X})$. Inequality (7.14) shows that if $x_1,x_2,\dots$ is eventually zero, then

$$\|s(\mathbf{x})\| = \lim_{n\to\infty} \|s(\mathbf{x}^{(n)})\| \le \sqrt{2}\lim_{n\to\infty} \|x^{(n)}\|_{h^2} = \sqrt{2}\|x\|_{h^2}.$$

The definition of h^2 implies that $\mathbf{x}^{(n)} \to \mathbf{x}$ in h^2 so the space of sequences that are eventually zero is dense in h^2. We have shown that the linear operator s is densely defined and bounded on h^2. It therefore extends uniquely to a bounded linear map on h^2, which we also denote by s.

The map
$$(\mathbf{x}, \mathbf{y}) \mapsto \frac{1}{2}\Big(\langle \mathbf{x}, \mathbf{y} \rangle_2 + \langle s(\mathbf{x}), s(\mathbf{y}) \rangle \Big)$$
is an inner product on h^2 and the norm it defines,
$$\mathbf{x} \mapsto \tfrac{1}{2} \left(\|\mathbf{x}\|_2^2 + \|s(\mathbf{x})\|^2 \right),$$
agrees with the norm in h^2 on a dense subset. Therefore, h^2 is an inner product space and (7.12) holds for all $\mathbf{x}, \mathbf{y} \in h^2$. □

Remark 7.3.8. If $\mathbf{X}$ is an inner product space then $\ell^2(\mathbf{X}) \neq h^2[\mathbf{X}]$. To see this, fix a unit vector $x \in \mathbf{X}$. The sequence $(x, x/2, x/3, \dots)$ is in $\ell^2(\mathbf{X})$ because the series $1^2 + (1/2)^2 + (1/3)^2 + \dots$ converges. However, for any m,
$$\sup_{m<n} \|(x/(m+1), \dots, x/n)\|_{h^2_{n-m}}^2 = \sup_{m<n} \frac{1}{2} \left(\sum_{k=m+1}^{n} \frac{1}{k^2} + \left(\sum_{k=m+1}^{n} \frac{1}{k} \right)^2 \right) = \infty.$$
By definition, $(x, x/2, x/3, \dots) \notin h^2$.

In the next example we construct an element of h^2 that is not in ℓ^1, showing that the inclusion $\ell^1 \subset h^2$ is strict.

Example 7.3.9 (Kikianty and Sinnamon [75])**.** When $\mathbf{X} = \mathbb{R}$, $h^2 \not\subset \ell^1$.

Proof. The sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots)$ is not in ℓ^1. However, if $m < n$ then
$$\left\| \left(\frac{(-1)^m}{m+1}, \dots, \frac{(-1)^{n-1}}{n} \right) \right\|_{h^2_{n-m}} = \frac{1}{2} \left(\sum_{j=m+1}^{n} \frac{1}{j^2} + \left| \sum_{j=m+1}^{n} \frac{(-1)^{j-1}}{j} \right|^2 \right)$$
can be made arbitrarily small by taking m sufficiently large. This shows that the sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots)$ is in h^2. □

The permutation invariance of the p-norms carries over from finite dimensional spaces to sequence spaces. In contrast, the permutation invariance of the norm on h_n^p may be lost in the transition to h^p. We have seen that $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots) \in h^2$ but it is a simple matter to rearrange the terms of the conditionally convergent series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ so that its partial sums are unbounded. The resulting sequence is not in h^2.

Chapter 8

Conclusion and future work

In this chapter, we summarize the work of this dissertation, recall the main achievements, discuss some open problems that are yet to be addressed and suggest the future research to be undertaken.

8.1 Summary

The p-HH-norm ($1 \leq p < \infty$) is defined on the Cartesian square $\mathbf{X}^2$ of a normed space $(\mathbf{X}, \|\cdot\|)$, by the following

$$\|(x, y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{\frac{1}{p}}, \quad (x, y) \in \mathbf{X}^2.$$

It is connected with the p-norms by the Hermite-Hadamard inequality

$$\left\| \frac{x+y}{2} \right\| \leq \|(x, y)\|_{p-HH} \leq \frac{1}{2^{\frac{1}{p}}} \|(x, y)\|_p, \quad (x, y) \in \mathbf{X}^2. \tag{8.1}$$

Together with the following inequality

$$\frac{1}{(2p+2)^{\frac{1}{p}}} \|(x, y)\|_p \leq \|(x, y)\|_{p-HH}, \quad (x, y) \in \mathbf{X}^2,$$

the Hermite-Hadamard inequality (8.1) shows that the p-HH-norms and the p-norms are equivalent in $\mathbf{X}^2$. However, they are geometrically different, as pointed out in Example 3.2.4 of Chapter 3.

As a consequence of the Hölder inequality, the relationship amongst the p-norms and the p-HH-norms is highlighted in the following inequalities, for $1 \leq p \leq q < \infty$,

$$\|(x,y)\|_{p-HH} \leq \|(x,y)\|_{q-HH} \leq \|(x,y)\|_{\infty} \leq \|(x,y)\|_{q} \leq \|(x,y)\|_{p},$$

for all $(x,y) \in \mathbf{X}^2$. In conjunction with the Hermite-Hadamard inequality (8.1), some quantitative comparisons between the p-HH-norm and the p-norm (for a fixed $1 \leq p < \infty$) are derived by some inequalities of Ostrowski type. These inequalities bound the counterpart of the second Hermite-Hadamard inequality, that is, the difference $\frac{1}{2^{\frac{1}{p}}}\|(x,y)\|_p - \|(x,y)\|_{p-HH}$. In the same spirit, the comparisons amongst the p-HH-norms are derived by some inequalities of Grüss type. These inequalities bound the Čebyšev difference $\|(x,y)\|_{p+q-HH}^{p+q} - \|(x,y)\|_{p-HH}^{p}\|(x,y)\|_{q-HH}^{q}$. Some of these inequalities are proven to be sharp.

When $\mathbf{X} = \mathbb{R}$, the p-HH-norm resembles a familiar concept, that is, the pth order generalized logarithmic mean of two positive numbers. Although the generalized logarithmic mean has been studied since its introduction in 1975 by Stolarsky [113], it is investigated from a different point of view in this dissertation, that is, by the norm structure given by it.

The properties of the p-HH-norms can be summarized as follows:

1. They preserve the completeness and reflexivity of $\mathbf{X}$;
2. They preserve the smoothness of $\mathbf{X}$ (note that in contrast to the 1-norm, the 1-HH-norm preserves the smoothness of $\mathbf{X}$);
3. Excluding the case of $p = 1$, they preserve the Fréchet smoothness, strict convexity and uniform convexity of $\mathbf{X}$.

The 2-HH-norm, in particular, is a Hilbert norm in $\mathbf{X}^2$, provided that $\mathbf{X}$ is an inner product space. Using this norm, some notions of orthogonality in normed spaces are defined:

1. HH-P-orthogonality: $x \perp_{HH-P} y$ if and only if

$$\int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2);$$

2. HH-I-orthogonality: $x \perp_{HH-I} y$ if and only if

$$\int_0^1 \|(1-t)x+ty\|^2 dt = \int_0^1 \|(1-t)x-ty\|^2 dt;$$

3. HH-C-orthogonality: $x \perp_{HH-C} y$ if and only if

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \; dt = 0,$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, $\sum_{i=1}^m \alpha_i \beta_i^2 = \sum_{i=1}^m \alpha_i \gamma_i^2 = 0$ and $\sum_{i=1}^m \alpha_i \beta_i \gamma_i = 1$.

They are shown to be closely related to the Pythagorean, Isosceles and Carlsson's orthogonalities, respectively, by the following relation:
"*If $x, y \in \mathbf{X}$ such that $(1-t)x \perp ty$ (P) for almost every $t \in (0,1)$, then $x \perp_{HH-P} y$.*" Similar statements hold for I-orthogonality and HH-I-orthogonality, as well as C-orthogonality and HH-C-orthogonality. The homogeneity, or the additivity (to the left), of these orthogonalities characterize inner product spaces. The uniqueness of HH-I-orthogonality characterizes strictly convex spaces.

The p-HH-norms are then extended to $\mathbf{X}^n$, by the following

$$\|\mathbf{x}\|_{p-HH} = \left(\frac{1}{|E_n|} \int_{E_n} \|u_1 x_1 + \cdots + u_n x_n\|^p \, du' \right)^{1/p}, \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}^n$$

where $|E_n| = \int_{E_n} du'$ is the measure of the set

$$E_n = \{(u_1, \ldots, u_{n-1}) \in (0,1)^{n-1} : u_1 + \cdots + u_{n-1} < 1\},$$

$(u_1, \ldots, u_{n-1}) \in E_n$, $u_n = 1 - u_1 - \cdots - u_{n-1}$ and $du' = du_{n-1} \ldots du_1$. The similar properties (of the p-HH-norms in $\mathbf{X}^2$) hold in this extension.

The p-HH-norms are extended from $\mathbf{X}^n$ to the spaces of sequences of elements in $\mathbf{X}$, that is,

$$h^p = h^p[\mathbf{X}] = \left\{ (x_1, x_2, \ldots) : \lim_{N\to\infty} \sup_{n>m\geq N} \|(x_{m+1}, \ldots, x_n)\|_{h^p_{n-m}} = 0 \right\}$$

where

$$\|(x_1, \ldots, x_n)\|_{h^p_n} = \left(\frac{\Gamma(p+n)}{\Gamma(p+1)\Gamma(n)} \right)^{1/p} \|(x_1, \ldots, x_n)\|_{p-HH}.$$

The h^p spaces are normed spaces with the following norm

$$\|(x_1, x_2, \dots)\|_{h^p} = \lim_{n\to\infty} \|(x_1, x_2, \dots, x_n)\|_{h_n^p}.$$

The resulting sequence spaces all lie between $\ell^1(\mathbf{X})$ and $\ell^\infty(\mathbf{X})$. The space $h^2[\mathbf{X}]$ is an inner product space, provided that $\mathbf{X}$ is an inner product space. However, the resemblance to $\ell^p(\mathbf{X})$ ends there. Some examples show that:

1. The 2-HH-norm extends to a sequence space that strictly contains ℓ^1;
2. These sequence spaces need not be lattices;
3. They need not be complete spaces;
4. They need not to be closed under a permutation of the terms of the sequence.

8.2 Main achievements

This section covers the main contributions of the work in the dissertation. The contributions include those in Banach space theory and classical analysis, such as the theory of inequalities and the theory of means.

The main contribution of the thesis is in Banach space theory, that is, the development of a new family of norms in the Cartesian power of a normed space. These norms behave 'nicely', in the sense that they preserve some of the metrical and geometrical properties of the underlying normed spaces. In particular, the 1-HH-norm preserves the smoothness, in contrast to the 1-norm, which is not a smooth norm, even when the underlying space is a smooth space. Using a limit of isometric embeddings, the norms are extended to spaces of bounded sequences that include all summable sequences.

The thesis has also contributed in the study of Banach space geometry. The 2-HH-norm is employed to define some notions of orthogonality in normed space. The homogeneity, as well as the additivity (to the left), of these orthogonalities characterizes inner product space. In particular, the uniqueness of HH-I-orthogonality characterizes strictly convex space.

In the theory of inequalities, this thesis contributes firstly in the refinement of the Hermite-Hadamard inequality for functions defined on linear spaces. This refinement generalizes the results by Dragomir [33, 34] and proves the sharpness of the inequalities that have not been addressed in the corresponding work. The results in Chapter 4 also contributes to the development of Ostrowski type inequalities in normed spaces. Lastly, the results in Chapter 5, gives some new bounds to the Čebyšev functional in the Riemann-Stieltjes integral approximation.

In the theory of means, the study of the p-HH-norms shows that the norms are related to the hypergeometric means (generalized logarithmic means, when $n = 2$), but are not restricted to the positive real numbers. Therefore, they extend the concept of hypergeometric means to a more general setting of vector space. In extending the p-HH-norms to the h^p spaces, these norms also extend the hypergeometric mean to infinite sequences, when the underlying space is the field of real numbers.

8.3 Future work

This section discusses some open problems and further research to be undertaken regarding the work of this thesis.

There are several open problems yet to be addressed. The matter of the sharpness of inequalities in Chapter 4 and Chapter 5 can be demonstrated in some particular cases, but the analytical proofs have not been given. The following are the lists of conjectures and open problems related to the sharpness of inequalities:

1. Conjectures 4.3.8, 4.4.1 and 4.4.2 of Chapter 4;

2. Open problems 5.2.5, 5.3.5 and 5.4.10 of Chapter 5.

In Chapter 6, we introduce the notions of orthogonality in normed space via the 2-HH-norm and investigate their properties. We are interested in further research on the HH type orthogonality, in particular, investigating their applications.

In extending the p-HH-norms to $\mathbf{X}^n$, it is natural to consider the analogous inequalities to those described in Chapters 4 and 5. These inequalities will give refinements and extensions to the multivariate Hermite-Hadamard inequalities and the multivariate Grüss type inequality.

It is also of interest to investigate the application of the p-HH-norms in interpolation spaces due to the following relation:

$$\ell^1 \subset h^q \subset h^p \subset \ell^\infty$$

for $1 \leq p \leq q < \infty$. We are also interested in investigating the application of the p-HH-norms in function spaces. A possible application in function spaces is described in Appendix A.

The h^p spaces are also of interest, which requires further development. The notion of completeness in h^p spaces are yet to be investigated. We may employ the 'standard' method of completing the space, but there is no guarantee that the resulting space is the space of sequences.

Appendix A

Possible applications of the p-HH-norms

A.1 Lebesgue space

Let $1 \leq p < \infty$ and f be an element of $L^p[-1,1]$. The usual Lebesgue norm is given by

$$\|f\|_{L^p[-1,1]} := \left(\int_{-1}^{1} |f(t)|^p \ dt \right)^{1/p}, \quad f \in L^p[-1,1].$$

Observe that for any $f \in L^p[-1,1]$, we have

$$\begin{aligned} \|f\|_{L^p[-1,1]} = \left(\int_{-1}^{1} |f(t)|^p dt \right)^{\frac{1}{p}} &= \left(\int_{-1}^{0} |f(t)|^p dt + \int_{0}^{1} |f(t)|^p dt \right)^{\frac{1}{p}} \\ &= \left(\int_{0}^{1} |f(-t)|^p dt + \int_{0}^{1} |f(t)|^p dt \right)^{\frac{1}{p}}. \end{aligned}$$

Now, let g be the restriction of f on the interval $[0,1]$ and h be a function such that $h(t) := f(-t)$ for any $t \in [0,1]$. Therefore,

$$\begin{aligned} \|f\|_{L^p[-1,1]} &= \left(\int_{0}^{1} |f(-t)|^p dt + \int_{0}^{1} |f(t)|^p dt \right)^{\frac{1}{p}} \\ &= \left(\int_{0}^{1} |h(t)|^p dt + \int_{0}^{1} |g(t)|^p dt \right)^{\frac{1}{p}} \\ &= \left(\|h\|_{L^p[0,1]}^p + \|g\|_{L^p[0,1]}^p \right)^{\frac{1}{p}}. \end{aligned} \tag{A.1}$$

By (A.1), we may identify f as an element of $L^p[0,1] \times L^p[0,1]$. Note that the space $L^p[0,1] \times L^p[0,1]$ together with the p-norm, is $L^p[-1,1]$ with the usual Lebesgue norm.

Now, define the following mapping $|\|\cdot\||_{L^p[-1,1]}$ on $L^p[-1,1]$ by utilizing the p-HH-norm, as follows

$$
\begin{aligned}
|\|f\||_{L^p[-1,1]} := \|(g,h)\|_{p-HH} &= \left(\int_0^1 \|(1-t)g + th\|_{L^p[0,1]}^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \int_0^1 |(1-t)g(s) + th(s)|^p ds dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \int_0^1 |(1-t)f(s) + tf(-s)|^p ds dt\right)^{\frac{1}{p}}.
\end{aligned}
$$

It can be easily shown that this mapping is also a norm on $L^p[-1,1]$.

Note that from inequality (3.3) and Theorem 3.3.2, we have the following identity

$$
\frac{1}{[2(p+1)]^{\frac{1}{p}}}\|f\|_{L^p[-1,1]} \le |\|f\||_{L^p[-1,1]} \le \frac{1}{2^{\frac{1}{p}}}\|f\|_{L^p[-1,1]}, \tag{A.2}
$$

for any $f \in L^p[-1,1]$.

The following example shows that the constants $\frac{1}{[2(p+1)]^{\frac{1}{p}}}$ and $\frac{1}{2^{\frac{1}{p}}}$ are best possible in (A.2).

Example A.1.1. Let $f \in L^p[-1,1]$ be an odd function. We have

$$
\begin{aligned}
|\|f\||_{L^p[-1,1]} &= \left[\int_0^1 \left(\int_0^1 |(1-t)f(s) + tf(-s)|^p ds\right) dt\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left(\int_0^1 |(1-t)f(s) - tf(s)|^p ds\right) dt\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left(\int_0^1 |1-2t|^p |f(s)|^p ds\right) dt\right]^{\frac{1}{p}} \\
&= \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \|f\|_{L^p[0,1]} = \left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \|f\|_{L^p[-1,1]}.
\end{aligned}
$$

This shows that the constant $\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$ is best possible in the first inequality of (A.2).

Let $g \in L^p[-1,1]$ be an even function. We have

$$\begin{aligned}
|||g|||_{[-1,1]} &= \left[\int_0^1 \left(\int_0^1 |(1-t)g(s)+tg(-s)|^p ds\right) dt\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left(\int_0^1 |(1-t)g(s)+tg(s)|^p ds\right) dt\right]^{\frac{1}{p}} \\
&= \left[\int_0^1 \left(\int_0^1 |g(s)|^p ds\right) dt\right]^{\frac{1}{p}} \\
&= \|g\|_{L^p[0,1]} = \frac{1}{2^{\frac{1}{p}}}\|g\|_{L^p[-1,1]}.
\end{aligned}$$

This shows that the constant $\frac{1}{2^{\frac{1}{p}}}$ is best possible in the second inequality of (A.2).

A.2 Linear operator

In this section, we extend the idea in Section A.1 to a more general setting.

Proposition A.2.1. *Let $(\mathbf{X}, \|\cdot\|)$ be a (real) normed space and $U : \mathbf{X} \to \mathbf{X}$ be a linear operator. Define*

$$\|x\|_{[p]} := \|(x, Ux)\|_{p-HH} = \left(\int_0^1 \|(1-t)x + tUx\|^p dt\right)^{\frac{1}{p}},$$

for any $x \in \mathbf{X}$. Then, $\|\cdot\|_{[p]}$ is a norm on $\mathbf{X}$.

Proof. Let $x = 0$, then $Ux = 0$ (by linearity), thus implies that $\|x\|_{[p]} = 0$. Conversely, let $\|x\|_{[p]} = \|(x, Ux)\|_{p-HH} = 0$, then $Ux = -x$, by the Hermite-Hadamard inequality. Now, we have $0 = \|(x, -x)\|_{p-HH} = \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \|x\|$. Therefore $x = 0$ since $\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \neq 0$.

For any $x \in \mathbf{X}$ and $\alpha \in \mathbb{R}$, we have

$$\|\alpha x\|_{[p]} = \|(\alpha x, U(\alpha x))\|_{p-HH} = \|(\alpha x, \alpha U(x))\|_{p-HH}$$

by linearity. Therefore,

$$\begin{aligned}\|\alpha x\|_{[p]} &= \left(\int_0^1 \|(1-t)\alpha x + t\alpha Ux\|^p dt\right)^{\frac{1}{p}} \\ &= |\alpha| \left(\int_0^1 \|(1-t)x + tUx\|^p dt\right)^{\frac{1}{p}} \\ &= |\alpha|\|x\|_{[p]}.\end{aligned}$$

We also have for any $x, y \in \mathbf{X}$

$$\begin{aligned}\|x+y\|_{[p]} &= \|(x+y, U(x+y))\|_{p-HH} \\ &= \left(\int_0^1 \|(1-t)(x+y) + tU(x+y)\|^p dt\right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \|(1-t)(x+y) + t(Ux+Uy)\|^p dt\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^1 \|(1-t)x + tUx\|^p dt\right)^{\frac{1}{p}} + \left(\int_0^1 \|(1-t)y + tUy\|^p dt\right)^{\frac{1}{p}} \\ &= \|x\|_{[p]} + \|y\|_{[p]},\end{aligned}$$

which proves the triangle inequality. □

Example A.2.2. Let $U : L^p[a,b] \to L^p[a,b]$, where $Uf(t) = f(a+b-t)$. Then,

$$\begin{aligned}\|f\|_{[p]} &:= \left(\int_0^1 \|(1-t)f + tUf\|^p_{L^p[a,b]} dt\right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \int_a^b |(1-t)f(s) + tf(a+b-s)|^p ds dt\right)^{\frac{1}{p}}.\end{aligned}$$

Particularly, when $a = -1$ and $b = 1$, we have

$$\begin{aligned}\|f\|_{[p]} &= \left(\int_0^1 \int_{-1}^1 |(1-t)f(s) + tf(-s)|^p ds dt\right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}} \left(\int_0^1 \int_0^1 |(1-t)f(s) + tf(-s)|^p ds dt\right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}} |||f|||_{L^p[-1,1]},\end{aligned}$$

where $|||\cdot|||_{L^p[-1,1]}$ is the norm defined in Section A.1.

Colophon

This thesis was made in $\LaTeX\,2_\varepsilon$ using the "hepthesis" class.

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