MONOGRAPH:
ON APPROXIMATING OF THE RIEMANN–STIELTJES DOUBLE INTEGRAL

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CHAPTER I

PREFACE

The approximation problem of the Riemann–Stieltjes integral in terms of Riemann–Stieltjes sums is very interesting problem. The most significant way to study this problem may be done through the well–known Ostrowski inequality. In recent years, several authors have studied the well–known Ostrowski inequality in one variable for various types of mappings such as: absolutely continuous, Lipschitzian and n-differentiable mappings as well as mappings of bounded variation. However, a small attention and a few works have been considered for mappings of two variables. Among others, Dragomir and his group have studied a very interesting inequalities for mapping of two independent variables.

The concept of Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$; where f is called the integrand, u is called the integrator, plays an important role in Mathematics. The approximation problem of the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ in terms of the Riemann–Stieltjes sums have been considered recently by many authors. However, a small attention and a few works have been considered for mappings of two variables; i.e., The approximation problem of the Riemann–Stieltjes double integral $\int_a^b \int_c^d f(t,s)\,d_s d_t u(t,s)$ in terms of the Riemann–Stieltjes double sums. This study is devoted to obtain several bounds for $\int_a^b \int_c^d f(t,s)\,d_s d_t u(t,s)$ under various assumptions on the integrand f and the integrator u. Mainly, the concepts of bounded variation and bi-variation are used at large in the context. Several proposed cubature formula are introduced to approximate such double integrals. For mappings of two variables several inequalities of Trapezoid, Grüss and Ostrowski type for mappings of bounded variation, bounded bi-variation, Lipschitzian and monotonic are introduced and discussed. Namely, Trapezoid-type rules for $\mathcal{RS}$-Double integrals are proved, and therefore the classical Hermite–Hadamard inequality for mappings of two variables is established. A Korkine type identity is used to obtain several Grüss type inequalities for integrable functions. Finally, approximating real functions of two variables which possess n-th partial derivatives of bounded bi-variation, Lipschitzian and absolutely continuous are established and investigated.

The main concern in this monograph is to study the approximation problem of the Riemann–Stieltjes double integral in terms of the Riemann–Stieltjes double sums. In fact, the Ostrowski inequality for mappings of two independent variables which are of bounded bi-variation, Hölder continuous and absolutely continuous are used to discuss this problem. In this way, an interesting study of the approximation problem of the Riemann–Stieltjes double integral is presented and therefore several proposed cubature rules for mappings of two variables are given.

The organization of this monograph is given as follows. The first chapter gives a general introduction of the research work where the motivation and objectives are defined.

In chapter II, some basic concepts of bounded variation, Riemann-Stieltjes integral including some of its properties are given. Some known inequalities of Ostrowski's type with some related refinements and generalizations in one and two variables are given.

In chapter III, several new inequalities of Ostrowski's type are introduced. Trapezoid and Midpoint type rules for double Riemann–Stieltjes double integral are proved. A generalization of the well known Beesack–Darst–Pollard inequality for double $\mathcal{RS}$–double integrals is also considered. Finally, as applications, two cubature formulae are proposed.

In chapter IV, in order to approximate the $\mathcal{RS}$–double integrals some functionals are introduced and therefore several representations of the errors are established. Finally, as application, a cubature formulae is proposed.

In chapter V, some related inequalities are proved. Namely, Grüss type inequalities are proved as well as an approximation of a real function of two variables which possess n-th partial derivatives of bounded bivariation are established. Finally Trapezoid-type rules for $\mathcal{RS}$–double integrals are provided are given.

CHAPTER II

LITERATURE REVIEW AND BACKGROUND

2.1 INTRODUCTION

This chapter shall be considered as a review of some famous, fundamental and basic concepts of mappings of bounded variation and bounded bi-variation in two variables with some of their properties. As well as, several inequalities of Ostrowski's and Simpson's type in one and two variables are reviewed.

2.2 FUNCTIONS OF BOUNDED VARIATION AND BIVARIATION

Let $D \subseteq \mathbb{R}^2$ and let $f : D \to R$ be any function. We say that f is bounded above on D if there is $\alpha \in \mathbb{R}$ such that $f(x, y) \leq \alpha$ for all $(x, y) \in D$; in this case, we say that f attains its upper bound on D if there is $(x_0, y_0) \in D$ such that $\sup f(x, y) : (x, y) \in D = f(x_0, y_0)$. Likewise, we say that f is bounded below on D if there is $\beta \in \mathbb{R}$ such that $f(x, y) \geq \beta$ for all $(x, y) \in D$; in this case, we say that f attains its lower bound on D if there is $(x_0, y_0) \in D$ such that $\inf f(x, y) : (x, y) \in D = f(x_0, y_0)$. Finally, we say that f is bounded on D if it is bounded above on D as well as bounded below on D; in this case, we say that f attains its bounds on D if it attains its upper bound on D and also attains its lower bound on D.

In general, we remark that for the order pairs $(x_1, y_1)\,(x_2, y_2) \in D$, we write $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. Now, let I and J be intervals in $\mathbb{R}$ such that $I \times J \subseteq D$. We say that

1. f is monotonically increasing on $I \times J$ if for all $(x_1, y_1), (x_2, y_2)$ in $I \times J$, we have $(x_1, y_1) \leq (x_2, y_2) \Longrightarrow f(x_1, y_1) \leq f(x_2, y_2)$

2. f is monotonically decreasing on $I \times J$ if for all $(x_1, y_1), (x_2, y_2)$ in $I \times J$, we have $(x_1, y_1) \leq (x_2, y_2) \Longrightarrow f(x_1, y_1) \geq f(x_2, y_2)$

3. f is bimonotonically increasing on $I \times J$ if for all $(x_1, y_1), (x_2, y_2)$ in $I \times J$, we have $(x_1, y_1) \leq (x_2, y_2) \Longrightarrow f(x_1, y_2) + f(x_2, y_1) \leq f(x_1, y_1) + f(x_2, y_2)$

4. f is bimonotonically decreasing on $I \times J$ if for all $(x_1, y_1), (x_2, y_2)$ in $I \times J$, we have $(x_1, y_1) \leq (x_2, y_2) \Longrightarrow f(x_1, y_2) + f(x_2, y_1) \geq f(x_1, y_1) + f(x_2, y_2)$

It may be noted that f is monotonically increasing on $I \times J$ if and only if it is (monotonically) increasing in each of the two variables. The following result gives conditions under which an increasing function in the variable x and an increasing function in the variable y can be added or multiplied to obtain a monotonic and/or bimonotonic function of two variables.

Proposition 2.2.1. *(Ghorpade & Limaye 2009) Let I, J be nonempty intervals in $\mathbb{R}$. Given any $\phi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$, consider $f : I \times J \to \mathbb{R}$ and $g : I \times J \to \mathbb{R}$ defined by $f(x, y) = \phi(x) + \psi(y)$ and $g(x, y) = \phi(x)\psi(y)$ for $(x, y) \in I \times J$. Then we have the following*

1. *f is monotonically increasing on $I \times J$ if and only if ϕ is increasing on I and ψ is increasing on J.*

2. *Assume that $\phi(x) \geq 0$ and $\psi(y) \geq 0$ for all $x \in I$, $y \in J$, and also that $\phi(x_0) > 0$ and $\psi(y_0) > 0$ for some $x_0 \in I$ and some $y_0 \in J$. Then g is monotonically increasing on $I \times J$ if and only if ϕ is increasing on I and ψ is increasing on J.*

3. *f is always bimonotonically increasing and also bimonotonically decreasing on $I \times J$.*

4. *If ϕ is monotonic on I and ψ is monotonic on J, then g is bimonotonic on $I \times J$. More specifically, if ϕ and ψ are both increasing or both decreasing, then g is bimonotonically increasing, whereas if ϕ is increasing and ψ is decreasing, or vice-versa, then g is bimonotonically decreasing.*

We recall that for I interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and for all $\alpha \in [0, 1]$, the inequality

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \tag{2.2.1}$$

holds. If (2.2.1) is strictly for all $x \neq y$ and $\alpha \in (0, 1)$, then f is said to be strictly convex. If the inequality in (2.2.1) is reversed, then f is said to be concave. If (2.2.1) is strictly for all $x \neq y$ and $\alpha \in (0, 1)$, then f is said to be strictly concave (see (Pečarić et al. 1992)).

The above proposition as well as the one below can be used to generate several examples of monotonic and bimonotonic functions.

Proposition 2.2.2. *(Ghorpade & Limaye 2009) Let I, J be nonempty intervals in $\mathbb{R}$. The set $I + J := \{x + y | x \in I, y \in J\} \subseteq \mathbb{R}$. Further, consider $\phi : I + J \to \mathbb{R}$ be any function and consider $f : I \times J \to \mathbb{R}$ defined by $f(x, y) = \phi(x + y)$, for $(x, y) \in I \times J$. Then we have the following*

1. *ϕ is increasing on $I + J \Longrightarrow f$ is monotonically increasing on $I \times J$.*
2. *ϕ is decreasing on $I + J \Longrightarrow f$ is monotonically decreasing on $I \times J$.*
3. *ϕ is convex on $I + J \Longrightarrow f$ is bimonotonically increasing on $I \times J$.*
4. *ϕ is concave on $I + J \Longrightarrow f$ is bimonotonically decreasing on $I \times J$.*

Example 2.2.3. *(Ghorpade & Limaye 2009)*

1. *Consider* $f : [-1, 1]^2 \to \mathbb{R}$ *defined by*

$$f(x, y) = \begin{cases} (x+1)(y+1), & x + y < 0 \\ (x+2)(y+2), & x + y \geq 0 \end{cases}.$$

If we fix $y_0 \in [-1, 1]$ *and consider the function* $\phi : [-1, 1] \to \mathbb{R}$ *defined by*

$$\phi(x) = \begin{cases} (y_0+1)(x+1), & x + y < 0 \\ (y_0+2)(x+2), & x + y \geq 0 \end{cases}$$

then it is easy to see that ϕ *is increasing on* $[-1, 1]$*. Similarly, if we fix* $x_0 \in [-1, 1]$ *and consider the function* $\psi : [-1, 1] \to \mathbb{R}$ *defined by*

$$\psi(y) = \begin{cases} (x_0+1)(y+1), & x + y < 0 \\ (x_0+2)(y+2), & x + y \geq 0 \end{cases}$$

then it is easy to see that ψ *is increasing on* $[-1, 1]$*. It follows that* f *is monotonically increasing on* $[-1, 1]^2$*. However,* f *is not bimonotonic on* $[-1, 1]^2$*. To see this note that* $(0, 0) \leq (1, 1)$ *and* $f(0, 1) + f(1, 0) = 6 + 6 < 4 + 9 = f(0, 0) + f(1, 1)$*, whereas* $(-1, 0) \leq (0, 1)$ *and* $f(-1, 1) + f(0, 0) = 3 + 4 > 0 + 6 = f(-1, 0) + f(0, 1)$.

2. *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) := \cos x + \sin y$*. Using Proposition 2.2.1, we readily see that* f *is bimonotonic, but not monotonic.*

A functions of bounded variation is an interesting class of functions that is very closely related to monotonic functions. Let us recall some facts about functions of bounded variation if $[a, b]$ is a compact interval , a set of points $P := \{x_0, x_1, \cdots, x_n\}$, satisfying the inequalities

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

is called a partition of $[a, b]$. The interval $[x_{k-1}, x_k]$ is called kth subinterval of P and we write $\Delta x_k = x_k - x_{k-1}$, so that $\sum_{k=1}^{n} \Delta x_k = b - a$. The collection of all possible partitions of $[a, b]$ will be denoted by $\mathcal{P}[a, b]$.

Definition 2.2.4. *(Apostol 1974) Let f be defined on $[a,b]$. If $P := \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a,b]$, write $\Delta f_k = f(x_k) - f(x_{k-1})$, for $k = 1, 2, \cdots, n$. If there exists a positive number M such that $\sum\limits_{k=1}^{n} |\Delta f_k| \leq M$ for all partition of $[a,b]$, then f is said to be of bounded variation on $[a,b]$. Moreover, if f is of bounded variation on $[a,b]$, and $\sum(P)$ denote the sum $\sum\limits_{k=1}^{n} |\Delta f_k|$ corresponding to the partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$. The number*

$$\bigvee_a^b (f) = \sup\left\{\sum(P) : P \in \mathcal{P}[a,b]\right\},$$

is called the total variation of f on the interval $[a,b]$.

In two variables or more, the concept of bounded variation is quite different. According to Clarkson and Adams (1933), several definitions have been given of conditions under which a function of two or more independent variables shall be said to be of bounded variation. Of these definitions six are usually associated with the names of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli respectively. A seventh has been formulated by Hahn and attributed by him to Pierpont; which are equivalent, and the proof of this fact was presented in the same paper. In general, some relations between these classes are discussed and investigated in the interesting paper Clarkson & Adams (1933).

In this work, we are interested in two of the above senses, which are; bounded variation in Arzelà and Vitali senses.

The monotonic mappings plays a main role in studying mappings of bounded variation, in two variables, the sum of two monotonic functions need not be monotonic. For example, $f : [0,1]^2 \to \mathbb{R}$ defined by $f(x,y) := x - y$ is a sum of monotonic functions (given by $(x,y) \mapsto x$ and $(x,y) \mapsto -y$, but it is neither increasing nor decreasing. On the other hand, since a monotonic function on a (closed) rectangle is bounded ($f : [a,b] \times [c,d] \to \mathbb{R}$ monotonic $\Longrightarrow$ the values of f lie between $f(a,c)$ and $f(b,d)$), sums of monotonic functions are bounded. In fact, they satisfy a stronger property defined below.

For $a,b,c,d \in \mathbb{R}$, we consider the subset $Q := Q_{a,c}^{b,d} = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$ of $\mathbb{R}^2$.

Definition 2.2.5. *(Clarkson & Adams 1933) If*

$$P := \{(x_i, y_i) : x_{i-1} \leq x \leq x_i\,;\; y_{i-1} \leq y \leq y_i\,;\; i = 1, \ldots, n\}$$

is a partition of Q, write

$$\Delta f\,(x_i, y_i) = f\,(x_i, y_i) - f\,(x_{i-1}, y_{i-1})$$

for $i = 1, 2, \cdots, n$. The function $f(x,y)$ is said to be of ***bounded variation in the Arzelà sense*** *(or simply bounded variation) if there exists a positive quantity M such that for every partition on Q we have $\sum\limits_{i=1}^{n} |\Delta f\,(x_i, y_i)| \leq M$.*

Therefore, one can define the concept of total variation of a function of two variables, as follows:

Let f be of bounded variation on Q, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n} |\Delta f\,(x_i, y_i)|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f) := \bigvee_{c}^{d}\bigvee_{a}^{b}(f) := \sup\left\{\sum P : P \in \mathcal{P}\,(Q)\right\},$$

is called the total variation of f on Q.

In the following, we point out some elementary properties of functions of bounded variation.

Proposition 2.2.6. *(Ghorpade & Limaye 2009) Let $f, g : Q \to \mathbb{R}$ and $r \in \mathbb{R}$. Then*

1. *f is bounded variation $\Longrightarrow$ f is bounded.*

2. *f is monotonic $\Longrightarrow$ f is of bounded variation.*

3. *f, g are of bounded variation $\Longrightarrow$ $f + g$, rf, fg are of bounded variation.*

Proposition 2.2.7. *(Ghorpade & Limaye 2009) Let I, J be nonempty intervals in $\mathbb{R}$. Given any $\phi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$, consider $f : I \times J \to \mathbb{R}$ and $g : I \times J \to \mathbb{R}$ defined by $f(x,y) = \phi(x) + \psi(y)$ and $g(x,y) = \phi(x)\psi(y)$ for $(x,y) \in I \times J$. Then we have the following:*

1. *f is of bounded variation on $[a,b] \times [c,d]$ if and only if ϕ is of bounded variation on $[a,b]$ and ψ is of bounded variation on $[c,d]$.*

2. *Assume that ϕ and ψ are not identically zero, then g is of bounded variation on $[a,b] \times [c,d]$ if and only if ϕ is of bounded variation on $[a,b]$ and ψ is of bounded variation on $[c,d]$.*

Definition 2.2.8. *(Clarkson & Adams 1933) If*

$$P := \{(x_i, y_j) : x_{i-1} \leq x \leq x_i\,;\, y_{j-1} \leq y \leq y_j\,;\, i = 1, \ldots, n\,;\, j = 1, \ldots, m\}$$

is a partition of Q, write

$$\Delta_{11} f(x_i, y_j) = f(x_{i-1}, y_{j-1}) - f(x_{i-1}, y_j) - f(x_i, y_{j-1}) + f(x_i, y_j)$$

*for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$. The function $f(x,y)$ is said to be of **bounded variation in the Vitali sense** (or simply bounded bivariation) if there exists a positive quantity M such that for every partition on Q we have $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} |\Delta_{11} f(x_i, y_j)| \leq M$.*

Therefore, one can define the concept of total bivariation of a function of two variables, as follows:

Let f be of bounded bivariation on Q, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f) := \bigvee_{c}^{d}\bigvee_{a}^{b}(f) := \sup\left\{\sum P : P \in \mathcal{P}(Q)\right\},$$

is called the total bivariation of f on Q.

In the following, we point out some elementary properties of functions of bounded bivariation.

Proposition 2.2.9. *(Ghorpade & Limaye 2009) Let* $f, g : Q \to \mathbb{R}$ *and* $r \in \mathbb{R}$*. Then*

1. *If* f *is of bounded bivariation and, in addition,* f *is bounded on any two adjacent sides of the rectangle* $[a, b] \times [c, d]$*, then* f *is bounded.*

2. *If* f *is bimonotonic, then* f *is of bounded bivariation*

3. *If* f *and* g *are of bounded bivariation, then so are* $f + g$ *and* rf*.*

Proposition 2.2.10. *(Ghorpade & Limaye 2009) Let* I, J *be nonempty intervals in* $\mathbb{R}$*. Given any* $\phi : I \to \mathbb{R}$ *and* $\psi : J \to \mathbb{R}$*, consider* $f : I \times J \to \mathbb{R}$ *and* $g : I \times J \to \mathbb{R}$ *defined by* $f(x, y) = \phi(x) + \psi(y)$ *and* $g(x, y) = \phi(x)\psi(y)$ *for* $(x, y) \in I \times J$*. Then we have the following:*

1. f *is always of bounded bivariation on* $[a, b] \times [c, d]$*.*

2. *Assume that* ϕ *and* ψ *are not constant functions, then* g *is of bounded bivariation on* $[a, b] \times [c, d]$ *if and only if* ϕ *is of bounded variation on* $[a, b]$ *and* ψ *is of bounded variation on* $[c, d]$*.*

For further properties of mappings of bounded variation and bivariation we refer the reader to the comprehensive book Ghorpade and Limaye (2009) .

2.3 INEQUALITIES FOR MAPPINGS OF ONE VARIABLE

2.3.1 Ostrowski type Inequalities

For a differentiable mapping f defined $[a, b]$ and f' be integrable on $[a, b]$, then the Montgomery identity

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P(x, t)\, f'(t)\,dt \tag{2.3.1}$$

holds, where $P(x,t)$ is the Peano kernel,

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\leq t\leq x,\\ \frac{t-b}{b-a}, & x<t\leq b.\end{cases}$$

In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 2.3.1. *(Ostrowski 1938) Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|\leq M$, then the following inequality,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M(b-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{2.3.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In 1992, Fink and earlier in 1976, Milovanović and Pečarić have obtained some interesting generalizations of (2.3.2) in the form

$$\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq C(n,p,x)\left\|f^{(n)}\right\|_\infty \tag{2.3.3}$$

where,

$$F_k(x)=\frac{n-k}{n!}\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a},$$

and, $\|\cdot\|_r$, $1\leq r\leq\infty$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|f\|_\infty:=ess\sup_{t\in[a,b]}|f(t)|,$$

and

$$\|f\|_r:=\left(\int_a^b|f(t)|^r\,dt\right)^{1/r},\ \ 1\leq r<\infty.$$

In fact, Milovanović and Pečarić (see also Mitrinović et al. (1994)) have proved that

$$C(n,\infty,x)=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(b-a)\,n\,(n+1)!},$$

while Fink proved that the inequality (2.3.3) holds provided $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_p[a,b]$, with

$$C(n,p,x) = \frac{\left[(x-a)^{nq+1} + (b-x)^{nq+1}\right]^{1/q}}{(b-a)\,n!}\beta^{1/q}\left((n-1)\,q+1, q+1\right),$$

for $1 < p \leq \infty$, β is the beta function, and

$$C(n,1,x) = \frac{(n-1)^{n-1}}{(b-a)\,n^n n!}\max\left\{(x-a)^n, (b-x)^n\right\}.$$

In 2001, Dragomir proved the following Ostrowski's inequality for mappings of bounded variation:

Theorem 2.3.2. *(Dragomir 2001b) Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right] \cdot \bigvee_a^b (f), \tag{2.3.4}$$

for any $x \in [a,b]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $\frac{1}{2}$ is best possible.

The following trapezoid type inequality for mappings of bounded variation holds:

Theorem 2.3.3. *(Cerone & Dragomir 2000) Let $f : [a,b] \to \mathbb{R}$, be a mapping of bounded variation on $[a,b]$, Then*

$$\left|\int_a^b f(t)\,dt - (b-a)\frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2}(b-a)\bigvee_a^b (f). \tag{2.3.5}$$

The constant $\frac{1}{2}$ is the best possible.

A generalization of (2.3.5) and 2.3.4 for mappings of bounded variation, was considered by Cerone et al. (2000), as follows:

$$\left|(b-x)\,f(b) + (x-a)\,f(a) - \int_a^b f(t)\,dt\right| \leq \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \cdot \bigvee_a^b (f) \tag{2.3.6}$$

for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

In the same way, the following midpoint type inequality for mappings of bounded variation was proved in Cerone and Dragomir (2000):

Theorem 2.3.4. *(Cerone & Dragomir 2000) Let $f : [a,b] \to \mathbb{R}$, be a mapping of bounded variation on $[a,b]$, Then*

$$\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \leq \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{2.3.7}$$

The constant $\frac{1}{2}$ is the best possible.

In the recent paper Tseng et al. (2008), the authors have proved the following weighted Ostrowski inequality for mappings of bounded variation, as follows:

Theorem 2.3.5. *(Tseng et al. 2008) Let $0 \leq \alpha \leq 1$, $g : [a,b] \to [0,\infty)$ continuous and positive on (a,b) and let $h : [a,b] \to \mathbb{R}$ be differentiable such that $h'(t) = g(t)$ on $[a,b]$. Let $c = h^{-1}\left(\left(1-\frac{\alpha}{2}\right)h(a) + \frac{\alpha}{2}h(b)\right)$ and $d = h^{-1}\left(\frac{\alpha}{2}h(a) + \left(1-\frac{\alpha}{2}\right)h(b)\right)$. Suppose that f is of bounded variation on $[a,b]$, then for all $x \in [c,d]$, we have*

$$\left|\int_a^b f(t)\,g(t)\,dt - \left[(1-\alpha) f(x) + \alpha \frac{f(a)+f(b)}{2}\right]\int_a^b g(t)\,dt\right| \ \leq\ K \cdot \bigvee_a^b (f) \tag{2.3.8}$$

where,

$$K = \begin{cases} \frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) + \frac{h(a)+h(b)}{2}\right|, & 0 \leq \alpha \leq \frac{1}{2} \\ \max\left\{\frac{1-\alpha}{2}\int_a^b g(t)\,dt + \left|h(x) + \frac{h(a)+h(b)}{2}\right|, \frac{\alpha}{2}\int_a^b g(t)\,dt\right\}, & \frac{1}{2} < \alpha < \frac{2}{3} \\ \frac{\alpha}{2}\int_a^b g(t)\,dt, & \frac{2}{3} \leq \alpha \leq 1 \end{cases}$$

and $\bigvee_a^b (f)$ is the total variation of f over $[a,b]$. The constant $\frac{1-\alpha}{2}$ for $0 \leq \alpha \leq \frac{1}{2}$ and the constant $\frac{\alpha}{2}$ for $\frac{2}{3} \leq \alpha \leq 1$ are the best possible.

Another new generalization of weighted Ostrowski type inequality for mappings of bounded variation has been obtained by Liu (2012), as follows:

Theorem 2.3.6. *(Liu 2012) Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation, $g : [a,b] \to [0,\infty)$ continuous and positive on (a,b). Then for any $x \in [a,b]$ and $\alpha \in [0,1]$, we have*

$$\left|\int_a^b f(t)\,g(t)\,dt - \left[(1-\alpha)\,f(x)\int_a^b g(t)\,dt + \alpha\left(f(a)\int_a^x g(t)\,dt + f(b)\int_x^b g(t)\,dt\right)\right]\right|$$
$$\leq \left[\frac{1}{2} + \left|\frac{1}{2} - \alpha\right|\right]\left[\frac{1}{2}\int_a^b g(t)\,dt + \left|\int_a^x g(t)\,dt - \frac{1}{2}\int_a^b g(t)\,dt\right|\right] \cdot \bigvee_a^b (f) \quad (2.3.9)$$

where, $\bigvee_a^b (f)$ denotes to the total variation of f over $[a,b]$. The constant $\left[\frac{1}{2} + \left|\frac{1}{2} - \alpha\right|\right]$ is the best possible.

In 2002, Guessab and Schmeisser, incorporate the mid-point and the trapezoid inequality together, and they have proved the following companion of Ostrowski's inequality:

Theorem 2.3.7. *(Guessab & Schmeisser 2002) Assume that the function $f : [a,b] \to \mathbb{R}$ is of r-H–Hölder type, where $r \in (0,1]$ and $H > 0$ are given, i.e.,*

$$|f(t) - f(s)| \leq H\,|t-s|^r,$$

for any $t, s \in [a,b]$. Then, for each $x \in [a, \frac{a+b}{2}]$, one has the inequality

$$\left|\frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq H\left[\frac{2^{r+1}(x-a)^{r+1} + (a+b-2x)^{r+1}}{2^r(r+1)(b-a)}\right]. \quad (2.3.10)$$

This inequality is sharp for each admissible x. Equality is obtained if and only if $f = \pm H f_* + c$, with $c \in \mathbb{R}$ and

$$f_*(t) = \begin{cases} (x-t)^r, & a \leq t \leq x \\ (t-x)^r, & x \leq t \leq \frac{a+b}{2} \\ f_*(a+b-x), & \frac{a+b}{2} \leq t \leq b \end{cases}.$$

Dragomir (2002), has proved the following companion of the Ostrowski inequality for mappings of bounded variation:

Theorem 2.3.8. *(Dragomir 2002) Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b(f)\,, \quad (2.3.11)$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $1/4$ is best possible.

Dragomir (2000), has introduced an Ostrowski type integral inequality for the Riemann-Stieltjes integral, as follows:

Theorem 2.3.9. *(Dragomir 2000) Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation and $u:[a,b]\to\mathbb{R}$ a function of r-H-Hölder type, i.e.,*

$$|u(x)-u(y)|\leq H\,|x-y|^r\,,\qquad\forall x,y\in[a,b]\,,$$

where $r\in(0,1]$ and $H>0$ are given. Then, for any $x\in[a,b]$,

$$\begin{aligned}&\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)\,du(t)\right|\\&\leq H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right] \qquad (2.3.12)\\&\leq H\times\begin{cases}[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\\ [(x-a)^{qr}+(b-x)^{qr}]^{1/q}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{1/p}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f)\end{cases}\end{aligned}$$

where, $\bigvee_c^d(f)$ denotes the total variation of f on the interval $[c,d]$.

For other results concerning inequalities for Stieltjes integrals, see Liu (2004) and Cerone and Dragomir (2002). In 2007, Cerone et al. established some Ostrowski type inequalities for the Stieltjes integral where the integrand is absolutely continuous while the integrator is of bounded variation. Also, the case when $|f'|$ is convex was explored.

Recently, Dragomir (2008) provided an approximation for the function f which possesses continuous derivatives up to the order $n-1$ $(n \geq 1)$ and has the n-th derivative of bounded variation, in terms of the chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ and some more terms which depend on the values of the k derivatives of the function taken at the end points a and b, where k is between 1 and n.

As pointed out by Dragomir, if $f : [a, b] \to \mathbb{R}$ is assumed to be bounded on $[a, b]$. The chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ has the equation

$$d_f : [a, b] \to \mathbb{R}, \quad d_f(\mathbf{x}) = \frac{1}{b-a}\left[f(a)(b-\mathbf{x}) + f(b)(\mathbf{x}-a)\right].$$

Before that in (2007) Dragomir was introduced the error in approximating the value of the function $f(\mathbf{x})$ by $d_f(\mathbf{x})$ with $\mathbf{x} \in [a, b]$ by $\Phi_f(\mathbf{x})$, i.e., $\Phi_f(\mathbf{x})$ is defined by:

$$\Phi_f(\mathbf{x}) := \frac{b-\mathbf{x}}{b-a} \cdot f(a) + \frac{\mathbf{x}-a}{b-a} \cdot f(b) - f(\mathbf{x}).$$

The following bounds for $\Phi_f(\mathbf{x})$ holds :

Theorem 2.3.10. *(Dragomir 2007) If $f : [a, b] \to \mathbb{R}$ is of bounded variation, then*

$$\begin{aligned} |\Phi_f(\mathbf{x})| &\leq \left(\frac{b-\mathbf{x}}{b-a}\right) \cdot \bigvee_a^x (f) + \left(\frac{\mathbf{x}-a}{b-a}\right) \cdot \bigvee_x^b (f) \\ &\leq \begin{cases} \left[\frac{1}{2} + \left|\frac{\mathbf{x}-\frac{a+b}{2}}{b-a}\right|\right] \cdot \bigvee_a^b (f) \\ \left[\left(\frac{b-\mathbf{x}}{b-a}\right)^p + \left(\frac{\mathbf{x}-a}{b-a}\right)^p\right]^{\frac{1}{p}} \cdot \left[\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right]^{\frac{1}{q}}; \\ \qquad p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right| \end{cases} \end{aligned} \tag{2.3.13}$$

The first inequality in (2.3.13) is sharp. The constant $\frac{1}{2}$ is best possible in the first and third branches.

Therefore, a generalization Theorem 2.3.10, was considered as follows:

Theorem 2.3.11. *(Dragomir 2008) Let I be a closed subinterval on $\mathbb{R}$, let $a, b \in I$ with $a < b$ and let n be a nonnegative integer. If $f : I \to \mathbb{R}$ is such that the n-th derivative*

$f^{(n)}$ is of bounded variation on the interval $[a,b]$, then for any $x \in [a,b]$ we have the representation

$$f(x) = \frac{1}{b-a}\left[(b-x) f(a) + (x-a) f(b)\right]$$
$$+ \frac{(b-x)(x-a)}{b-a} \cdot \sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)^{k-1} f^{(k)}(a) + (-1)^{k}(b-x)^{k-1} f^{(k)}(b)\right]$$
$$+ \frac{1}{b-a}\int_a^b S_n(x,t)\, d\left(f^{(n)}(t)\right), \quad (2.3.14)$$

where the kernel $S_n : [a,b]^2 \to \mathbb{R}$ is given by

$$S_n(x,t) = \begin{cases} (x-t)^n (b-x), & a \le t \le x \\ \\ (-1)^{n+1}(t-x)^n (x-a), & a \le t \le x \end{cases}$$

and the integral in the remainder is taken in the Riemann–Stieltjes sense.

After that, on utilizing the notations

$$D_n(f;x,a,b) := \frac{1}{b-a}\left[(b-x) f(a) + (x-a) f(b)\right]$$
$$+ \frac{(b-x)(x-a)}{b-a} \cdot \sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)^{k-1} f^{(k)}(a) + (-1)^{k}(b-x)^{k-1} f^{(k)}(b)\right]$$
$$(2.3.15)$$

and

$$E_n(f;x,a,b) := \frac{1}{b-a}\int_a^b S_n(x,t)\, d\left(f^{(n)}(t)\right) \quad (2.3.16)$$

under the assumptions of Theorem 2.3.11, Dragomir approximated the function f utilizing the polynomials $D_n(f;x,a,b)$ with the error $E_n(f;x,a,b)$. In other words, we have $f(x) = D_n(f;x,a,b) + E_n(f;x,a,b)$ for any $x \in [a,b]$.

More recently, Dragomir (2009) introduced an approximation for the Riemann–Stieltjes integral $\int_a^b f(t)\, du(t)$ by the use of some generalized trapezoid-type rules. To be more specific, we investigate the error bounds in approximating $\int_a^b f(t)\, du(t)$ by the simpler quantities:

$$f(b)\left[\frac{1}{b-a}\int_a^b u(t)\, dt - u(a)\right] + f(a)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\, dt\right] \quad (2.3.17)$$

and

$$[u(b)-u(a)]\left[f(b)+f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right] \tag{2.3.18}$$

provided the Riemann integral $\int_a^b f(t)\,dt$ exists and can be either computed exactly or can be accurately approximated by the use of various classical quadrature rules.

For a function $g:[a,b]\to\mathbb{R}$, Dragomir defined $\psi_g:[a,b]\to\mathbb{R}$ by

$$\psi_g(t):=g(t)-\frac{(t-a)\,g(a)+(b-t)\,g(b)}{b-a}$$

and he gave the following representation

Theorem 2.3.12. *(Dragomir 2009) If $f,u:[a,b]\to\mathbb{R}$ are bounded on $[a,b]$ and such that the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist, then*

$$\int_a^b f(t)\,du(t)-\left\{f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right]+f(a)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right]\right\}$$
$$=\int_a^b \psi_f(t)\,du(t) \tag{2.3.19}$$

and

$$[u(b)-u(a)]\left[f(b)+f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt\right]-\int_a^b f(t)\,du(t)$$
$$=\int_a^b \psi_u(t)\,df(t) \tag{2.3.20}$$

Therefore, several error inequalities of approximating the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ by the generalized trapezoid formulae (2.3.17) and (2.3.18) under various assumptions were obtained in the same paper.

Mercer (2008) has introduced a midpoint and a trapezoid type rules for the Riemann–Stieltjes integral which engender a natural generalization of Hadamard's integral inequality. Error terms are then obtained for this Riemann-Stieltjes Trapezoid Rule and other related quadrature rules.

Theorem 2.3.13. *(Mercer 2008) Let g be continuous and increasing, let c satisfy*

$$\int_a^b g(t)\,dt = (c-a)\,g(a) + (b-c)\,g(b)\,,$$

and let

$$G = \frac{1}{b-a}\int_a^b g(t)\,dt$$

If $f'' \geq 0$, then we have

$$f(c)\,[g(b)-g(a)] \leq \int_a^b f dg \leq [G-g(a)]\,f(a) + [g(b)-G]\,f(b) \tag{2.3.21}$$

2.3.2 Simpson's Type Inequalities

The Simpson's inequality was known in the literature, as follows:

Theorem 2.3.14. *(Davis & Rabinowitz 1976) Suppose $f : [a,b] \to$ **R** is four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty := \sup\limits_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. The following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int\limits_a^b f(x)\,dx\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty \tag{2.3.22}$$

holds.

In 1999, Dragomir proved the Simpson's inequality for functions of bounded variation, as follows:

Theorem 2.3.15. *(Dragomir 1999) Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality:*

$$\left|\int\limits_a^b f(x)\,dx - \frac{(b-a)}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)}{3}\bigvee_a^b (f)\,, \tag{2.3.23}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{3}$ is the best possible.

In 2000, Pečarić and Varošanec, obtained some inequalities of Simpson's type for functions whose n-th derivative, $n \in \{0,1,2,3\}$ is of bounded variation, as follow:

Theorem 2.3.16. *(Pečarić & Varošanec 2000) Let $n \in \{0,1,2,3\}$. Let f be a real function on $[a,b]$ such that $f^{(n)}$ is function of bounded variation. Then*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq C_n (b-a)^{n+1} \bigvee_a^b \left(f^{(n)}\right), \quad (2.3.24)$$

where,

$$C_0 = \frac{1}{3},\ C_1 = \frac{1}{24},\ C_2 = \frac{1}{324},\ C_3 = \frac{1}{1152},$$

and $\bigvee_a^b \left(f^{(n)}\right)$ is the total variation of $f^{(n)}$ on the interval $[a,b]$.

In recent years, many authors have considered Simpson's like inequalities and therefore several bounds are introduced, for details see Dedic et al. (2000), Dedić et al. (2001), Dedić et al. (2001), Pečarić and Varošanec (2001), Dedić et al. (2005), Pečarić and Franjić (2006) and Franjić et al. (2006)

2.4 INEQUALITIES FOR MAPPINGS OF TWO VARIABLES

2.4.1 Ostrowski and Grüss type inequalities

In the recent papers Barnett and Dragomir (2001) and Dragomir et al. (2003), the authors have proved the following inequality of Ostrowski type for double integrals:

Theorem 2.4.1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$. Then for all*

$(x,y) \in [a,b] \times [c,d]$*, we have the inequality*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, dsdt + f(x,y) - \frac{1}{(d-c)} \int_c^d f(x,s)\, ds - \frac{1}{(b-a)} \int_a^b f(t,y)\, dt \right|$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] \left[\frac{1}{4} + \left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right] (b-a)(d-c) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty, \quad if \;\; \frac{\partial^2 f}{\partial t \partial s} \in L_\infty(Q); \\ \\ \left[\frac{(b-a)(d-c)}{(1+q)^2}\right]^{\frac{1}{q}} \left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}} \left[\left(\frac{y-c}{d-c}\right)^{q+1} + \left(\frac{d-y}{d-c}\right)^{q+1}\right]^{\frac{1}{q}} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p, \\ if \;\; \frac{\partial^2 f}{\partial t \partial s} \in L_p(Q), p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \left[\frac{1}{2} + \left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right] \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1, \quad if \;\; \frac{\partial^2 f}{\partial t \partial s} \in L_1(Q) \end{cases} , \tag{2.4.1}$$

for all $(x,y) \in Q$*, where,*

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty := \sup_{(t,s) \in Q} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|$$

and

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|^p dsdt \right)^{\frac{1}{p}}, \;\; p \geq 1.$$

The best inequality we can get from (2.4.1) is the one for which $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. For some applications of the above results in numerical integration for cubature formulae see Barnett and Dragomir (2001) and Dragomir et al. (2003).

In order to approximate the double integral $\int_a^b \int_c^d f(t,s)\, dsdt$, Dragomir et al. (2000) introduced the following representation:

Theorem 2.4.2. *(Dragomir et al. 2000) Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(t,s)}{\partial t}, \frac{\partial f(t,s)}{\partial s}, \frac{\partial^2 f(t,s)}{\partial t \partial s}$ *exist and are continuous on* $[a,b] \times [c,d]$*. Then* *for all* $(x,y) \in [a,b] \times [c,d]$. *Then, we have the representation*

$$\begin{aligned}
f(x,y) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t) \frac{\partial f}{\partial t}(t,s)\,dsdt \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d q(y,s) \frac{\partial f}{\partial s}(t,s)\,dsdt \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(x,t)\, q(y,s) \frac{\partial^2 f}{\partial t \partial s}(t,s)\,dsdt, \quad (2.4.2)
\end{aligned}$$

where, $p : [a,b]^2 \to \mathbb{R}$ *and* $q : [c,d]^2 \to \mathbb{R}$ *and are given by*

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ \\ t-b, & t \in (x,b] \end{cases}$$

and

$$q(y,s) = \begin{cases} s-c, & s \in [c,y] \\ \\ s-d, & s \in (y,d] \end{cases}.$$

An interesting particular case for which $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ may be deduced to get a midpoint representation. Also, by letting $(x,y) = (a,c), (a,d), (b,c)$ and (b,d) in (2.4.1), then summing the obtained identities and do the required computations we obtain successively, we obtain the following a trapezoid type identity

$$\begin{aligned}
\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} = \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d f(t,s)\,dsdt \right. \\
+ \int_a^b \int_c^d \left(t - \frac{a+b}{2}\right) \frac{\partial f}{\partial t}(t,s)\,dsdt + \int_a^b \int_c^d \left(s - \frac{c+d}{2}\right) \frac{\partial f}{\partial s}(t,s)\,dsdt \\
\left. + \int_a^b \int_c^d \left(t - \frac{a+b}{2}\right)\left(s - \frac{c+d}{2}\right) \frac{\partial^2 f}{\partial t \partial s}(t,s)\,dsdt \right] \quad (2.4.3)
\end{aligned}$$

After that Dragomir et al. pointed out an inequality of Ostrowski type for mapping of two independent variables, as follows:

Theorem 2.4.3. *(Dragomir et al. 2000) Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(t,s)}{\partial t}, \frac{\partial f(t,s)}{\partial s}, \frac{\partial^2 f(t,s)}{\partial t \partial s}$ *exist and are continuous on* $[a,b] \times [c,d]$. *Then* *for all* $(x,y) \in [a,b] \times [c,d]$, *we have*

$$\left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds \right| \leq M_1(x) + M_2(y) + M_3(x,y), \tag{2.4.4}$$

where,

$$M_1(x) = \begin{cases} \frac{\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]}{b-a} \left\| \frac{\partial f}{\partial t} \right\|_\infty, & \frac{\partial f(t,s)}{\partial t} \in L_\infty(Q); \\ \frac{\left(\frac{(b-x)^{q_1+1}+(x-a)^{q_1+1}}{q_1+1}\right)^{\frac{1}{q_1}}}{(b-a)(d-c)^{\frac{1}{p_1}}} \left\| \frac{\partial f}{\partial t} \right\|_{p_1}, & \frac{\partial f(t,s)}{\partial t} \in L_{p_1}(Q), \\ & p_1 > 1, \frac{1}{p_1} + \frac{1}{q_1} = 1; \\ \frac{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]}{(b-a)(d-c)} \left\| \frac{\partial f}{\partial t} \right\|_1, & \frac{\partial f(t,s)}{\partial t} \in L_1(Q). \end{cases}$$

$$M_2(y) = \begin{cases} \frac{\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right]}{d-c} \left\| \frac{\partial f}{\partial t} \right\|_\infty, & \frac{\partial f(t,s)}{\partial s} \in L_\infty(Q); \\ \frac{\left(\frac{(d-y)^{q_2+1}+(y-c)^{q_2+1}}{q_2+1}\right)^{\frac{1}{q_2}}}{(d-c)(b-a)^{\frac{1}{p_2}}} \left\| \frac{\partial f}{\partial s} \right\|_{p_2}, & \frac{\partial f(t,s)}{\partial s} \in L_{p_2}(Q), \\ & p_2 > 1, \frac{1}{p_2} + \frac{1}{q_2} = 1; \\ \frac{\left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]}{(b-a)(d-c)} \left\| \frac{\partial f}{\partial s} \right\|_1, & \frac{\partial f(t,s)}{\partial s} \in L_1(Q). \end{cases}$$

and

$$M_3(x,y) = \begin{cases} \frac{\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \cdot \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]}{(b-a)(d-c)} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty, & \frac{\partial^2 f(t,s)}{\partial t \partial s} \in L_\infty(Q); \\ \frac{\left(\frac{(b-x)^{q_3+1}+(x-a)^{q_3+1}}{q_3+1}\right)^{\frac{1}{q_3}} \cdot \left(\frac{(d-y)^{q_3+1}+(y-c)^{q_3+1}}{q_3+1}\right)^{\frac{1}{q_3}}}{(b-a)(d-c)} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_{p_3}, & \frac{\partial^2 f(t,s)}{\partial t \partial s} \in L_{p_3}(Q), \\ & p_3 > 1, \frac{1}{p_3} + \frac{1}{q_3} = 1; \\ \frac{\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \cdot \left[\frac{1}{2}(d-c) + \left|y - \frac{c+d}{2}\right|\right]}{(b-a)(d-c)} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1, & \frac{\partial^2 f(t,s)}{\partial t \partial s} \in L_1(Q). \end{cases} .$$

Therefore, several special cases, e.g., midpoint and trapezoid type inequalities were obtained in Dragomir et al. (2000), as well as applications to cubature formulae are considered.

Hanna et al. (2002a) have obtained some generalizations of an Ostrowski type inequality in two dimensions for n-time differentiable mappings

Theorem 2.4.4. *(Hanna et al. 2002a) Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be continuous on* $[a,b] \times [c,d]$ *, and assume that* $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ *exist on* $(a,b) \times (c,d)$. *Further, consider* $K_n : [a,b]^2 \to \mathbb{R}$ *and* $S_m : [c,d]^2 \to \mathbb{R}$ *given by*

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!}, & t \in [a,x] \\ \\ \frac{(t-b)^n}{n!}, & t \in (x,b] \end{cases}, \qquad S_m(y,s) = \begin{cases} \frac{(s-c)^m}{m!}, & s \in [c,y] \\ \\ \frac{(s-d)^m}{m!}, & s \in (y,d] \end{cases}.$$

Then we have the inequality

$$\begin{aligned} &\left| \int_a^b \int_c^d f(t,s)\,ds dt - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} X_k(x) \cdot Y_l(y) \frac{\partial^{k+l} f(x,y)}{\partial t^k \partial s^l} \right. \\ &\qquad - (-1)^m \sum_{k=0}^{n-1} X_k(x) \int_c^d S_m(y,s) \frac{\partial^{k+m} f(x,s)}{\partial t^k \partial s^m} ds \\ &\qquad \left. - (-1)^n \sum_{l=0}^{m-1} Y_l(y) \int_c^d K_n(x,t) \frac{\partial^{n+l} f(t,y)}{\partial t^n \partial s^l} dt \right| \\ &\leq \begin{cases} \frac{1}{(n+1)!(m+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1}\right] \left[(y-c)^{m+1} + (d-y)^{m+1}\right] \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty, \\ if \;\; \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty(Q) \\ \\ \frac{1}{n!m!} \left[\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1} \right]^{\frac{1}{q}} \left[\frac{(d-c)^{mq+1} + (d-y)^{mq+1}}{mq+1} \right]^{\frac{1}{q}} \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p, \\ if \;\; \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_p(Q), p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{4n!m!} \left[(x-a)^n + (b-x)^n + |(x-a)^n - (b-x)^n|\right] \\ \times \left[(y-c)^m + (d-y)^m + |(y-c)^m - (d-y)^m|\right] \cdot \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_1; \;\; if \;\; \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_1(Q) \end{cases} \end{aligned} \tag{2.4.5}$$

for all $(x, y) \in Q$, *where,*

$$\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty := \sup_{(t,s)\in Q} \left| \frac{\partial^{n+m} f(t, s)}{\partial t^n \partial s^m} \right|,$$

$$\left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^{n+m} f(t, s)}{\partial t^n \partial s^m} \right|^p dsdt \right)^{\frac{1}{p}}, \; p \geq 1,$$

and

$$X_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}, \quad Y_l(y) = \frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)!}.$$

Keeping in mind that x and y are free parameters, then one can produce “mid-point” and “boundary-point” type results by choosing appropriate values for x and y. In addition choosing values for n and m will re-capture the earlier results of Hanna et al. (2000) and Dragomir et al. (2000).

In order to compare the integral mean of the product with the product of the integral means, Grüss (1935) have considered the Čebyšev functional defined by

$$C(f, g) = \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt, \qquad (2.4.6)$$

and he has proved that for two integrable mappings f, g such that $\phi \leq f(x) \leq \Phi$ and $\gamma \leq f(x) \leq \Gamma$, the inequality

$$|C(f, g)| \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma) \qquad (2.4.7)$$

holds, and the constant $\frac{1}{4}$ is the best possible.

The proof of (2.4.7) may be done by applying the Cauchy–Bunyakovsky–Schwarz integral inequality for double integrals, on the right hand side of the well-known Korkine identity,

$$(b-a) \int_a^b f(x) g(x) dx - \int_a^b f(x) dx \cdot \int_a^b g(x) dx = \frac{1}{2} \int_a^b \int_a^b [f(x) - f(y)] [g(x) - g(y)] dxdy \qquad (2.4.8)$$

After that, and in order to represent the remainder of the Taylor formula in an integral form which will allow a better estimation using the Grüss type inequalities, Hanna et al. (2002b), generalized the above Korkine identity (2.4.8), for double integrals and therefore Grüss type inequalities were proved. Namely, they proved

Theorem 2.4.5. *(Hanna et al. 2002b) We assume that*

$$|f(x,y)-f(u,v)| \leq M_1 |x-u|^{\alpha_1} + M_2 |x-u|^{\alpha_2}$$

where, $M_1, M_2 > 0$*,* $\alpha_1, \alpha_2 \in (0,1]$ *and*

$$|f(x,y)-f(u,v)| \leq N_1 |x-u|^{\beta_1} + N_2 |x-u|^{\beta_2}$$

where, $N_1, N_2 > 0$*,* $\beta_1, \beta_2 \in (0,1]$ *for all* $(x,y),(u,v) \in [a,b]\times[c,d]$*, then we have the following inequality:*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\quad -\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \cdot \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx\Bigg| \\
&\leq M_1N_1\frac{(b-a)^{\alpha_1+\beta_1}}{(\alpha_1+\beta_1+1)(\alpha_1+\beta_1+2)} + M_1N_2\frac{2(b-a)^{\alpha_1}(d-c)^{\beta_2}}{(\alpha_1+1)(\alpha_1+2)(\beta_2+1)(\beta_2+2)} \\
&+ M_2N_1\frac{2(b-a)^{\alpha_2}(d-c)^{\beta_1}}{(\alpha_2+1)(\alpha_2+2)(\beta_1+1)(\beta_1+2)} + M_2N_2\frac{(b-a)^{\alpha_2+\beta_2}}{(\alpha_2+\beta_2+1)(\alpha_2+\beta_2+2)}
\end{aligned} \tag{2.4.9}$$

Corollary 2.4.6. *When* $\alpha_1 = \alpha_2 = 1$ *and* $\beta_1 = \beta_2 = 1$*, then*

$$|f(x,y)-f(u,v)| \leq L_1|x-u| + L_2|x-u|,$$

$$|f(x,y)-f(u,v)| \leq K_1|x-u| + K_2|x-u|$$

where, $L_1, L_2, K_1, K_2 > 0$*,* $\beta_1, \beta_2 \in (0,1]$ *and then (2.4.9) becomes*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx \\
&\quad -\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \cdot \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx\Bigg| \\
&\leq L_1K_1\frac{(b-a)^2}{12} + L_1K_2\frac{(b-a)(d-c)}{18} + L_2K_1\frac{(b-a)(d-c)}{18} + L_2K_2\frac{(d-c)^2}{12}.
\end{aligned} \tag{2.4.10}$$

For approximation of multiple integrals see the classical book Stroud (1971). Several multivariate and univariate higher order Ostrowski type inequalities over Euclidean domains as well as multivariate Fink type identity were recently proved in Anastassiou (1995), Anastassiou (1997), Anastassiou (2002), Pachpatte (2002b), Pachpatte (2001), Pachpatte (2002a) Anastassiou (2007) and Anastassiou and Goldstein (2007) and (2008). For recent comprehensive list of refinements, counterparts and generalizations of Ostrowski integral inequality see, Mitrinović et al. (1993) and Dragomir and Rassias (2002), as well as the recent PhD study, Hanna (2009) where the author introduced new multivariate approximations from a generalized Taylor perspective in terms of Appell type polynomials.

2.4.2 Simpson's Type Inequalities

Zhongxue (2008), has proved the following Simpson type inequality for mappings of two independent variables:

Theorem 2.4.7. *(Zhongxue 2008) Let $f : [a, c] \times [b, d] \to R$ be an absolutely continuous function, whose partial derivative of order 2 is $f'' \in L^2([a, c] \times [b, d])$. Then*

$$\Bigg| \frac{f\left(a, \frac{b+d}{2}\right) + f\left(\frac{a+c}{2}, b\right) + f\left(\frac{a+c}{2}, d\right) + f\left(c, \frac{b+d}{2}\right) + 4f\left(\frac{a+c}{2}, \frac{b+d}{2}\right)}{9} + \frac{f(a,b) + f(a,d) + f(c,b) + f(c,d)}{36} - \frac{\int_a^c \left[f(s,b) + 4f\left(s, \frac{b+d}{2}\right) + f(s,d)\right] ds}{6(c-a)} - \frac{\int_b^d \left[f(a,t) + 4f\left(\frac{a+c}{2}, t\right) + f(c,t)\right] dt}{6(d-b)} + \frac{\int_a^c \int_b^d f(s,t)\, dsdt}{(c-a)(d-b)} \Bigg| \le \frac{\left[(c-a)(d-b)\right]^{1/2}}{36} \sqrt{\sigma(f'')}, \quad (2.4.11)$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{(c-a)(d-b)} \left(\int_a^c \int_b^d f(s,t)\, dsdt \right)^2, \quad (2.4.12)$$

and

$$\|f\|_2^2 = \left(\int_a^c \int_b^d |f(s,t)|^2\, dsdt \right)^{1/2}.$$

The inequality (2.4.11) is sharp in the sense that the constant $1/36$ *cannot be replaced by a smaller one.*

Another interesting result was considered by Zhongxue (2008), as follows:

Theorem 2.4.8. *(Zhongxue 2008) Under the assumptions of Theorem 1, for any* $(x,y) \in [a,c] \times [b,d]$*, we have*

$$\Bigg|(c-a)(d-b) f(x,y) - (d-b)\int_a^c f(s,y)\,ds - (c-a)\int_b^d f(x,t)\,dt - \left(x - \frac{a+c}{2}\right)\left(y - \frac{b+d}{2}\right)[f(c,d) - f(a,d) - f(c,b) + f(a,b)] + \int_a^c \int_b^d f(s,t)\,dsdt\Bigg| \leq \frac{[7(c-a)(d-b)]^{\frac{3}{2}}}{12}\sqrt{\sigma(f''_{st})}, \quad (2.4.13)$$

where $\sigma(f''_{st})$ *is defined above. The Inequality (2.4.13) is sharp in the sense that the constant* $\frac{7\sqrt{7}}{12}$ *cannot be replaced by a smaller one.*

In his recent work, Liu (2010) has derived a new sharp inequality with a parameter for the absolutely continuous function $f : [a,c] \times [b,d] \to \mathbb{R}$ whose partial derivative of order 2 is $f''_{st} \in L^2([a,c] \times [b,d])$ via the new sharp bound (2.4.12), which will not only provide a generalization of inequalities (2.4.11) and (2.4.13), but also gives some other interesting sharp inequalities as special cases.

A generalization of (2.4.11) is considered recently by Liu (2010), as follows:

Theorem 2.4.9. *(Liu 2010) Let the assumptions of Theorem 1 hold. Then for any* $\theta \in [0,1]$ *and* $(x,y) \in [a,c] \times [b,d]$*, we have*

$$\Bigg|(c-a)(d-b)\left\{(1-\theta)^2 f(x,y) + \frac{\theta(1-\theta)}{2}[f(a,y) + f(c,y) + f(x,b) + f(x,d)] + \frac{\theta^2}{4}[f(a,b) + f(a,d) + f(c,b) + f(c,d)]\right\} - (1-\theta)^2\left(x - \frac{a+c}{2}\right)\left(y - \frac{b+d}{2}\right)[f(c,d) - f(a,d) - f(c,b) + f(a,b)] - \frac{d-b}{2}\int_a^c [\theta f(s,b) + 2(1-\theta) f(s,y) + \theta f(s,d)]\,ds - \frac{c-a}{2}\int_b^d [\theta f(a,t) + 2(1-\theta) f(x,t) + \theta f(c,t)]\,ds + \int_a^c \int_b^d f(s,t)\,dsdt\Bigg|$$

$$\leq \left\{ \theta (2-\theta)(1-\theta)^2 (c-a)(d-b)\left(x-\frac{a+c}{2}\right)^2\left(y-\frac{b+d}{2}\right)^2 \right.$$
$$+\frac{(1-\theta)(1-3\theta-3\theta^2)}{12}(c-a)(d-b)$$
$$\times\left[(d-b)^2\left(x-\frac{a+c}{2}\right)^2+(c-a)^2\left(y-\frac{b+d}{2}\right)^2\right]$$
$$\left. +\frac{(1-3\theta-3\theta^2)^2}{144}(c-a)^3(d-b)^3\right\}^{\frac{1}{2}}\sqrt{\sigma\left(f''_{st}\right)} \qquad (2.4.14)$$

where $\sigma\left(f''_{st}\right)$ is defined above. The inequality (2.4.14) is sharp in the sense that the coefficient constant 1 of the right-hand side cannot be replaced by a smaller one.

In special case, if we set $x=\frac{a+c}{2}$ and $y=\frac{b+d}{2}$ with $\theta=\frac{1}{3}$, we get

$$\left|\frac{f\left(a,\frac{b+d}{2}\right)+f\left(\frac{a+c}{2},b\right)+f\left(\frac{a+c}{2},d\right)+f\left(c,\frac{b+d}{2}\right)+2f\left(\frac{a+c}{2},\frac{b+d}{2}\right)}{8}\right.$$
$$+\frac{f(a,b)+f(a,d)+f(c,b)+f(c,d)}{16}-\frac{\int_a^c\left[f(s,b)+2f\left(s,\frac{b+d}{2}\right)+f(s,d)\right]ds}{4(c-a)}$$
$$\left.-\frac{\int_b^d\left[f(a,t)+2f\left(\frac{a+c}{2},t\right)+f(c,t)\right]dt}{4(d-b)}+\frac{\int_a^c\int_b^d f(s,t)\,dsdt}{(c-a)(d-b)}\right|$$
$$\leq\frac{\left[(c-a)(d-b)\right]^{1/2}}{48}\sqrt{\sigma\left(f''\right)}, \qquad (2.4.15)$$

CHAPTER III

OSTROWSKI'S TYPE INEQUALITIES

3.1 INTRODUCTION

In this chapter, several new inequalities of Ostrowski's type are introduced. Trapezoid and Midpoint type rules for double $\mathcal{RS}$–double integral are proved. A generalization of the well known Beesack–Darst–Pollard inequality for double $\mathcal{RS}$–double integrals is also considered. Finally, as applications, two cubature formulae are proposed.

3.2 PRELIMINARIES AND LEMMAS

In this section, we introduce some fundamental inequalities concerning Riemann–Stieltjes double integrals. Namely, we first prove integration by parts formula for the Riemann–Stieltjes double integral and then using the concept of bounded bi-variation, bi-monotonic and Lipschitz mappings to generalize some basic and well–known inequalities for double integrals in the Riemann–Stieltjes sense.

Fréchet (1910) has given the following characterization for the double Riemann–Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle

$$Q : (a \le x \le b\ ;\ c \le y \le d);$$

let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x = x_i$, $y = y_j$,

$$a = x_0 < x_1 < \cdots < x_n = b, \quad and \quad c = y_0 < y_1 < \cdots < y_m = d;$$

let ζ_i, η_j be any numbers satisfying the inequalities $x_{i-1} \leq \zeta_i \leq x_i$, $y_{j-1} \leq \eta_j \leq y_j$, $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$; and for all i, j let

$$\Delta_{11}\alpha\left(x_i, y_j\right) = \alpha\left(x_{i-1}, y_{j-1}\right) - \alpha\left(x_{i-1}, y_j\right) - \alpha\left(x_i, y_{j-1}\right) + \alpha\left(x_i, y_j\right).$$

Then if the sum

$$S = \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(\zeta_i, \eta_j\right)\Delta_{11}\alpha\left(x_i, y_j\right)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f\left(x, y\right) d_x d_y \alpha\left(x, y\right). \tag{3.2.1}$$

If in the above formulation S is replaced by the sum

$$S^* = \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(\zeta_{ij}, \eta_{ij}\right)\Delta_{11}\alpha\left(x_i, y_j\right),$$

where ζ_{ij}, η_{ij} are any numbers satisfying the inequalities $x_{i-1} \leq \zeta_{ij} \leq x_i$, $y_{j-1} \leq \eta_{ij} \leq y_j$, we call the limit, when it exists, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f\left(x, y\right) d_x d_y \alpha\left(x, y\right). \tag{3.2.2}$$

The existence of (3.2.2) implies both the existence of (3.2.1) and its equality to (3.2.2). On the other hand, Clarkson (1933) has shown that the existence of (3.2.1) does not imply the existence of (3.2.2) (see Clarkson (1933)).

Lemma 3.2.1. *(Integration by parts) If $f \in \mathcal{RS}(\alpha)$ on Q, then $\alpha \in \mathcal{RS}(f)$ on Q, and we have*

$$\int_c^d\int_a^b f\left(t, s\right) d_t d_s \alpha\left(t, s\right) + \int_c^d\int_a^b \alpha\left(t, s\right) d_t d_s f\left(t, s\right)$$
$$= f\left(b, d\right)\alpha\left(b, d\right) - f\left(b, c\right)\alpha\left(b, c\right) - f\left(a, d\right)\alpha\left(a, d\right) + f\left(a, c\right)\alpha\left(a, c\right). \tag{3.2.3}$$

Proof. Let $\epsilon > 0$ be given. Since $\int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s)$ exists, there is a partition P_ϵ of Q such that for every P' finer than P_ϵ, we have

$$\left| S(P', f, \alpha) - \int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s) \right| < \epsilon. \tag{3.2.4}$$

Consider an arbitrary Riemann–Stieltjes sum for the integral $\alpha(t,s)\, d_t d_s f(t,s)$, say

$$\begin{aligned} S(P, f, \alpha) &= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i, s_j)\, \Delta_{11} f(x_i, y_j) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i, s_j)\, f(x_{i-1}, y_{j-1}) - \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i, s_j)\, f(x_{i-1}, y_j) \\ &\quad - \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i, s_j)\, f(x_i, y_{j-1}) + \sum_{j=1}^{m} \sum_{i=1}^{n} \alpha(t_i, s_j)\, f(x_i, y_j), \end{aligned}$$

where P finer than P_ϵ. Writing

$$A = f(b,d)\,\alpha(b,d) - f(b,c)\,\alpha(b,c) - f(a,d)\,\alpha(a,d) + f(a,c)\,\alpha(a,c),$$

we have the identity

$$\begin{aligned} A &= \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1}, y_{j-1})\, \alpha(x_{i-1}, y_{j-1}) - \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1}, y_j)\, \alpha(x_{i-1}, y_j) \\ &\quad - \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_{j-1})\, \alpha(x_i, y_{j-1}) + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j)\, \alpha(x_i, y_j). \end{aligned}$$

Subtracting the last two displayed equations, we find

$$\begin{aligned} A - S(P, f, \alpha) &= \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1}, y_{j-1}) \left[\alpha(x_{i-1}, y_{j-1}) - \alpha(t_i, s_j)\right] \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{i-1}, y_j) \left[\alpha(t_i, s_j) - \alpha(x_{i-1}, y_j)\right] \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_{j-1}) \left[\alpha(t_i, s_j) - \alpha(x_i, y_{j-1})\right] \\ &\quad + \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j) \left[\alpha(x_i, y_j) - \alpha(t_i, s_j)\right]. \end{aligned}$$

The sums on the right can be combined into a single sum of the form $S(P', f, \alpha)$, where P' is that partition of Q obtained by taking the points (t_i, s_j), (x_i, y_j) together. Then P' is finer than P and hence finer than P_ϵ. Therefore the inequality (3.2.4) is valid and this means that we have

$$\left| A - S(P, f, \alpha) - \int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s) \right| < \epsilon,$$

whenever P is finer than P_ϵ. But this is exactly the statement $\int_c^d \int_a^b \alpha\left(t,s\right) d_t d_s f\left(t,s\right)$ exists and equals $A - \int_c^d \int_a^b f\left(t,s\right) d_t d_s \alpha\left(t,s\right)$. □

Lemma 3.2.2. *If f is continuous on Q and if α is of bounded bivariation on Q, then $f \in \mathcal{RS}(\alpha)$.*

Proof. First of all, we note that, by Lemma 3.2.1, a second sufficient condition can be obtained by interchanging f and α in the hypothesis. It suffices to prove the theorem when α is bi-monotonically increasing with $\alpha\left(a,\cdot\right) \le \alpha\left(b,\cdot\right)$, $\alpha\left(\cdot,c\right) \le \alpha\left(\cdot,d\right)$ and $\alpha\left(t,s\right) \le \alpha\left(x,y\right)$, for all $t < x$ and $s < y$ in Q. Since f is continuous on Q then f is uniformly continuous on Q, i.e., $\forall \epsilon > 0$ there exists $\delta > 0$, such that

$$\left|f\left(x,y\right) - f\left(t,s\right)\right| < \frac{\epsilon}{A} \qquad \textit{whenever} \qquad \left\|\left(x-t, y-s\right)\right\| < \delta,$$

where $A = 4\left[\alpha\left(b,d\right) - \alpha\left(b,c\right) - \alpha\left(a,d\right) + \alpha\left(a,c\right)\right]$. If P_ϵ is a partition of Q with $\|P_\epsilon\| < \delta$, then for P finer than P_ϵ we must have $M_{ij}\left(f\right) - m_{ij}\left(f\right) \le \epsilon/A$, where

$$M_{ij}\left(f\right) - m_{ij}\left(f\right) = \sup\left\{f\left(x,y\right) - f\left(x,s\right) - f\left(t,y\right) + f\left(t,s\right) : \left(x,y\right), \left(t,s\right) \in Q\right\}.$$

Multiplying the inequality by $\Delta\alpha_{11}$ and summing, we find

$$U\left(P,f,\alpha\right) - L\left(P,f,\alpha\right) \le \frac{\epsilon}{A} \sum_{j=1}^{m} \sum_{i=1}^{n} \Delta_{11}\alpha = \frac{\epsilon}{4} < \epsilon.$$

Hence, $f \in \mathcal{RS}(\alpha)$ on Q. □

Lemma 3.2.3. *Assume that $g \in \mathcal{RS}(\alpha)$ on Q and α is of bounded bivariation on Q, then*

$$\left|\int_c^d \int_a^b g\left(x,y\right) d_x d_y \alpha\left(x,y\right)\right| \le \sup_{(x,y)\in Q} \left|g\left(x,y\right)\right| \cdot \bigvee_Q \left(\alpha\right). \tag{3.2.5}$$

Proof. The existence of $\int_c^d \int_a^b g\left(x,y\right) d_x d_y \alpha\left(x,y\right)$ follows from Lemma 3.2.4. Let $\Delta_n := a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_m := c = y_0 < y_1 < \cdots < y_m = d$ be a partitions of $[a,b]$ and $[c,d]$; respectively. Let

$$\Delta_{n,m} := \left\{ \left(x_0,y_0\right), \cdots, \left(x_0,y_m\right), \left(x_1,y_0\right), \cdots, \left(x_1,y_m\right), \cdots, \left(x_n,y_0\right), \cdots, \left(x_n,y_m\right) \right\}$$

be a partition of Q and $l\left(\Delta_{n,m}\right) := \max\limits_{i,j}\left\{x_{i+1}-x_i, y_{j+1}-y_j\right\}$ be the length of Q, therefore,

$$\begin{aligned}\left|\int_c^d\int_a^b g\left(x,y\right)d_xd_y\alpha\left(x,y\right)\right| &= \left|\lim_{l(\Delta_{n,m})\to 0}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1} g\left(\xi_i^{(n)},\eta_j^{(m)}\right)\Delta_{11}\alpha\left(x_i,y_j\right)\right| \\ &\le \lim_{l(\Delta_{n,m})\to 0}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)},\eta_j^{(m)}\right)\right|\left|\Delta_{11}\alpha\left(x_i,y_j\right)\right| \\ &\le \sup_{(x,y)\in Q}\left|g\left(x,y\right)\right|\cdot\sup_{\Delta_{n,m}}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left|\Delta_{11}\alpha\left(x_i,y_j\right)\right| \\ &= \sup_{(x,y)\in Q}\left|g\left(x,y\right)\right|\cdot\bigvee_Q\left(\alpha\right),\end{aligned}$$

which is required. □

Lemma 3.2.4. *Let g be a continuous mapping on $Q_{a,c}^{b,d}$ and α is bi-monotonic non-decreasing on $Q_{a,c}^{b,d}$, then*

$$\left|\int_c^d\int_a^b g\left(x,y\right)d_xd_y\alpha\left(x,y\right)\right| \le \int_c^d\int_a^b\left|g\left(x,y\right)\right|d_xd_y\alpha\left(x,y\right) \tag{3.2.6}$$

Proof. Let

$$\Delta_{nm} := \left\{\left(x_0,y_0\right),\cdots,\left(x_0,y_m\right),\left(x_1,y_0\right),\cdots,\left(x_1,y_m\right),\cdots,\left(x_n,y_0\right),\cdots,\left(x_n,y_m\right)\right\}$$

be a partition of Q and $l\left(\Delta_{nm}\right) := \max\limits_{i,j}\left\{x_{i+1}-x_i, y_{j+1}-y_j\right\}$ be the length of Q, therefore,

$$\begin{aligned}\left|\int_c^d\int_a^b g\left(x,y\right)d_xd_y\alpha\left(x,y\right)\right| &= \left|\lim_{l(\Delta_{n,m})\to 0}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1} g\left(\xi_i^{(n)},\eta_j^{(m)}\right)\Delta_{11}\alpha\left(x_i,y_j\right)\right| \\ &\le \lim_{l(\Delta_{n,m})\to 0}\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)},\eta_j^{(m)}\right)\right|\left|\Delta_{11}\alpha\left(x_i,y_j\right)\right|\end{aligned}$$

which is required. □

Lemma 3.2.5. *Let $g,\alpha : Q \to \mathbb{R}$, be such that g is L–Lipschitz on Q and α is Riemann-integrable on Q then the Riemann–Stieltjes integral $\int_c^d\int_a^b g\left(x,y\right)d_xd_y\alpha\left(x,y\right)$ exists and the inequality*

$$\left|\int_c^d\int_a^b g\left(x,y\right)d_xd_y\alpha\left(x,y\right)\right| \le L\int_c^d\int_a^b\left|g\left(x,y\right)\right|dxdy, \tag{3.2.7}$$

holds

Proof. The existence of $\int_c^d \int_a^b g(x,y)\, d_x d_y \alpha(x,y)$ follows from Lemma 3.2.2. Let $\Delta_n := a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_m := c = y_0 < y_1 < \cdots < y_m = d$ be a partitions of $[a,b]$ and $[c,d]$; respectively. Let

$$\Delta_{nn} := \{(x_0,y_0),(x_1,y_1),\cdots,(x_n,y_0),\cdots,(x_n,y_n)\}$$

be a partition of Q and $l(\Delta_{nn}) := \max_{i,j}\{x_{i+1}-x_i, y_{j+1}-y_j\}$ be the length of Q, therefore,

$$\begin{aligned}\left|\int_c^d \int_a^b g(x,y)\, d_x d_y \alpha(x,y)\right| &= \left|\lim_{l(\Delta_{nn})\to 0}\sum_{j=0}^{n-1}\sum_{i=0}^{n-1} g\left(\xi_i^{(n)},\eta_j^{(n)}\right)\left[\alpha(x_i,y_i)-\alpha(x_{i-1},y_{i-1})\right]\right| \\ &\le \lim_{l(\Delta_{nn})\to 0}\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)},\eta_j^{(n)}\right)\right|\left|\alpha(x_i,y_i)-\alpha(x_{i-1},y_{i-1})\right| \\ &\le L\lim_{l(\Delta_{nn})\to 0}\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\left|g\left(\xi_i^{(n)},\eta_j^{(n)}\right)\right|\left|(x_i,y_i)-(x_{i-1},y_{i-1})\right| \\ &= L\int_c^d \int_a^b |g(x,y)|\, dxdy,\end{aligned}$$

which is required. □

3.3 OSTROWSKI INEQUALITY FOR MAPPINGS BOUNDED BIVARIATION

In this section and in order to approximate the Riemann–Stieltjes double integral, some of Ostrowski, trapezoid and Simpson type inequalities are proved.

We begin with the following generalization of (2.3.4):

Theorem 3.3.1. *Let $f : Q \to \mathbb{R}$ be a mapping of bounded bivariation on Q. Then for all $(x,y) \in Q$, we have the inequality*

$$\begin{aligned}&\left|(b-a)(d-c)f(x,y)-\int_c^d\int_a^b f(t,s)\,dtds\right| \\ &\qquad\le\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\cdot\bigvee_Q(f),\end{aligned}\tag{3.3.1}$$

where $\bigvee_Q(f)$ denotes the total (double) bivariation of f on Q.

Proof. From Lemma 3.2.1, we have

$$\int_c^y \int_a^x (t-a)(s-c)\, d_t d_s f(t,s) = (x-a)(y-c) f(x,y) - \int_c^y \int_a^x f(t,s)\, dt ds,$$

$$\int_y^d \int_a^x (t-a)(s-d)\, d_t d_s f(t,s) = (x-a)(d-y) f(x,y) - \int_y^d \int_a^x f(t,s)\, dt ds,$$

$$\int_c^y \int_x^b (t-b)(s-c)\, d_t d_s f(t,s) = (b-x)(y-c) f(x,y) - \int_c^y \int_x^b f(t,s)\, dt ds,$$

and

$$\int_y^d \int_x^b (t-b)(s-d)\, d_t d_s f(t,s) = (x-b)(y-d) f(x,y) - \int_y^d \int_x^b f(t,s)\, dt ds.$$

Adding the above equalities, we get

$$\int_c^d \int_a^b P(x,t;y,s)\, d_t d_s f(t,s) = (b-a)(d-c) f(x,y) - \int_c^d \int_a^b f(t,s)\, dt ds$$

where,

$$P(x,t;y,s) = \begin{cases} (t-a)(s-c), & (x,y) \in [a,x] \times [c,y] \\ (t-a)(s-d), & (x,y) \in [a,x] \times (y,d] \\ (t-b)(s-c), & (x,y) \in (x,b] \times [c,y] \\ (t-b)(s-d), & (x,y) \in (x,b] \times (y,d] \end{cases}$$

for all $(t,s) \in Q$.

Now, applying Lemma 3.2.3, by letting $g = P$ and $\alpha = f$, we get

$$\left|\int_c^d \int_a^b P(x,t;y,s)\, d_t d_s f(t,s)\right|$$
$$\leq \sup_{(x,y)\in Q} |P(x,t;y,s)| \cdot \bigvee_Q (f)$$
$$= \max_{x,y} \{(x-a)(y-c), (x-a)(d-y), (b-x)(y-c), (b-x)(d-y)\} \cdot \bigvee_Q (f),$$

but,

$$
\begin{aligned}
M &= \max_{x,y}\left\{(x-a)(y-c),(x-a)(d-y),(b-x)(y-c),(b-x)(d-y)\right\} \\
&= \max_{x}\left\{\max_{y}\left\{(x-a)(y-c),(x-a)(d-y),(b-x)(y-c),(b-x)(d-y)\right\}\right\},
\end{aligned}
$$

and since $\max\limits_{y}$ is independent of x, we have

$$
\begin{aligned}
M &= \max_{x}\left\{(x-a)\cdot\max_{y}\left\{(y-c),(d-y)\right\},(b-x)\cdot\max_{y}\left\{(y-c),(d-y)\right\}\right\} \\
&\leq \max_{x}\left\{(x-a),(b-x)\right\}\cdot\max_{y}\left\{(y-c),(d-y)\right\} \\
&= \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right],
\end{aligned}
$$

it follows that,

$$
\begin{aligned}
&\left|\int_c^d\int_a^b P(x,t;y,s)\,d_t d_s f(t,s)\right| \\
&\qquad\qquad \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\cdot\bigvee_{Q}(f),
\end{aligned}
$$

which completes the proof. □

Corollary 3.3.2. *In Theorem 3.3.1. Let $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we have*

$$
\left|(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\int_c^d\int_a^b f(t,s)\,dtds\right|\leq\frac{(b-a)(d-c)}{4}\cdot\bigvee_{Q}(f),
$$

Remark 3.3.3. *Similar inequalities can be found if we assume that u is monotonous on Q, we left the details to the interested reader.*

Corollary 3.3.4. *In Theorem 3.3.1. Assume $[a,b]=[c,d]$, we get*

$$
\begin{aligned}
&\left|(b-a)^2 f(x,y)-\int_a^b\int_a^b f(t,s)\,dtds\right| \\
&\qquad\qquad \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{b-a}{2}+\left|y-\frac{a+b}{2}\right|\right]\cdot\bigvee_{Q}(f).
\end{aligned}
$$

A generalization of the trapezoid inequality (2.3.5) for mappings of two variables may be stated as follows:

Theorem 3.3.5. *Let $f : Q \to \mathbb{R}$ be a mapping of bounded bivariation on Q. Then for all $(x, y) \in Q$, we have the inequality*

$$\left| \frac{(b-a)(d-c)}{4} \cdot [f(b,d) - f(b,c) - f(a,d) + f(a,c)] - \int_c^d \int_a^b f(t,s)\, dtds \right| \leq \frac{(b-a)(d-c)}{4} \cdot \bigvee_Q (f), \quad (3.3.2)$$

The constant $\frac{1}{4}$ is best possible value.

Proof. From Lemma 3.2.1, we have

$$\int_c^d \int_a^b R(t,s)\, d_t d_s f(t,s) = \frac{(b-a)(d-c)}{4} \cdot [f(b,d) - f(b,c) - f(a,d) + f(a,c)] - \int_c^d \int_a^b f(t,s)\, dtds,$$

where, $R(t,s) = \left(t - \frac{a+b}{2}\right)\left(s - \frac{c+d}{2}\right), \quad a \leq t \leq b \ ; \ c \leq s \leq d$.

Now, applying Lemma 3.2.3, by letting $g = R$ and $\alpha = f$, we get

$$\left| \int_c^d \int_a^b R(t,s)\, d_t d_s f(t,s) \right| \leq \sup_{(t,s) \in Q} |R(t,s)| \cdot \bigvee_c^d \bigvee_a^b (f) = \frac{(b-a)(d-c)}{4} \cdot \bigvee_Q (f),$$

which is required. □

The following theorem generalize the inequality (3.3.2).

Theorem 3.3.6. *Let $f : Q \to \mathbb{R}$ be a mapping of bounded bivariation on Q. Then for all $(x, y) \in Q$, we have the inequality*

$$\left| (f\alpha)(b,d) - (f\alpha)(b,c) - (f\alpha)(a,d) + (f\alpha)(a,c) - \int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s) \right| \leq \sup_{(t,s) \in Q} |\alpha(t,s)| \cdot \bigvee_Q (f). \quad (3.3.3)$$

Proof. From Lemma 3.2.1, we have

$$\int_c^d \int_a^b \alpha(t,s)\, d_t d_s f(t,s) + \int_c^d \int_a^b f(t,s)\, d_t d_s \alpha(t,s) = f(b,d)\alpha(b,d) - f(b,c)\alpha(b,c) - f(a,d)\alpha(a,d) + f(a,c)\alpha(a,c).$$

Now, applying Lemma 3.2.3, by letting $g = R$ and $\alpha = f$, we get

$$\left| \int_c^d \int_a^b \alpha(t,s)\, d_t d_s f(t,s) \right| \le \sup_{(t,s)\in Q} |\alpha(t,s)| \cdot \bigvee_Q (f),$$

which is required. $\square$

The following result holds

Theorem 3.3.7. *Let $u : Q \to \mathbb{R}$ be a function of bounded bivariation and $f : Q \to \mathbb{R}$ a function such that there exists the constants $m, M \in \mathbb{R}$ with $m \le f(t,s) \le M$, for each $(t,s) \in Q$, and the Stieltjes integral $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ exists. Then, by defining the error functional*

$$\begin{aligned} &\omega(f,u,m,M;Q) \\ &\quad := \int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) - \frac{m+M}{2}\left[u(b,d) - u(b,c) - u(a,d) + u(a,c)\right] \end{aligned} \tag{3.3.4}$$

we have the bound

$$\omega(f,u,m,M;Q) \le \frac{1}{2}(M-m) \cdot \bigvee_c^d \bigvee_a^b (u) \tag{3.3.5}$$

Proof. Since, obviously, the function $f - \frac{m+M}{2}$ satisfies the inequality

$$\left| f(t,s) - \frac{m+M}{2} \right| \le \frac{1}{2}(M-m),\ \forall (t,s) \in Q$$

and the Stieltjes integral $\int_c^d \int_a^b \left(f(t,s) - \frac{m+M}{2}\right) d_t d_s u(t,s)$ exists, then

$$\begin{aligned} &\left| \int_c^d \int_a^b \left(f(t,s) - \frac{m+M}{2} \right) d_t d_s u(t,s) \right| \\ &\qquad \le \sup_{(t,s)\in Q} \left(f(t,s) - \frac{m+M}{2} \right) \cdot \bigvee_c^d \bigvee_a^b (u) \le \frac{1}{2}(M-m) \cdot \bigvee_c^d \bigvee_a^b (u) \end{aligned}$$

and the inequality (3.3.5) is proved. $\square$

Now, we consider an Ostrowski type inequality for (β_1, β_2)–Hölder type mapping on the co-ordinate.

Definition 3.3.8. *A function $f : Q_{a,c}^{b,d} \to \mathbb{R}$ is to be of (β_1, β_2)–Hölder type mapping on the co-ordinate, if for all $(t_1, s_1), (t_1, s_1) \in Q_{a,c}^{b,d}$, there exist $H_1, H_2 > 0$ and $\beta_1, \beta_2 > 0$ such that*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq H_1 |t_1 - t_2|^{\beta_1} + H_2 |s_1 - s_2|^{\beta_2}.$$

If $\beta_1 = \beta_2 = 1$, then f is called (L_1, L_2)–Lipschitz on the co-ordinate, i.e.,

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2|.$$

Theorem 3.3.9. *Let $f : Q_{a,c}^{b,d} \to \mathbb{R}$ be a (β_1, β_2)–Hölder type mapping on the co-ordinate, i.e., for all $(t_1, s_1), (t_1, s_1) \in Q_{a,c}^{b,d}$, there exist $H_1, H_2 > 0$ and $\beta_1, \beta_2 > 0$ such that*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq H_1 |t_1 - t_2|^{\beta_1} + H_2 |s_1 - s_2|^{\beta_2},$$

and $u : Q_{a,c}^{b,d} \to \mathbb{R}$ be a mapping of bounded bivariation on $Q_{a,c}^{b,d}$. Then for all $(x, y) \in Q_{a,c}^{b,d}$, we have the inequality

$$\begin{aligned} &\left|[u(b,d) - u(b,c) - u(a,d) + u(a,c)] f(x,y) - \int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)\right| \\ &\leq \left(H_1 \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\beta_1} + H_2 \left[\frac{d-c}{2} + \left|y - \frac{c+d}{2}\right|\right]^{\beta_2}\right) \bigvee_{Q_{a,c}^{b,d}} (u). \quad (3.3.6) \end{aligned}$$

Proof. From Lemma 3.2.3, we have

$$\begin{aligned} &\left|f(x,y)(u(b,d) - u(b,c) - u(a,d) + u(a,c)) - \int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)\right| \\ &= \left|\int_c^d \int_a^b (f(x,y) - f(t,s))\, d_t d_s u(t,s)\right| \\ &\leq \sup_{(t,s) \in Q_{a,c}^{b,d}} |f(x,y) - f(t,s)| \cdot \bigvee_{Q_{a,c}^{b,d}} (u). \end{aligned}$$

Now, since f is a (β_1, β_2)–Hölder on the co-ordinate, we have

$$\begin{aligned} \sup_{(t,s) \in Q_{a,c}^{b,d}} |f(x,y) - f(t,s)| &\leq \sup_{(t,s) \in Q_{a,c}^{b,d}} \left(H_1 |x - t|^{\beta_1} + H_2 |y - s|^{\beta_2}\right) \\ &= H_1 \sup_{t \in [a,b]} |x - t|^{\beta_1} + H_2 \sup_{s \in [c,d]} |y - s|^{\beta_2} \\ &= H_1 \max\left\{(x-a)^{\beta_1}, (b-x)^{\beta_1}\right\} + H_2 \max\left\{(y-c)^{\beta_1}, (d-y)^{\beta_2}\right\} \\ &= H_1 \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^{\beta_1} + H_2 \left[\frac{d-c}{2} + \left|y - \frac{c+d}{2}\right|\right]^{\beta_2} \end{aligned}$$

which follows that,

$$\left|[u(b,d)-u(b,c)-u(a,d)+u(a,c)]\,f(x,y)-\int_c^d\int_a^b f(t,s)\,d_td_su(t,s)\right|$$
$$\leq\left(H_1\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^{\beta_1}+H_2\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]^{\beta_2}\right)\bigvee_{Q_{a,c}^{b,d}}(u),$$

which completes the proof. □

Corollary 3.3.10. *Let* u *as in Theorem 3.3.13 and let* $f : Q_{a,c}^{b,d} \to \mathbb{R}$ *be an* L*–Lipschitzian mapping on the co-ordinate on* $Q_{a,c}^{b,d}$*, i.e., for all* $(t_1,s_1),(t_1,s_1)\in Q_{a,c}^{b,d}$*, there exist* $L_1, L_2 > 0$ *such that*

$$|f(t_1,s_1)-f(t_2,s_2)|\leq L_1|t_1-t_2|+L_2|s_1-s_2|.$$

Then for all $(x,y)\in Q_{a,c}^{b,d}$*, we have the inequality*

$$\left|[u(b,d)-u(b,c)-u(a,d)+u(a,c)]\,f(x,y)-\int_c^d\int_a^b f(t,s)\,d_td_su(t,s)\right|$$
$$\leq L_1\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]+L_2\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\quad(3.3.7)$$

Theorem 3.3.11. *Let* $f : Q_{a,c}^{b,d}\to\mathbb{R}$ *be a mapping of bounded bivariation on* $Q_{a,c}^{b,d}$ *and* $u : Q_{a,c}^{b,d}\to\mathbb{R}$ *be a* (β_1,β_2)*–Hölder mapping on the co-ordinate. Then for all* $(x,y)\in Q_{a,c}^{b,d}$*, we have the inequality*

$$\begin{aligned}|\Theta(f,u;x,a,b;y,c,d)|\leq{}&\left[H_1(x-a)^{\beta_1}+H_2(y-c)^{\beta_2}\right]\cdot\bigvee_c^y\bigvee_a^x(f)\\&+\left[H_1(b-x)^{\beta_1}+H_2(y-c)^{\beta_2}\right]\cdot\bigvee_c^y\bigvee_x^b(f)\\&+\left[H_1(x-a)^{\beta_1}+H_2(d-y)^{\beta_2}\right]\cdot\bigvee_y^d\bigvee_a^x(f)\\&+\left[H_1(b-x)^{\beta_1}+H_2(d-y)^{\beta_2}\right]\cdot\bigvee_y^d\bigvee_x^b(f),\qquad(3.3.8)\end{aligned}$$

where,

$$\begin{aligned}&\Theta(f,u;x,a,b;y,c,d)\\&\quad:=[u(b,d)-u(b,c)-u(a,d)+u(a,c)]\,f(x,y)-\int_c^d\int_a^b f(t,s)\,d_td_su(t,s)\end{aligned}$$

is the Ostrowski's functional associated to f *and* u *as above.*

Proof. As u is continuous and f is of bounded bivariation on $Q_{a,c}^{b,d}$, the following double Riemann–Stieltjes integrals exist and, by the integration by parts formula, we can state that

$$\begin{aligned} I_1 &:= \int_c^y \int_a^x (u(t,s) - u(a,c))\, d_t d_s f(t,s) \\ &= [u(x,y) - u(x,c) - u(a,y) + u(a,c)]\, f(x,y) - \int_c^y \int_a^x f(t,s)\, d_t d_s u(t,s), \end{aligned}$$

$$\begin{aligned} I_2 &:= \int_c^y \int_x^b (u(t,s) - u(b,c))\, d_t d_s f(t,s) \\ &= [u(b,y) - u(b,c) - u(x,y) + u(x,c)]\, f(x,y) - \int_c^y \int_x^b f(t,s)\, d_t d_s u(t,s), \end{aligned}$$

$$\begin{aligned} I_3 &:= \int_y^d \int_a^x (u(t,s) - u(a,d))\, d_t d_s f(t,s) \\ &= [u(x,d) - u(x,y) - u(a,d) + u(a,y)]\, f(x,y) - \int_y^d \int_a^x f(t,s)\, d_t d_s u(t,s), \end{aligned}$$

and

$$\begin{aligned} I_4 &:= \int_y^d \int_x^b (u(t,s) - u(b,d))\, d_t d_s f(t,s) \\ &= [u(b,d) - u(b,y) - u(x,d) + u(x,y)]\, f(x,y) - \int_y^d \int_x^b f(t,s)\, d_t d_s u(t,s). \end{aligned}$$

If we add the above identities, we obtain

$$\begin{aligned} \Theta(f,u;x,a,b;y,c,d) = &\int_c^y \int_a^x (u(t,s) - u(a,c))\, d_t d_s f(t,s) \\ &+ \int_c^y \int_x^b (u(t,s) - u(b,c))\, d_t d_s f(t,s) \\ &+ \int_y^d \int_a^x (u(t,s) - u(a,d))\, d_t d_s f(t,s) \\ &+ \int_y^d \int_x^b (u(t,s) - u(b,d))\, d_t d_s f(t,s) \end{aligned}$$

Now, using the properties of modulus, we have:

$$
\begin{aligned}
|\Theta(f,u;x,a,b;y,c,d)| \leq{} & \left|\int_c^y \int_a^x (u(t,s)-u(a,c))\, d_t d_s f(t,s)\right| \\
& + \left|\int_c^y \int_x^b (u(t,s)-u(b,c))\, d_t d_s f(t,s)\right| \\
& + \left|\int_y^d \int_a^x (u(t,s)-u(a,d))\, d_t d_s f(t,s)\right| \\
& + \left|\int_y^d \int_x^b (u(t,s)-u(b,d))\, d_t d_s f(t,s)\right| \\
\leq{} & \sup_{(t,s)\in Q_{a,c}^{x,y}} |u(t,s)-u(a,c)| \cdot \bigvee_c^y \bigvee_a^x (f) \\
& + \sup_{(t,s)\in Q_{x,c}^{b,y}} |u(t,s)-u(b,c)| \cdot \bigvee_c^y \bigvee_x^b (f) \\
& + \sup_{(t,s)\in Q_{a,y}^{x,d}} |u(t,s)-u(a,d)| \cdot \bigvee_y^d \bigvee_a^x (f) \\
& + \sup_{(t,s)\in Q_{x,y}^{b,d}} |u(t,s)-u(b,d)| \cdot \bigvee_y^d \bigvee_x^b (f).
\end{aligned}
$$

However,

$$
|u(t,s)-u(a,c)| \leq H_1 |t-a|^{\beta_1} + H_2 |s-c|^{\beta_2},
$$

so that,

$$
\begin{aligned}
\sup_{(t,s)\in Q_{a,c}^{x,y}} |u(t,s)-u(a,c)| &\leq \sup_{(t,s)\in Q_{a,c}^{x,y}} \left(H_1 |t-a|^{\beta_1} + H_2 |s-c|^{\beta_2}\right) \\
&= H_1 (x-a)^{\beta_1} + H_2 (y-c)^{\beta_2}.
\end{aligned}
$$

Similarly, for

$$
\sup_{(t,s)\in Q_{x,c}^{b,y}} |u(t,s)-u(b,c)| \leq H_1 (b-x)^{\beta_1} + H_2 (y-c)^{\beta_2},
$$

$$
\sup_{(t,s)\in Q_{a,y}^{x,d}} |u(t,s)-u(a,d)| \leq H_1 (x-a)^{\beta_1} + H_2 (d-y)^{\beta_2}
$$

and

$$
\sup_{(t,s)\in Q_{x,y}^{b,d}} |u(t,s)-u(b,d)| \leq H_1 (b-x)^{\beta_1} + H_2 (d-y)^{\beta_2},
$$

in obtaining the above inequalities we get the required result. □

Remark 3.3.12. *In the above results, if one chooses* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, we get inequalities of midpoint type for mappings of two independent variables.*

In the following, we generalize the inequality (2.3.11) which is a companion of Ostrowski's inequality for mappings of two variables:

Theorem 3.3.13. *Let* $f : Q_{a,c}^{b,d} \to \mathbb{R}$ *be a mapping of bounded bivariation on* $Q_{a,c}^{b,d}$*. Then for all* $(x,y) \in Q_{a,c}^{\frac{a+b}{2},\frac{c+d}{2}}$*, we have the inequality*

$$\left|\frac{(b-a)(d-c)}{4}\left[f(x,y)+f(a+b-x,y)+f(x,c+d-y)+f(a+b-x,c+d-y)\right] - \int_c^d\int_a^b f(t,s)\,dtds\right| \leq \left[\frac{b-a}{4}+\left|x-\frac{3a+b}{4}\right|\right]\cdot\left[\frac{d-c}{4}+\left|y-\frac{3c+d}{4}\right|\right]\cdot\bigvee_c^d\bigvee_a^b(f). \quad (3.3.9)$$

Proof. From Lemma 3.2.1, we have

$$\int_c^y\int_a^x (t-a)(s-c)\,d_td_sf(t,s) = (x-a)(y-c)f(x,y) \quad (3.3.10)$$

$$\int_c^y\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)(s-c)\,d_td_sf(t,s) = \left(\frac{a+b}{2}-x\right)(y-c)f(a+b-x,y) - \left(x-\frac{a+b}{2}\right)(y-c)f(x,y) \quad (3.3.11)$$

$$\int_c^y\int_{a+b-x}^b (t-b)(s-c)\,d_td_sf(t,s) = (x-a)(y-c)f(a+b-x,y) \quad (3.3.12)$$

$$\int_y^{c+d-y}\int_a^x (t-a)\left(s-\frac{c+d}{2}\right)d_td_sf(t,s) = f(x,c+d-y)(x-a)\left(\frac{c+d}{2}-y\right) + f(x,y)(x-a)\left(\frac{c+d}{2}-y\right) \quad (3.3.13)$$

$$\int_{y}^{c+d-y}\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)d_t d_s f(t,s)$$
$$=f(a+b-x,c+d-y)\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)$$
$$+f(a+b-x,y)\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)$$
$$+f(x,c+d-y)\left(x-\frac{a+b}{2}\right)\left(\frac{c+d}{2}-y\right)$$
$$+f(x,y)\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) \quad (3.3.14)$$

$$\int_{y}^{c+d-y}\int_{a+b-x}^{b}(t-b)\left(s-\frac{c+d}{2}\right)d_t d_s f(t,s)$$
$$=f(a+b-x,c+d-y)(x-a)\left(\frac{c+d}{2}-y\right)+f(a+b-x,y)(a-x)\left(y-\frac{c+d}{2}\right) \quad (3.3.15)$$

$$\int_{c+d-y}^{d}\int_{a}^{x}(t-a)(s-d)\,d_t d_s f(t,s)=(x-a)(y-c)\,f(x,c+d-y) \quad (3.3.16)$$

$$\int_{c+d-y}^{d}\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)(s-d)\,d_t d_s f(t,s)$$
$$=\left(\frac{a+b}{2}-x\right)(d-y)\,f(a+b-x,c+d-y)+\left(x-\frac{a+b}{2}\right)(c-y)\,f(x,c+d-y) \quad (3.3.17)$$

$$\int_{c+d-y}^{d}\int_{a+b-x}^{b}(t-b)(s-d)\,d_t d_s f(t,s)=f(a+b-x,c+d-y)(x-a)(d-y) \quad (3.3.18)$$

Adding the above equalities, we get

$$\frac{(b-a)(d-c)}{4}\left[f(x,y)+f(a+b-x,y)+f(x,c+d-y)+f(a+b-x,c+d-y)\right]$$
$$-\int_{c}^{d}\int_{a}^{b}f(t,s)\,dtds$$
$$=\int_{c}^{d}\int_{a}^{b}K_1(x,t)\,K_2(y,s)\,d_t d_s f(t,s),$$

where,

$$K_1(x,t)=\begin{cases} t-a, & t\in[a,x] \\ t-\frac{a+b}{2}, & t\in(x,a+b-x] \\ t-b, & t\in(a+b-x,b] \end{cases},$$

and

$$K_2(y,s) = \begin{cases} s-c, & s \in [a,y] \\ s-\frac{c+d}{2}, & s \in (y, c+d-y] \\ s-d, & s \in (c+d-y, d] \end{cases}$$

Now, by Lemma 3.2.2

$$\left| \frac{(b-a)(d-c)}{4} [f(x,y) + f(a+b-x,y) + f(x,c+d-y) + f(a+b-x,c+d-y)] \right.$$
$$\left. - \int_c^d \int_a^b f(t,s)\, dtds \right|$$
$$= \left| \int_c^d \int_a^b K_1(x,t) K_2(y,s)\, d_t d_s f(t,s) \right| \le \sup_{(x,y)\in Q_{a,c}^{b,d}} |K_1(x,t) K_2(y,s)| \cdot \bigvee_{Q_{a,c}^{b,d}} (f).$$

Since

$$\sup_{(x,y)\in Q_{a,c}^{b,d}} |K_1(x,t) K_2(y,s)| = \max_x \left\{ (x-a), \left(\frac{a+b}{2} - x \right) \right\} \cdot \max_y \left\{ (y-c), \left(\frac{c+d}{2} - y \right) \right\}$$
$$= \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] \cdot \left[\frac{d-c}{4} + \left| y - \frac{3c+d}{4} \right| \right].$$

Combining the above identities we get the required results. □

Corollary 3.3.14. *In the above theorem, choose*

1. $x = a$ *and* $y = c$*, we get*

$$\left| \frac{(b-a)(d-c)}{4} [f(a,c) + f(b,c) + f(a,d) + f(b,d)] - \int_c^d \int_a^b f(t,s)\, dtds \right|$$
$$\le \frac{(b-a)(d-c)}{4} \cdot \bigvee_{Q_{a,c}^{b,d}} (f). \quad (3.3.19)$$

2. $x = \frac{3a+b}{4}$ *and* $y = \frac{3c+d}{4}$*, we get*

$$\left| \frac{(b-a)(d-c)}{4} \left[f\left(\frac{3a+b}{4}, \frac{3c+d}{4} \right) + f\left(\frac{a+3b}{4}, \frac{3c+d}{4} \right) + f\left(\frac{3a+b}{4}, \frac{c+3d}{4} \right) \right. \right.$$
$$\left. \left. + f\left(\frac{a+3b}{4}, \frac{c+3d}{4} \right) \right] - \int_c^d \int_a^b f(t,s)\, dtds \right| \le \frac{(b-a)(d-c)}{16} \cdot \bigvee_{Q_{a,c}^{b,d}} (f).$$
$$(3.3.20)$$

3. $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, we get*

$$\left|(b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \int_c^d \int_a^b f(t,s)\, dtds\right| \le \frac{(b-a)(d-c)}{4} \cdot \bigvee_{Q_{a,c}^{b,d}} (f). \tag{3.3.21}$$

A generalization of the Simpson inequality (2.3.23) for mappings of bounded bivariation, is considered as follows:

Theorem 3.3.15. *Let* $f : Q \to \mathbb{R}$ *be a mapping of bounded bivariation on* Q*. Then for all* $(x, y) \in Q$*, we have the inequality*

$$\begin{aligned}
&\left|\frac{(b-a)(d-c)}{36}\left[f(b,d) - f(b,c) - f(a,d) + f(a,c)\right]\right. \\
&\quad + \frac{(b-a)(d-c)}{9}\left[f\left(\frac{a+b}{2}, d\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right. \\
&\quad\quad \left.\left. - f\left(\frac{a+b}{2}, c\right) - f\left(a, \frac{c+d}{2}\right)\right] - \int_a^b \int_c^d f(s,t)\, dtds\right| \\
&\quad\quad\quad \le \frac{(b-a)(d-c)}{9} \cdot \bigvee_Q (f),
\end{aligned} \tag{3.3.22}$$

where $\bigvee_Q (f)$ *denotes the total (double) bivariation of* f *on* Q*.*

Proof. From Lemma 3.2.1, we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} \left(s - \frac{5a+b}{6}\right)\left(t - \frac{5c+d}{6}\right) d_t d_s f(t,s) \\
&= \frac{(b-a)(d-c)}{9} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
&\quad - \frac{(b-a)(d-c)}{18}\left[f\left(\frac{a+b}{2}, c\right) + f\left(a, \frac{c+d}{2}\right)\right] \\
&\quad + \frac{(b-a)(d-c)}{36} f(a,c) - \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(s,t)\, dtds
\end{aligned}$$

$$\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\left(s-\frac{5a+c}{6}\right)\left(t-\frac{c+5d}{6}\right)d_t d_s f(t,s)$$
$$=\frac{(b-a)(d-c)}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$-\frac{(b-a)(d-c)}{18}\left[f\left(a,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},d\right)\right]$$
$$-\frac{(b-a)(d-c)}{36}f(a,d)-\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(s,t)\,dtds$$

$$\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\left(s-\frac{a+5b}{6}\right)\left(t-\frac{5c+d}{6}\right)d_t d_s f(t,s)$$
$$=\frac{(b-a)(d-c)}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$+\frac{(b-a)(d-c)}{18}\left[f\left(b,\frac{c+d}{2}\right)-f\left(\frac{a+b}{2},c\right)\right]$$
$$-\frac{(b-a)(d-c)}{36}f(b,c)-\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} f(s,t)\,dtds$$

and

$$\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left(s-\frac{a+5b}{6}\right)\left(t-\frac{c+5d}{6}\right)d_t d_s f(t,s)$$
$$=\frac{(b-a)(d-c)}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$+\frac{(b-a)(d-c)}{18}\left[f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)\right]$$
$$+\frac{(b-a)(d-c)}{36}f(b,d)-\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(s,t)\,dtds$$

Adding the above equalities, we get

$$\int_a^b\int_c^d K(s,t)\,d_t d_s f(s,t)$$
$$=\frac{(b-a)(d-c)}{36}\left[f(b,d)-f(b,c)-f(a,d)+f(a,c)\right]$$
$$+\frac{(b-a)(d-c)}{9}\left[f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.-f\left(\frac{a+b}{2},c\right)-f\left(a,\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(s,t)\,dtds$$

where,

$$K(s,t)=\begin{cases}\left(s-\frac{5a+b}{6}\right)\left(t-\frac{5c+d}{6}\right), & a\le s\le \frac{a+b}{2}, c\le t\le \frac{c+d}{2},\\[2ex] \left(s-\frac{5a+b}{6}\right)\left(t-\frac{c+5d}{6}\right), & a\le s\le \frac{a+b}{2}, \frac{c+d}{2}< t\le d,\\[2ex] \left(s-\frac{a+5b}{6}\right)\left(t-\frac{5c+d}{6}\right), & \frac{a+b}{2}< s\le b, c\le t\le \frac{c+d}{2},\\[2ex] \left(s-\frac{a+5b}{6}\right)\left(t-\frac{c+5d}{6}\right), & \frac{a+b}{2}< s\le b, \frac{c+d}{2}< t\le d.\end{cases}$$

for all $(t,s)\in Q$.

Now, applying Lemma 3.2.3, by letting $g=P$ and $\alpha=f$, we get

$$\left|\int_c^d\int_a^b K(s,t)\,d_td_sf(t,s)\right|\le \sup_{(x,y)\in Q}|K(s,t)|\cdot\bigvee_c^d\bigvee_a^b(f)=\frac{(b-a)(d-c)}{9}\cdot\bigvee_c^d\bigvee_a^b(f),$$

which completes the proof. □

3.4 QUADRATURE RULES FOR $\mathcal{RS}$–DOUBLE INTEGRAL

In this section and using Mercer approach to prove Theorem 2.3.13, we introduce the following quadrature rule for Riemann–Stieltjes double integral. Looking for a trapezoidal rule for the double RS-integral, we seek numbers A, B, C and D such that

$$\int_a^b\int_c^d f(x,y)\,d_yd_xg(x,y)\cong Af(a,c)+Bf(a,d)+Cf(b,c)+Df(b,d)$$

is equality for $f(x,y)=1$,$f(x,y)=x$,$f(x,y)=y$ and $f(x,y)=xy$. That is,

$$\int_a^b\int_c^d 1d_yd_xg(x,y)=A+B+C+D,$$

$$\int_a^b\int_c^d xd_yd_xg(x,y)=Aa+Ba+Cb+Db,$$

$$\int_a^b\int_c^d yd_yd_xg(x,y)=Ac+Bd+Cc+Dd,$$

and

$$\int_a^b \int_c^d xy d_y d_x g(x,y) = Aac + Bad + Cbc + Dbd.$$

Solving these equations for A, B, C and D, we obtain *our double RS-trapezoidal rule*:

$$\begin{aligned}
\int_a^b \int_c^d f(x,y)\, d_y d_x g(x,y) &= \left[g(a,c) - \frac{1}{b-a}\int_a^b g(x,c)\,dx - \frac{1}{d-c}\int_c^d g(a,y)\,dy \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y) dy dx \right] f(a,c) \\
&- \left[g(a,d) - \frac{1}{b-a}\int_a^b g(x,d)\,dx - \frac{1}{d-c}\int_c^d g(a,y)\,dy \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y) dy dx \right] f(a,d) \\
&- \left[g(b,c) - \frac{1}{b-a}\int_a^b g(x,c)\,dx - \frac{1}{d-c}\int_c^d g(b,y)\,dy \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y) dy dx \right] f(b,c) \\
&+ \left[g(b,d) - \frac{1}{b-a}\int_a^b g(x,d)\,dx - \frac{1}{d-c}\int_c^d g(b,y)\,dy \right. \\
&\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y) dy dx \right] f(b,d).
\end{aligned}$$

Looking for a midpoint rule for the double RS-integral, we seek $A \in \mathbf{R}$ and $(t,s) \in [a,b] \times [c,d]$ such that

$$\int_a^b \int_c^d f(x,y)\, d_y d_x g(x,y) \cong Af(t,s),$$

is equality for $f(x,y) = 1$, $f(x,y) = x$, and $f(x,y) = y$. That is,

$$\int_a^b \int_c^d 1 d_y d_x g(x,y) = A, \quad \int_a^b \int_c^d x d_y d_x g(x,y) = tA, \quad and \quad \int_a^b \int_c^d y d_y d_x g(x,y) = sA.$$

Solving these equations for A, t and s, we obtain *our double RS-midpoint rule*:

$$\int_a^b \int_c^d f(x,y)\, d_y d_x g(x,y) \cong [g(b,d) - g(a,d) - g(b,c) + g(a,c)]\, f(t,s)$$

where,

$$t=\frac{b\left[g\left(b,d\right)-g\left(b,c\right)\right]-a\left[g\left(a,d\right)-g\left(a,c\right)\right]-\int\limits_a^b\left[g\left(x,d\right)-g\left(x,c\right)\right]dx}{g\left(b,d\right)-g\left(a,d\right)-g\left(b,c\right)+g\left(a,c\right)},$$

and

$$s=\frac{d\left[g\left(b,d\right)-g\left(a,d\right)\right]-c\left[g\left(b,c\right)-g\left(a,c\right)\right]-\int\limits_c^d\left[g\left(b,y\right)-g\left(a,y\right)\right]dy}{g\left(b,d\right)-g\left(a,d\right)-g\left(b,c\right)+g\left(a,c\right)}.$$

Theorem 3.4.1. *Let $g:\boldsymbol{R}^2\to\boldsymbol{R}$ be continuous and increasing, let $(t,s)\in[a,b]\times[c,d]$ satisfy*

$$\int\limits_a^b\left[g\left(x,d\right)-g\left(x,c\right)\right]dx=\left(b-t\right)\left[g\left(b,d\right)-g\left(b,c\right)\right]+\left(t-a\right)\left[g\left(a,d\right)-g\left(a,c\right)\right],$$

and

$$\int\limits_c^d\left[g\left(b,y\right)-g\left(a,y\right)\right]dy=\left(d-s\right)\left[g\left(b,d\right)-g\left(a,d\right)\right]+\left(s-c\right)\left[g\left(b,c\right)-g\left(a,c\right)\right].$$

If $\frac{\partial^2 f}{\partial x\partial y}\geq 0$ and g has continuous second partial derivatives, then we have

$$\begin{aligned}
&\left[g\left(b,d\right)-g\left(a,d\right)-g\left(b,c\right)+g\left(a,c\right)\right]f\left(t,s\right)\\
\leq\;&\int\limits_a^b\int\limits_c^d f\left(x,y\right)d_yd_xg\left(x,y\right) \qquad (3.4.1)\\
\leq\;&\left[g\left(a,c\right)-\frac{1}{b-a}\int\limits_a^b g\left(x,c\right)dx-\frac{1}{d-c}\int\limits_c^d g\left(a,y\right)dy+G\right]f\left(a,c\right)\\
&-\left[g\left(a,d\right)-\frac{1}{b-a}\int\limits_a^b g\left(x,d\right)dx-\frac{1}{d-c}\int\limits_c^d g\left(a,y\right)dy+G\right]f\left(a,d\right)\\
&-\left[g\left(b,c\right)-\frac{1}{b-a}\int\limits_a^b g\left(x,c\right)dx-\frac{1}{d-c}\int\limits_c^d g\left(b,y\right)dy+G\right]f\left(b,c\right)\\
&+\left[g\left(b,d\right)-\frac{1}{b-a}\int\limits_a^b g\left(x,d\right)dx-\frac{1}{d-c}\int\limits_c^d g\left(b,y\right)dy+G\right]f\left(b,d\right).
\end{aligned}$$

where,

$$G=\frac{1}{\left(b-a\right)\left(d-c\right)}\int\limits_a^b\int\limits_c^d g\left(x,y\right)dydx.$$

Proof. We begin with the right hand inequality. Let

$$h(u,v) = g(u,v) - G_1(u) - G_2(v) + G,$$

where

$$G_1(u) = \int_c^d g(u,y)\,dy\,, \quad G_2(v) = \int_a^b g(x,v)\,dx,$$

so that

$$H(t,s) = \int_a^t \int_c^s h(u,v)\,dvdu$$

satisfies $H(a,c) = H(a,d) = H(b,c) = H(b,d) = 0$. Therefore,

$$\begin{aligned} I &= \int_a^b \int_c^d f(u,v)\,d_v d_u g(u,v) - \left.\left.[g(u,v) - G_1(u) - G_2(v) + G]\,f(u,v)\right|_a^b\right|_c^d \\ &= \int_a^b \int_c^d f(u,v)\,d_v d_u h(u,v) - \left.\left.h(u,v)\,f(u,v)\right|_a^b\right|_c^d, \end{aligned}$$

therefore using integration by parts twice, then using $H(a,c) = H(a,d) = H(b,c) = H(b,d) = 0$, we see that

$$I = \int_a^b \int_c^d H(u,v)\,\frac{\partial^2 f}{\partial u \partial v}(u,v)\,dvdu.$$

We claim that $H \leq 0$. Then by hypothesis $\frac{\partial^2 f}{\partial u \partial v}(u,v) \geq 0$ and so $I \leq 0$, which would prove the right-hand inequality.

To prove the claim, let $\tau := (\tau_1, \tau_2) \in \Delta$ be provided by the First Mean Value Theorem for Double integrals: $g(\tau_1, \tau_2) = G$, where τ is unique because g is increasing. For $(x,y) \in [a,\tau_1] \times [c,\tau_2]$ we have

$$H(x,y) = \int_a^x \int_c^y h(u,v)\,dvdu \leq 0,$$

since g is increasing. For $(x,y) \in [\tau_1, b] \times [\tau_2, d]$ we have

$$\begin{aligned}
H(x,y) &= \int_a^{\tau_1}\int_c^{\tau_2} h(u,v)\,dvdu + \int_{\tau_1}^{x}\int_{\tau_2}^{y} h(u,v)\,dvdu \\
&= -\int_{\tau_1}^{b}\int_c^{\tau_2} h(u,v)\,dvdu + \int_{\tau_1}^{x}\int_{\tau_2}^{y} h(u,v)\,dvdu \\
&= -\int_x^b \left[\int_c^{\tau_2} h(u,v)\,dv + \int_{\tau_2}^{y} h(u,v)\,dv\right] du \\
&= -\int_x^b \left[\int_c^{\tau_2} \left(g(u,v) - \int_c^d g(u,y)\,dy - \int_a^b g(x,v)\,dx + G\right) dv\right. \\
&\quad \left. + \int_{\tau_2}^{y} \left(g(u,v) - \int_c^d g(u,y)\,dy - \int_a^b g(x,v)\,dx + G\right) dv\right] du \\
&= -\int_c^{\tau_2} \left[\int_x^b \left(g(u,v) - \int_c^d g(u,y)\,dy - \int_a^b g(x,v)\,dx + G\right) du\right. \\
&\quad \left. - \int_{\tau_2}^{y}\int_x^b \left(g(u,v) - \int_c^d g(u,y)\,dy - \int_a^b g(x,v)\,dx + G\right) du\right] dv \\
&= \int_c^{\tau_2} \left[-\int_x^b g(u,v)\,du + \int_x^b\int_c^d g(u,y)\,dydu + \int_x^b\int_a^b g(x,v)\,dxdu - \int_x^b G du\right] dv \\
&\quad + \int_{\tau_2}^{y} \left[-\int_x^b g(u,v)\,du + \int_x^b\int_c^d g(u,y)\,dydu + \int_x^b\int_a^b g(x,v)\,dxdu - \int_x^b G du\right] dv \\
&= \int_c^{\tau_2} \left[-\int_x^b g(u,v)\,du + \int_x^b\int_c^d g(u,y)\,dydu + (b-x)\int_a^b g(x,v)\,dx - (b-x)\,G\right] dv \\
&\quad + \int_{\tau_2}^{y} \left[-\int_x^b g(u,v)\,du + \int_x^b\int_c^d g(u,y)\,dydu + (b-x)\int_a^b g(x,v)\,dx - (b-x)\,G\right] dv \\
&\leq \int_c^{\tau_2} [-(b-x)\,g(x,v) + (b-x)\,g(x,d) + (b-x)\,g(b,v) - (b-x)\,G]dv \\
&\quad + \int_{\tau_2}^{y} [-(b-x)\,g(x,v) + (b-x)\,g(x,d) + (b-x)\,g(b,v) - (b-x)\,G]dv
\end{aligned}$$

$$\begin{aligned}
&= -\int_{\tau_2}^{d} [-(b-x)\,g(x,v)+(b-x)\,g(x,d)+(b-x)\,g(b,v)-(b-x)\,G]dv \\
&+ \int_{\tau_2}^{y} [-(b-x)\,g(x,v)+(b-x)\,g(x,d)+(b-x)\,g(b,v)-(b-x)\,G]dv \\
&= -\int_{y}^{d} [-(b-x)\,g(x,v)+(b-x)\,g(x,d)+(b-x)\,g(b,v)-(b-x)\,G]dv \\
&= \int_{y}^{d} (b-x)\,g(x,v)dv-(b-x)\int_{y}^{d} g(x,d)dv \\
&\quad -(b-x)\int_{y}^{d} g(b,v)dv+(d-y)\,(b-x)\,G \\
&\leq (b-x)\,(d-y)\,g(x,y)-2\,(b-x)\,(d-y)\,g(b,d)+(d-y)\,(b-x)\,G \\
&= (b-x)\,(d-y)\cdot[g(x,y)-2g(b,d)+G] \\
&\leq 0,
\end{aligned}$$

so the claim is proved because g is increasing.

For the left-hand inequality, we begin instead with

$$h_c^d(x)=\begin{cases} g(x,d)-g(x,c)-g(a,d)+g(a,c)\,, & x\in[a,t] \\ g(x,d)-g(x,c)-g(b,d)+g(b,c)\,, & x\in(t,b] \end{cases}$$

Here again,

$$H_c^d(x)=\int_a^x h_c^d(u)\,du,$$

clearly $H_c^d(a)=0$. Now, we have

$$\begin{aligned}
H_c^d(b) &= \int_a^t [g(x,d)-g(x,c)-(g(a,d)-g(a,c))]\,dx \\
&\quad +\int_t^b [g(x,d)-g(x,c)-(g(b,d)-g(b,c))]\,dx \\
&= \int_a^b [g(x,d)-g(x,c)]\,dx-(b-t)\,[g(b,d)-g(b,c)] \\
&\quad -(t-a)\,[g(a,d)-g(a,c)]=0.
\end{aligned}$$

by our choice of t. Similarly, we have

$$h_a^b(y) = \begin{cases} g(b,y) - g(a,y) - g(b,c) + g(a,c), & y \in [c,s] \\ g(b,y) - g(a,y) - g(b,d) + g(a,d), & y \in (s,d] \end{cases}$$

Here again,

$$H_a^b(y) = \int_c^y h_a^b(v)\,dv,$$

clearly $H_c^d(c) = 0$. Now, we have

$$\begin{aligned} H_a^b(d) &= \int_c^s [g(b,y) - g(a,y) - (g(b,c) - g(a,c))]\,dx \\ &\quad + \int_s^d [g(b,y) - g(a,y) - (g(b,d) - g(a,d))]\,dx \\ &= \int_c^d [g(b,y) - g(a,y)]\,dy - (d-s)[g(b,d) - g(a,d)] \\ &\quad - (s-c)[g(b,c) - g(a,c)] = 0 \end{aligned}$$

by our choice of s.

Define $h(x,y) = h_c^d(x)\,h_a^b(y)$ on $[a,b] \times [c,d]$, and therefore

$$H(x,y) = \int_a^x \int_c^y H_c^d(u)\,H_a^b(v)\,dvdu.$$

We use integration by parts (twice), and $H(a,c) = H(a,d) = H(b,c) = H(b,d) = 0$, to obtain

$$\begin{aligned} &\int_a^b \int_c^d f(x,y)\,d_y d_x g(x,y) - [g(b,d) - g(a,d) - g(b,c) + g(a,c)]\,f(t,s) \\ = &\int_a^b \int_c^d H(x,y)\,\frac{\partial^2 f}{\partial x \partial y}(x,y)\,dydx. \end{aligned}$$

We claim that $H \geq 0$, and then by hypothesis $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ and so

$$\int_a^b \int_c^d f(x,y)\,d_y d_x g(x,y) - [g(b,d) - g(a,d) - g(b,c) + g(a,c)]\,f(t,s) \geq 0,$$

which would prove the left hand-side inequality.

To prove the claim, that is $H \geq 0$ iff $0 \leq H_c^d(x) H_a^b(y)$ which implies either $H_c^d \geq 0$ and $H_a^b \geq 0$, or $H_c^d \leq 0$ and $H_a^b \leq 0$.

We will show that $H_c^d \geq 0$ and $H_a^b \geq 0$ and hence the other case does not hold.

Let $x \in [a, t]$, since g is increasing, then

$$H_c^d(x) = \int_a^x [g(u,d) - g(u,c) - g(a,d) + g(a,c)]\, du.$$

For $x \in (z, b]$, we have

$$\begin{aligned}
H_c^d(x) &= \int_a^t [g(u,d) - g(u,c) - g(a,d) + g(a,c)]\, du \\
&\quad + \int_t^x [g(u,d) - g(u,c) - g(b,d) + g(b,c)]\, du \\
&= \int_a^x [g(u,d) - g(u,c)]\, du - (x-t)[g(b,d) - g(b,c)] \\
&\quad - (t-a)[g(a,d) - g(a,c)] \\
&= \int_a^b [g(u,d) - g(u,c)]\, du - (t-a)[g(a,d) - g(a,c)] \\
&\quad - \int_x^b [g(u,d) - g(u,c)]\, du - (x-t)[g(b,d) - g(b,c)] \\
&= (b-t)[g(b,d) - g(b,c)] - \int_x^b [g(u,d) - g(u,c)]\, du \\
&\quad - (x-t)[g(b,d) - g(b,c)]
\end{aligned}$$

by our choice of t, which gives

$$H_c^d(x) = (b-x)[g(b,d) - g(b,c)] - \int_x^b [g(u,d) - g(u,c)]\, du \geq 0,$$

again since g is increasing. Similarly one can prove that $H_a^b(y) \geq 0$, and therefore our second claim is proved, which completes the proof of the theorem. $\square$

The quadrature rule in Theorem 3.4.1 requires knowledge of G. In many applications this is not an obstacle, however if G is indeed unknown we may use instead its classical trapezoid rule approximation .

3.5 APPROXIMATIONS VIA BEESACK–DARST–POLLARD INEQUALITY

We start by establishing the Beesack–Darst–Pollard inequality in two real dimensional space $\mathbb{R}^2$.

Theorem 3.5.1. *Let $f, u : Q_{a,c}^{b,d} \to \mathbb{R}$ be such that f is of bounded bivariation on $Q_{a,c}^{b,d}$, u is continuous on $Q_{a,c}^{b,d}$ and $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ exists. Then, we have*

$$\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) \le A \cdot \inf_{(t,s)\in Q_{a,c}^{b,d}} f(t,s) + S\left(u; Q_{a,c}^{b,d}\right) \cdot \bigvee_{Q_{a,c}^{b,d}} (f) \tag{3.5.1}$$

where,

$$A := [u(b,d) - u(b,c) - u(a,d) + u(a,c)]$$

and

$$S\left(u; Q_{a,c}^{b,d}\right) := \sup_{\substack{a \le \alpha_1 < \beta_1 \le b \\ c \le \alpha_2 < \beta_2 \le d}} [u(\beta_1, \beta_2) - u(\beta_1, \alpha_2) - u(\alpha_1, \beta_2) + u(\alpha_1, \alpha_2)].$$

By replacing u with $(-u)$ in (3.5.1), we can also obtain the "dual" Beesack inequality

$$\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) \ge A \cdot \inf_{(t,s)\in Q_{a,c}^{b,d}} f(t,s) + s\left(u; Q_{a,c}^{b,d}\right) \cdot \bigvee_{Q_{a,c}^{b,d}} (f) \tag{3.5.2}$$

where,

$$s\left(u; Q_{a,c}^{b,d}\right) := \inf_{\substack{a \le \alpha_1 < \beta_1 \le b \\ c \le \alpha_2 < \beta_2 \le d}} [u(\beta_1, \beta_2) - u(\beta_1, \alpha_2) - u(\alpha_1, \beta_2) + u(\alpha_1, \alpha_2)].$$

Proof. We observe first that it is enough to prove the inequality in the case $\inf f = 0$, when it becomes

$$\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) \le S(u; Q) \cdot \bigvee_Q (f). \tag{3.5.3}$$

For the general case can be obtained from (3.5.3) by replacing h in it by $f - \inf f$. Clearly we may also suppose that for some (ξ_1, ξ_2), $f(\xi_1, \xi_2) = 0$. Since

$$\begin{aligned}\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) = \int_c^d \int_a^{\xi_1} f(t,s)\, d_s d_t u(t,s) \\ + \int_c^d \int_{-b}^{-\xi_1} f(-t,s)\, d_s d_t \left[-u(-t,s)\right] \\ + \int_c^{\xi_2} \int_a^b f(t,s)\, d_t d_s u(t,s) \\ + \int_{-d}^{-\xi_2} \int_a^b f(t,-s)\, d_t d_s \left[-u(t,-s)\right],\end{aligned}$$

and

$$\sup_{-b \le \beta_1 < \alpha_1 \le -\xi_1} \left\{\left[-u(-\beta_1, \cdot)\right] - \left[-u(-\alpha_1, \cdot)\right]\right\} = S\left(u; Q_{\xi_1, c}^{b,d}\right),$$

$$\sup_{-d \le \beta_2 < \alpha_2 \le -\xi_2} \left\{\left[-u(\cdot, -\beta_2)\right] - \left[-u(\cdot, \alpha_2)\right]\right\} = S\left(u; Q_{a, \xi_2}^{b,d}\right),$$

therefore, we need only to show that, if $f \ge 0$ and $f(b, \cdot) = 0$, then (3.5.3) holds.

To observe that let us assume that $u(a, \cdot) = 0 = u(\cdot, c)$. Define $\phi(t,s) := \inf_{\substack{a \le \xi_1 \le t \\ c \le \xi_2 \le s}} u(\xi_1, \xi_2)$ and

$$\psi(t,s) := u(t,s) - \phi(t,s) = \sup_{\substack{a \le \xi_1 \le t \\ c \le \xi_2 \le s}} \left\{u(t,s) - u(\xi_1, \xi_2)\right\} \le S\left(u, Q_{a,c}^{t,s}\right).$$

Then, ϕ is non-increasing, $\phi(t, \cdot) = 0 = \phi(\cdot, c)$, and $0 \le \psi(t,s) \le S\left(u, Q_{a,c}^{t,s}\right)$. Moreover, we have

$$\begin{aligned}\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) &= \int_c^d \int_a^b f(t,s)\, d_t d_s \phi(t,s) + \int_c^d \int_a^b f(t,s)\, d_t d_s \psi(t,s) \\ &\le 0 + \int_c^d \int_a^b f(t,s)\, d_t d_s \psi(t,s) \\ &= -\int_c^d \int_a^b \psi(t,s)\, d_t d_s f(t,s) \\ &\le \|\psi\|_\infty \cdot \bigvee(f) \\ &\le S\left(u, Q_{a,c}^{b,d}\right) \cdot \bigvee(f),\end{aligned}$$

which proves (3.5.1). Similarly, one can obtain (3.5.2) by replacing u with $(-u)$ in (3.5.1), and thus the proof is completely established. $\square$

As in one variable, a careful examination of the above proof, shows that the continuity of u was only used at two points of the proof: first, to justify the assumption that $f(\xi_1, \xi_2) = 0$ for some $(\xi_1, \xi_2) \in Q$ in the second reduction step of the proof; finally, to justify the existence of the integral $\int_c^d \int_a^b f(t,s)\, d_t d_s \psi(t,s)$ (since the continuity of ψ follows from the continuity of u). In the following we show that the bounds (3.5.1), (3.5.2) remain valid even if u is not continuous on Q, provided only that u is bounded on Q and $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ exists.

We observe first that when u is bounded with $u(a,\cdot) = 0 = u(\cdot, c)$, and if $\phi(t,s) := \inf\limits_{\substack{a \le \xi_1 \le t \\ c \le \xi_2 \le s}} u(\xi_1, \xi_2)$, for all $t \in [a,b]$ and $s \in [c,d]$, it follows that, ϕ is decreasing on Q. Now, it remains to observe that $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ exists. In order to complete the proof of our assertion, it suffices to rearrange the proof of Theorem 3.5.1 somewhat in order to avoid the necessity of assuming that f vanishes at some point of Q when $m = \inf f = 0$. As in the proof of Theorem 3.5.1, the general case of (3.5.1) follows from the case $m = 0$, so we are to prove that (3.5.3) holds, when $\inf f = 0$. Given an integer $n \ge 1$ there exists $(\xi_1^n, \xi_2^n) \in Q$ such that $f(\xi_1^n, \xi_2^n) = 0$. Writing

$$
\begin{aligned}
\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) = \int_c^d \int_a^{\xi_1^n} & f(t,s)\, d_s d_t u(t,s) \\
& + \int_c^d \int_{-b}^{-\xi_1^n} f(-t,s)\, d_s d_t \left[-u(-t,s)\right] \\
+ \int_c^{\xi_2^n} & \int_a^b f(t,s)\, d_t d_s u(t,s) \\
& + \int_{-d}^{-\xi_2^n} \int_a^b f(t,-s)\, d_t d_s \left[-u(t,-s)\right],
\end{aligned}
$$

and note that $f(t,s)$, $f(-t,s)$ and $f(t,-s)$ are nonnegative on their respective intervals of integration and

$$
\bigvee_c^d \bigvee_{-b}^{-\xi_1^n} (f(-t,s)) = \bigvee_c^d \bigvee_{\xi_1^n}^b (f(t,s))
$$

and

$$
\bigvee_{-d}^{-\xi_2^n} \bigvee_a^b (f(t,-s)) = \bigvee_{\xi_2^n}^d \bigvee_a^b (f(t,s)),
$$

also, $S\left(-u\left(-t,s\right),Q_{-b,c}^{-\xi_1^n,d}\right) = S\left(u\left(t,s\right),Q_{\xi_1^n,c}^{b,d}\right)$ and $S\left(-u\left(t,-s\right),Q_{a,-d}^{b,-\xi_2^n}\right) = S\left(u\left(t,s\right),Q_{a,\xi_2^n,c}^{b,d}\right)$.

Since $u\left(a,\cdot\right)=0=u\left(\cdot,c\right)$, then

$$\int_c^d\int_a^b f\left(t,s\right)d_td_su\left(t,s\right)=\int_c^d\int_a^b f\left(t,s\right)d_td_s\left(u\left(t,s\right)-u\left(a,s\right)-u\left(t,c\right)\right),$$

and defining $\phi\left(t,s\right):=\inf\limits_{\substack{a\le\xi_1\le t\\ c\le\xi_2\le s}}u\left(\xi_1,\xi_2\right)$ as above and $\psi\left(t,s\right):=u\left(t,s\right)-\phi\left(t,s\right)$, it follows that

$$\begin{aligned}\int_c^{\xi_2^n}\int_a^{\xi_1^n} f\left(t,s\right)d_td_su\left(t,s\right)&\le f\left(\xi_1^n,\xi_2^n\right)\psi\left(\xi_1^n,\xi_2^n\right)+S\left(u,Q_{a,c}^{\xi_1^n,\xi_2^n}\right)\cdot\bigvee_c^{\xi_2^n}\bigvee_a^{\xi_1^n}\left(f\right)\\&\le f\left(\xi_1^n,\xi_2^n\right)S\left(u,Q_{a,c}^{\xi_1^n,\xi_2^n}\right)+S\left(u,Q_{a,c}^{\xi_1^n,\xi_2^n}\right)\cdot\bigvee_c^{\xi_2^n}\bigvee_a^{\xi_1^n}\left(f\right)\end{aligned}$$

Proceeding in the same way on $Q_{-b,-d}^{\xi_1^n,\xi_2^n}:=\left[-b,-\xi_1^n\right]\times\left[-d,-\xi_2^n\right]$, we similarly obtain

$$\begin{aligned}\int_{-d}^{-\xi_2^n}\int_{-b}^{-\xi_1^n} f\left(t,s\right)d_td_su\left(t,s\right)&\le f\left(-b,-d\right)\psi\left(-b,-d\right)+S\left(u,Q_{-b,-d}^{-\xi_1^n,-\xi_2^n}\right)\cdot\bigvee_{-d}^{\xi_2^n}\bigvee_{-b}^{\xi_1^n}\left(f\right)\\&\le f\left(-\xi_1^n,-\xi_2^n\right)S\left(-u\left(-t,-s\right),Q_{-b,-d}^{-\xi_1^n,-\xi_2^n}\right)\\&\quad+S\left(-u\left(-t,-s\right),Q_{-b,-d}^{-\xi_1^n,-\xi_2^n}\right)\cdot\bigvee_c^{\xi_2^n}\bigvee_a^{\xi_1^n}\left(f\right)\\&=S\left(u\left(t,s\right),Q_{\xi_1^n,\xi_2^n}^{b,d}\right)\cdot\left[f\left(\xi_1^n,\xi_2^n\right)+\bigvee_{\xi_2^n}^{d}\bigvee_{\xi_1^n}^{b}\left(f\right)\right].\end{aligned}$$

It follows that for each $n\ge 1$,

$$\int_c^d\int_a^b f\left(t,s\right)d_td_su\left(t,s\right)\le S\left(u,Q_{a,c}^{b,d}\right)\cdot\left[2f\left(\xi_1^n,\xi_2^n\right)+\bigvee_c^d\bigvee_a^b\left(f\right)\right],$$

so that (3.5.1) follows on letting $n\to\infty$. Therefore, we just have proved the following fact:

Theorem 3.5.2. *Let $f,u:Q_{a,c}^{b,d}\to\mathbb{R}$ be such that f is of bounded bivariation on $Q_{a,c}^{b,d}$, u is bounded on $Q_{a,c}^{b,d}$ and $\int_c^d\int_a^b f\left(t,s\right)d_td_su\left(t,s\right)$ exists. Then, (3.5.1) and (3.5.2) hold.*

In the following, by use of the Beesack–Darst-Pollard inequalities (1.1) and (1.3), we provide other error bounds for the functionals $\omega\left(f,u,m,M;Q\right)$ and ..... .

3.6 APPLICATIONS TO QUADRATURE FORMULA

In this section, we apply some of the above obtained inequalities to give a sample of proposed quadrature rules for Riemann–Stieltjes integral. Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and $J_m : c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$, where $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, 1, \cdots, n-1$) and $\eta_j \in [y_j, y_{j+1}]$ ($j = 0, 1, \cdots, m-1$) are intermediate points. Consider the Riemann sum

$$R\left(f, I_n, J_m, \xi, \eta\right) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \left(x_{i+1} - x_i\right)\left(y_{j+1} - y_j\right) f\left(\xi_i, \eta_j\right) \tag{3.6.1}$$

Using Theorem 3.3.1, we can state the following theorem

Theorem 3.6.1. *Let f as in Theorem 3.3.1. Then we have*

$$\int_c^d \int_a^b f\left(t, s\right) dtds = R\left(f, I_n, J_m, \xi, \eta\right) + E\left(f, I_n, J_m, \xi, \eta\right), \tag{3.6.2}$$

where $R\left(f, I_n, J_m, \xi, \eta\right)$ is the Riemann sum defined in (3.6.1) and the remainder the through the approximation $E\left(f, I_n, J_m, \xi, \eta\right)$ satisfies the bound

$$\begin{aligned} &\left|E\left(f, I_n, J_m, \xi, \eta\right)\right| \\ &\le \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{x_{i+1} - x_i}{2} + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \cdot \left[\frac{y_{j+1} - y_j}{2} + \left|\eta_j - \frac{y_j + y_{j+1}}{2}\right|\right] \cdot \bigvee_{y_j}^{y_{j+1}} \bigvee_{x_i}^{x_{i+1}} (f). \end{aligned} \tag{3.6.3}$$

Proof. Applying Theorem 3.3.1 on the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get the required result. □

Similarly, we can give the following estimation for the Simpson's rule for mappings of bounded variation in two independent variables:

Theorem 3.6.2. *Let f as in Theorem 3.3.15. Then we have*

$$\int_c^d \int_a^b f\left(t, s\right) dtds = R_S\left(f, I_n, J_m, \xi, \eta\right) + E_S\left(f, I_n, J_m, \xi, \eta\right), \tag{3.6.4}$$

where $R_S(f, I_n, J_m, \xi, \eta)$ *is the Riemann sum defined such as*

$$\begin{aligned} &R_S(f, I_n, J_m, \xi, \eta) \\ &= \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{36} [f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j) \\ &\quad + 4f\left(\frac{x_i + x_{i+1}}{2}, y_{j+1}\right) + 4f\left(x_{i+1}, \frac{y_j + y_{j+1}}{2}\right) - 4f\left(\frac{x_i + x_{i+1}}{2}, y_j\right) \\ &\quad + 4f\left(x_i, \frac{y_j + y_{j+1}}{2}\right) + 16f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) - \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(t, s)\, dtds, \end{aligned}$$

and the remainder the through the approximation $E_S(f, I_n, J_m, \xi, \eta)$ *satisfies the bound*

$$|E_S(f, I_n, J_m, \xi, \eta)| \leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{9} \cdot \bigvee_{y_j}^{y_{j+1}} \bigvee_{x_i}^{x_{i+1}} (f) \qquad (3.6.5)$$

Proof. Applying Theorem 3.3.15 on the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get the required result. □

CHAPTER IV

ON AN OSTROWSKI TYPE FUNCTIONAL

4.1 INTRODUCTION

This chapter is devoted to introduce some functionals related with the Ostrowski integral inequality for mappings of two variables and therefore several representations of the errors are established. Therefore, inequalities of Trapezoid and Ostrowski type are discussed. Finally, as application, a cubature formula is given.

4.2 A FUNCTIONAL RELATED TO THE OSTROWSKI INEQUALITY

Theorem 4.2.1. *Let $f, g : Q \to \mathbb{R}$ be such that f is (β_1, β_2)–Hölder type mapping, where $H_1, H_2 > 0$ and $\beta_1, \beta_2 > 0$ are given, and g is a mapping of bounded bivariation on Q. Then we have the inequality*

$$\left|\int_c^d \int_a^b f(x,y)\, d_x d_y g(x,y) - \left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right] \cdot \int_c^d \int_a^b f(t,s)\, dt ds\right| \\ \leq \left[H_1 \frac{(b-a)^{\beta_1}}{2^{\beta_1+1}(\beta_1+1)} + H_2 \frac{(d-c)^{\beta_2}}{2^{\beta_2+1}(\beta_2+1)}\right] \cdot \bigvee_{Q_{c,d}^{a,b}} (g), \quad (4.2.1)$$

where $\bigvee_Q (g)$ denotes the total bivariation of g on Q.

Proof. As g is of bounded bivariation on Q, by Lemma 3.2.3 we have

$$\begin{aligned}
&\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|\\
&=\left|\int_c^d\int_a^b\left[f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right]d_xd_yg(x,y)\right|\\
&\leq\sup_{(x,y)\in Q_{c,d}^{a,b}}\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right|\cdot\bigvee_{Q_{c,d}^{a,b}}(g)\\
&\leq\frac{1}{(b-a)(d-c)}\sup_{(x,y)\in Q_{c,d}^{a,b}}\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|\cdot\bigvee_{Q_{c,d}^{a,b}}(g)\,. \qquad (4.2.2)
\end{aligned}$$

Now, as f is of (β_1,β_2)–Hölder type mapping, then we have

$$\begin{aligned}
&\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|\\
&\leq\int_c^d\int_a^b|f(x,y)-f(t,s)|\,dtds\\
&\leq\int_c^d\int_a^b\Big(H_1|x-t|^{\beta_1}+H_2|y-s|^{\beta_2}\Big)dtds\\
&=H_1(d-c)\int_a^b|x-t|^{\beta_1}dt+H_2(b-a)\int_c^d|y-s|^{\beta_2}\,ds\\
&=H_1(d-c)\,\frac{(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}}{\beta_1+1}+H_2(b-a)\,\frac{(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}}{\beta_2+1}
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\sup_{(x,y)\in Q_{c,d}^{a,b}}\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|\\
&\leq\sup_{(x,y)\in Q_{c,d}^{a,b}}\left[H_1(d-c)\,\frac{(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}}{\beta_1+1}+H_2(b-a)\,\frac{(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}}{\beta_2+1}\right]\\
&\leq H_1(d-c)\sup_{(x,y)\in Q_{c,d}^{a,b}}\left[\frac{(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}}{\beta_1+1}\right]\\
&\qquad+H_2(b-a)\sup_{(x,y)\in Q_{c,d}^{a,b}}\left[\frac{(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}}{\beta_2+1}\right]\\
&=H_1(d-c)\,\frac{(b-a)^{\beta_1+1}}{2^{\beta_1+1}(\beta_1+1)}+H_2(b-a)\,\frac{(d-c)^{\beta_2+1}}{2^{\beta_2+1}(\beta_2+1)}.
\end{aligned}$$

Using (5.4.2), we get

$$\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|$$
$$\leq\frac{1}{(b-a)(d-c)}\sup_{(x,y)\in Q_{c,d}^{a,b}}\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|\cdot\bigvee_{Q_{c,d}^{a,b}}(g)$$
$$\leq\left[H_1\frac{(b-a)^{\beta_1}}{2^{\beta_1+1}(\beta_1+1)}+H_2\frac{(d-c)^{\beta_2}}{2^{\beta_2+1}(\beta_2+1)}\right]\cdot\bigvee_{Q_{c,d}^{a,b}}(g),$$

as required. □

Theorem 4.2.2. *Let $f,g:Q\to\mathbb{R}$ be such that f f is (β_1,β_2)–Hölder type mapping, where $H_1,H_2>0$ and $\beta_1,\beta_2>0$ are given, and g is bimonotonic nondecreasing on Q. Then we have the inequality*

$$\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|$$
$$\leq\left[\frac{H_1(b-a)^{\beta_1}}{(\beta_1+1)}+\frac{H_2(d-c)^{\beta_2}}{(\beta_2+1)}\right]\cdot[g(b,d)-g(b,c)-g(a,d)+g(a,c)]. \quad (4.2.3)$$

Proof. As g is bimonotonic nondecreasing on Q, by Lemma 3.2.4 we have

$$\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|$$
$$=\left|\int_c^d\int_a^b\left[f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right]d_xd_yg(x,y)\right|$$
$$\leq\int_c^d\int_a^b\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right|d_xd_yg(x,y)$$
$$=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|d_xd_yg(x,y) \quad (4.2.4)$$

Now, as f is of (β_1,β_2)–Hölder type mapping, then we have

$$\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|$$
$$\leq H_1(d-c)\frac{(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}}{\beta_1+1}+H_2(b-a)\frac{(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}}{\beta_2+1}$$

as we shown in Theorem 4.2.1. Therefore,

$$\begin{aligned}
&\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|\\
&\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\left|\int_c^d\int_a^b[f(x,y)-f(t,s)]\,dtds\right|d_xd_yg(x,y)\\
&\le\frac{H_1}{(\beta_1+1)(b-a)}\int_c^d\int_a^b\left[(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}\right]d_xd_yg(x,y)\\
&\qquad+\frac{H_2}{(\beta_2+1)(d-c)}\int_c^d\int_a^b\left[(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}\right]d_xd_yg(x,y).
\end{aligned}$$

Using Riemann–Stieltjes double integral, we may deduce that

$$\begin{aligned}
&\int_c^d\int_a^b\left[(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}\right]d_xd_yg(x,y)\\
&=(b-a)^{\beta_1+1}[g(b,d)-g(b,c)-g(a,d)+g(a,c)]\\
&\qquad-\int_c^d\int_a^b g(x,y)\,d\left[(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}\right]dy\\
&=(b-a)^{\beta_1+1}[g(b,d)-g(b,c)-g(a,d)+g(a,c)]\\
&\qquad-(\beta_1+1)\int_c^d\int_a^b(x-a)^{\beta_1}g(x,y)\,dxdy\\
&\qquad+(\beta_1+1)\int_c^d\int_a^b(b-x)^{\beta_1}g(x,y)\,dxdy
\end{aligned}\tag{4.2.5}$$

and

$$\begin{aligned}
&\int_c^d\int_a^b\left[(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}\right]d_xd_yg(x,y)\\
&=(d-c)^{\beta_2+1}[g(b,d)-g(b,c)-g(a,d)+g(a,c)]\\
&\qquad-\int_c^d\int_a^b g(x,y)\,dxd\left[(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}\right]\\
&=(d-c)^{\beta_2+1}[g(b,d)-g(b,c)-g(a,d)+g(a,c)]\\
&\qquad-(\beta_2+1)\int_c^d\int_a^b(y-c)^{\beta_2}g(x,y)\,dxdy\\
&\qquad+(\beta_2+1)\int_c^d\int_a^b(d-y)^{\beta_2}g(x,y)\,dxdy
\end{aligned}\tag{4.2.6}$$

Now, on utilizing the bimonotonicity property of g on Q, we have

$$\begin{aligned}
\int_c^d\int_a^b(x-a)^{\beta_1}g(x,y)\,dxdy&\ge\left(\int_c^d g(a,y)\,dy\right)\left(\int_a^b(x-a)^{\beta_1}dx\right)\\
&\ge(d-c)\,g(a,c)\,\frac{(b-a)^{\beta_1+1}}{\beta_1+1},
\end{aligned}$$

$$\int_c^d \int_a^b (b-x)^{\beta_1} g(x,y)\,dxdy \leq \left(\int_c^d g(b,y)\,dy\right)\left(\int_a^b (b-x)^{\beta_1}\,dx\right)$$
$$\leq (d-c)\,g(b,d)\,\frac{(b-a)^{\beta_1+1}}{\beta_1+1},$$

$$\int_c^d \int_a^b (y-c)^{\beta_2} g(x,y)\,dxdy \geq \left(\int_c^d g(x,c)\,dy\right)\left(\int_a^b (y-c)^{\beta_2}\,dx\right)$$
$$\geq (b-a)\,g(a,c)\,\frac{(d-c)^{\beta_2+1}}{\beta_2+1},$$

and

$$\int_c^d \int_a^b (d-y)^{\beta_2} g(x,y)\,dxdy \leq \left(\int_c^d g(x,d)\,dy\right)\left(\int_a^b (d-y)^{\beta_2}\,dx\right)$$
$$\leq (b-a)\,g(b,d)\,\frac{(d-c)^{\beta_2+1}}{\beta_2+1}.$$

Substituting in (4.2.5) and (4.2.6), we get

$$\begin{aligned}
&\int_c^d \int_a^b \left[(x-a)^{\beta_1+1} + (b-x)^{\beta_1+1}\right] d_x d_y g(x,y) \\
&= (b-a)^{\beta_1+1}\left[g(b,d) - g(b,c) - g(a,d) + g(a,c)\right] \\
&\qquad - (\beta_1+1)\int_c^d \int_a^b (x-a)^{\beta_1} g(x,y)\,dxdy \\
&\qquad + (\beta_1+1)\int_c^d \int_a^b (b-x)^{\beta_1} g(x,y)\,dxdy \\
&\leq (b-a)^{\beta_1+1}\left[g(b,d) - g(b,c) - g(a,d) + g(a,c)\right]
\end{aligned} \tag{4.2.7}$$

and

$$\begin{aligned}
&\int_c^d \int_a^b \left[(y-c)^{\beta_2+1} + (d-y)^{\beta_2+1}\right] d_x d_y g(x,y) \\
&= (d-c)^{\beta_2+1}\left[g(b,d) - g(b,c) - g(a,d) + g(a,c)\right] \\
&\qquad - (\beta_2+1)\int_c^d \int_a^b (y-c)^{\beta_2} g(x,y)\,dxdy \\
&\qquad + (\beta_2+1)\int_c^d \int_a^b (d-y)^{\beta_2} g(x,y)\,dxdy \\
&\leq (d-c)^{\beta_2+1}\left[g(b,d) - g(b,c) - g(a,d) + g(a,c)\right]
\end{aligned} \tag{4.2.8}$$

which gives that

$$\begin{aligned}
&\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|\\
&\le\frac{H_1}{(\beta_1+1)(b-a)}\int_c^d\int_a^b\left[(x-a)^{\beta_1+1}+(b-x)^{\beta_1+1}\right]d_xd_yg(x,y)\\
&\qquad+\frac{H_2}{(\beta_2+1)(d-c)}\int_c^d\int_a^b\left[(y-c)^{\beta_2+1}+(d-y)^{\beta_2+1}\right]d_xd_yg(x,y)\\
&\le\left[\frac{H_1(b-a)^{\beta_1}}{(\beta_1+1)}+\frac{H_2(d-c)^{\beta_2}}{(\beta_2+1)}\right]\cdot[g(b,d)-g(b,c)-g(a,d)+g(a,c)],
\end{aligned}$$

which completes the proof. □

Theorem 4.2.3. *Let $f,g:Q\to\mathbb{R}$ be such that f is continuous on Q and f,g are of bounded bivariation on Q. Then we have the inequality*

$$\begin{aligned}
&\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|\\
&\qquad\le\bigvee_Q(f)\cdot\bigvee_Q(g), \qquad (4.2.9)
\end{aligned}$$

where $\bigvee_Q(f)$ denotes the total bivariation of f on Q.

Proof. As g is of bounded bivariation on Q, by Lemma 3.2.3 we have

$$\begin{aligned}
&\left|\int_c^d\int_a^b f(x,y)\,d_xd_yg(x,y)-\left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot\int_c^d\int_a^b f(t,s)\,dtds\right|\\
&=\left|\int_c^d\int_a^b\left[f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right]d_xd_yg(x,y)\right|\\
&\le\sup_{(x,y)\in Q}\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(t,s)\,dtds\right|\cdot\bigvee_Q(g)\\
&\le\frac{\bigvee_Q(f)\cdot\bigvee_Q(g)}{(b-a)(d-c)}\sup_{(x,y)\in Q}\left|\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\right|\\
&\le\bigvee_Q(f)\cdot\bigvee_Q(g).
\end{aligned}$$

Since

$$\sup_{(x,y)\in Q}\left|\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot\left[\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right]\right|=(b-a)(d-c)$$

which completes the proof. □

4.3 INTEGRAL REPRESENTATION OF ERROR

For a function $g : Q_{a,c}^{b,d} \to \mathbb{R}$, we define $\phi_g, \psi_g : Q_{a,c}^{b,d} \to \mathbb{R}$ by

$$\begin{aligned}\phi_g(t,s) :&= (t-a)\left[(s-c)\,g(a,c) + (d-s)\,g(a,d)\right] \\ &\quad + (b-t)\left[(d-s)\,g(b,d) + (s-c)\,g(b,c)\right]\end{aligned}$$

and

$$\psi_g(t,s) := g(t,s) - \frac{\phi_g(t,s)}{(b-a)(d-c)}. \tag{4.3.1}$$

We can state the following result

Theorem 4.3.1. *If $f, u : Q_{a,c}^{b,d} \to \mathbb{R}$ are bounded on $Q_{a,c}^{b,d}$ and such that the Riemann–Stieltjes double integral $\int_c^d \int_a^b f(t,s)\,d_t d_s u(t,s)$ and the Riemann double integral $\int_c^d \int_a^b u(t,s)\,dtds$ exist, then*

$$\begin{aligned}&\int_c^d \int_a^b \psi_f(t,s)\,d_t d_s u(t,s) \\ &= \left[\frac{f(a,c) - f(a,d) - f(b,c) + f(b,d)}{(b-a)(d-c)}\right] \cdot \int_c^d \int_a^b u(t,s)\,dtds - \int_c^d \int_a^b u(t,s)\,d_t d_s f(t,s)\end{aligned} \tag{4.3.2}$$

Proof. By assumptions, we have

$$\begin{aligned}&\int_c^d \int_a^b \psi_f(t,s)\,d_t d_s u(t,s) \\ &\qquad = \int_c^d \int_a^b f(t,s)\,d_t d_s u(t,s) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \phi_f(t,s)\,d_t d_s u(t,s).\end{aligned}$$

Integrating by parts in the Riemann–Stieltjes double integral (see Lemma 3.2.1), we also have

$$\begin{aligned}&\int_c^d \int_a^b \phi_f(t,s)\,d_t d_s u(t,s) \\ &= \phi_f(b,d)\,u(b,d) - \phi_f(b,c)\,u(b,c) - \phi_f(a,d)\,u(a,d) + \phi_f(a,c)\,u(a,c) \\ &\qquad - \int_c^d \int_a^b u(t,s)\,d_t d_s \phi_f(t,s) \\ &= (b-a)(d-c)\left[f(a,c)\,u(b,d) - f(a,d)\,u(b,c) - f(b,c)\,u(a,d) + f(b,d)\,u(a,c)\right] \\ &\qquad - \left[f(a,c) - f(a,d) - f(b,c) + f(b,d)\right] \cdot \int_c^d \int_a^b u(t,s)\,dtds,\end{aligned}$$

which gives that

$$\begin{aligned}
&\int_c^d \int_a^b \psi_f(t,s)\, d_t d_s u(t,s) \\
&= \int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s) + \left[\frac{f(a,c)-f(a,d)-f(b,c)+f(b,d)}{(b-a)(d-c)}\right] \cdot \int_c^d \int_a^b u(t,s)\, dt ds \\
&\qquad - \left[f(a,c)u(b,d) - f(a,d)u(b,c) - f(b,c)u(a,d) + f(b,d)u(a,c)\right] \\
&= \left[\frac{f(a,c)-f(a,d)-f(b,c)+f(b,d)}{(b-a)(d-c)}\right] \cdot \int_c^d \int_a^b u(t,s)\, dt ds - \int_c^d \int_a^b u(t,s)\, d_t d_s f(t,s)
\end{aligned}$$

which completes the proof. □

Theorem 4.3.2. *Let f, u, ψ_f as above. If f is continuous on $Q_{a,c}^{b,d}$ and u is of bounded bivariation on $Q_{a,c}^{b,d}$. Then we have the inequality*

$$\begin{aligned}
&\left|\int_c^d \int_a^b \psi_f(t,s)\, d_t d_s u(t,s)\right| \\
&\qquad\qquad \le \max\{|f(b,d)-f(a,c)|, |f(b,c)-f(a,d)|, \\
&\qquad\qquad\qquad |f(a,d)-f(b,d)|, |f(a,c)-f(b,c)|\} \cdot \bigvee_{Q_{a,c}^{b,d}}(u), \qquad (4.3.3)
\end{aligned}$$

where $\bigvee_{Q_{a,c}^{b,d}}(u)$ denotes the total bivariation of u on $Q_{a,c}^{b,d}$.

Proof. As u is of bounded bivariation on $Q_{a,c}^{b,d}$, we have

$$\left|\int_c^d \int_a^b \psi_f(t,s)\, d_t d_s u(t,s)\right| \le \sup_{(t,s)\in Q_{a,c}^{b,d}} |\psi_f(t,s)| \cdot \bigvee_{Q_{a,c}^{b,d}}(u)$$

But

$$\begin{aligned}
&\sup_{(t,s)\in Q_{a,c}^{b,d}} |\psi_f(t,s)| \\
&= \max\{|\psi_f(b,d)|, |\psi_f(b,c)|, |\psi_f(a,d)|, |\psi_f(a,c)|\} \\
&= \max\{|f(b,d)-f(a,c)|, |f(b,c)-f(a,d)|, |f(a,d)-f(b,d)|, |f(a,c)-f(b,c)|\}
\end{aligned}$$

which gives

$$\begin{aligned}
\left|\int_c^d \int_a^b \psi_f(t,s)\, d_t d_s u(t,s)\right| &\le \bigvee_{Q_{a,c}^{b,d}}(u) \times \max\{|f(b,d)-f(a,c)|, |f(b,c)-f(a,d)|, \\
&\qquad |f(a,d)-f(b,d)|, |f(a,c)-f(b,c)|\}
\end{aligned}$$

as required. □

Remark 4.3.3. *With assumptions of Theorem 4.3.2, if we assume in addition*

1. f *is increasing on* $Q_{a,c}^{b,d}$*, then we have*

$$\left|\int_c^d \int_a^b \psi_f (t,s)\, d_t d_s u(t,s)\right| \le \max\left\{\left|f(b,d) - f(a,c)\right|, \left|f(a,c) - f(b,c)\right|\right\} \cdot \bigvee_{Q_{a,c}^{b,d}} (u), \quad (4.3.4)$$

2. f *is decreasing on* $Q_{a,c}^{b,d}$*, then we have*

$$\left|\int_c^d \int_a^b \psi_f (t,s)\, d_t d_s u(t,s)\right| \le \max\left\{\left|f(b,c) - f(a,d)\right|, \left|f(a,d) - f(b,d)\right|\right\} \cdot \bigvee_{Q_{a,c}^{b,d}} (u), \quad (4.3.5)$$

The following result holds:

Theorem 4.3.4. *Let* $f, u : Q \to \mathbb{R}$ *be such that* f *is* (β_1, β_2)*–Hölder type mapping, where* $H_1, H_2 > 0$ *and* $\beta_1, \beta_2 > 0$ *are given, and* u *is a mapping of bounded bivariation on* Q*. Then we have the inequality*

$$\left|\int_c^d \int_a^b \psi_f (t,s)\, d_t d_s u(t,s)\right| \le \left[H_1 \frac{(b-a)^{\beta_1}}{\beta_1 + 1} + H_2 \frac{(d-c)^{\beta_2}}{\beta_2 + 1}\right] \cdot \bigvee_{Q_{c,d}^{a,b}} (u), \quad (4.3.6)$$

where $\bigvee_Q (u)$ *denotes the total bivariation of* u *on* Q*.*

Proof. The proof may be done by applying Theorems 4.2.1 and 4.3.1 directly. □

Now, if we denote the error of approximating the Riemann-Stieltjes double integral $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ by the representation of error $\psi_f(t,s)$ by $E\left(f, u; Q_{a,c}^{b,d}\right)$, which is given as follows:

$$
\begin{aligned}
&E\left(f,u;Q_{a,c}^{b,d}\right) \\
&=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b (t-a)(s-c)\left[f(t,s)-f(t,c)-f(a,s)+f(a,c)\right]d_t d_s u(t,s) \\
&+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b (t-a)(s-d)\left[f(t,d)-f(t,s)-f(a,d)+f(a,s)\right]d_t d_s u(t,s) \\
&+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b (t-b)(s-c)\left[f(b,s)-f(b,c)-f(t,s)+f(t,c)\right]d_t d_s u(t,s) \\
&+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b (t-b)(s-d)\left[f(b,d)-f(b,s)-f(t,d)+f(t,s)\right]d_t d_s u(t,s),
\end{aligned}
$$

then we can state the following result.

Corollary 4.3.5. *With the assumptions of Theorem 4.3.2, we have*

$$
\begin{aligned}
&E\left(f,u;Q_{a,c}^{b,d}\right) \\
&\quad=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\left(\int_c^d\int_a^b T(t,r_1,s,r_2)\,d_{r_1}d_{r_2}f(r_1,r_2)\right)d_t d_s u(t,s) \\
&\quad\quad=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\left(\int_c^d\int_a^b T(t,r_1,s,r_2)\,d_t d_s u(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)
\end{aligned}
\tag{4.3.7}
$$

where,

$$
T(t,r_1,s,r_2)=\begin{cases}
(t-a)(s-c), & a\le r_1\le t\le b,\ \ c\le r_2\le s\le d \\
(t-a)(s-d), & a\le r_1\le t\le b,\ \ c\le s\le r_2\le d \\
(t-b)(s-c), & a\le t\le r_1\le b,\ \ c\le r_2\le s\le d \\
(t-b)(s-d), & a\le t\le r_1\le b,\ \ c\le s\le r_2\le d
\end{cases}
$$

Proof. If f is bounded on $Q_{a,c}^{b,d}$, then for any $(t,s)\in Q_{a,c}^{b,d}$ the Riemann–Stieltjes double integrals $\int_c^s\int_a^t d_{r_1}d_{r_2}f(r_1,r_2)$, $\int_s^d\int_a^t d_{r_1}d_{r_2}f(r_1,r_2)$, $\int_c^s\int_t^b d_{r_1}d_{r_2}f(r_1,r_2)$ and $\int_s^d\int_t^b d_{r_1}d_{r_2}f(r_1,r_2)$ are exist and

$$
\int_c^s\int_a^t d_{r_1}d_{r_2}f(r_1,r_2)=f(t,s)-f(t,c)-f(a,s)+f(a,c),
$$

$$
\int_s^d\int_a^t d_{r_1}d_{r_2}f(r_1,r_2)=f(t,d)-f(t,s)-f(a,d)+f(a,s),
$$

$$\int_c^s \int_t^b d_{r_1} d_{r_2} f(r_1, r_2) = f(b, s) - f(b, c) - f(t, s) + f(t, c),$$

and

$$\int_s^d \int_t^b d_{r_1} d_{r_2} f(r_1, r_2) = f(b, d) - f(b, s) - f(t, d) + f(t, s).$$

Therefore,

$$\begin{aligned}\int_c^d \int_a^b T(r_1, r_2)\, d_{r_1} d_{r_2} f(r_1, r_2) = (t-a)(s-c) \int_c^s \int_a^t d_{r_1} d_{r_2} f(r_1, r_2) \\ + (t-a)(d-s) \int_s^d \int_a^t d_{r_1} d_{r_2} f(r_1, r_2) \\ + (b-t)(s-c) \int_c^s \int_t^b d_{r_1} d_{r_2} f(r_1, r_2) \\ + (b-t)(d-s) \int_s^d \int_t^b d_{r_1} d_{r_2} f(r_1, r_2) \\ = (b-a)(d-c)\, \psi_f(t, s),\end{aligned}$$

and by (4.3.2) we deduce the first and the second equalities in (4.3.4). The last part follows by the Fubini–type theorem for the Riemann–Stieltjes double integral. The details are omitted □

Remark 4.3.6. *One can obtain another error approximation for the Riemann–Stieltjes double integral defined above by considering the dual error of* $E\left(f, u; Q_{a,c}^{b,d}\right)$ *i.e., define*

$$\begin{aligned}&F\left(f, u; Q_{a,c}^{b,d}\right) \\ &= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b (t-a)(s-c)\left[u(t, s) - u(t, c) - u(a, s) + u(a, c)\right] d_t d_s f(t, s) \\ &+ \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b (t-a)(d-s)\left[u(t, d) - u(t, s) - u(a, d) + u(a, s)\right] d_t d_s f(t, s) \\ &+ \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b (b-t)(s-c)\left[u(b, s) - u(b, c) - u(t, s) + u(t, c)\right] d_t d_s f(t, s) \\ &+ \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b (b-t)(d-s)\left[u(b, d) - u(b, s) - u(t, d) + u(t, s)\right] d_t d_s f(t, s).\end{aligned}$$

Therefore, with the assumptions of Theorem 4.3.2, we have

$$\begin{aligned}&F\left(f, u; Q_{a,c}^{b,d}\right) \\ &= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \left(\int_c^d \int_a^b T(t, r_1, s, r_2)\, d_{r_1} d_{r_2} u(r_1, r_2)\right) d_t d_s f(t, s) \\ &= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \left(\int_c^d \int_a^b T(t, r_1, s, r_2)\, d_t d_s f(t, s)\right) d_{r_1} d_{r_2} u(r_1, r_2)\end{aligned} \tag{4.3.8}$$

where, $T(t, r_1, s, r_2)$ *is defined above.*

4.4 ERROR BOUNDS

The following result may be stated:

Corollary 4.4.1. *Assume that* $f, u : [a, b] \to \mathbb{R}$ *are bounded.*

1. *If* u *(respectively* f*) is of bounded bivariation and* f *(respectively* u*) is continuous, then*

$$|E(f, u; Q)|\,(|F(f, u; Q)|) \leq \sup_{(t,s)\in Q} |\psi_f(t, s)| \cdot \bigvee_Q (u) \left(\sup_{(t,s)\in Q} |\psi_u(t, s)| \cdot \bigvee_Q (f) \right) \quad (4.4.1)$$

2. *If* u *(respectively* f*) is bimonotonic nondecreasing and* f *(respectively* u*) is Riemann integrable on* Q*, then*

$$|E(f, u; Q)|\,(|F(f, u; Q)|) \leq \int_c^d \int_a^b |\psi_f(t, s)|\, d_t d_s u(t, s) \left(\int_c^d \int_a^b |\psi_u(t, s)|\, d_t d_s f(t, s) \right) \quad (4.4.2)$$

Proof. The proof follows, using Lemmas 3.2.3 and 3.2.4; respectively. □

Theorem 4.4.2. *Assume that* $f, u : [a, b] \to \mathbb{R}$ *are bounded on* $[a, b]$ *and the Riemann–Stieltjes (double) integral* $\int_c^d \int_a^b f(t, s)\, d_t d_s u(t, s)$ *exists. If* $-\infty < m_1 \leq f(t, \cdot) \leq M_1 < \infty$*, for all* $t \in [a, b]$ *and* $-\infty < m_2 \leq f(\cdot, s) \leq M_2 < \infty$*, for all*

$s \in [c,d]$, and u is bimonotonic nondecreasing on Q. Then, we have

$$\begin{aligned} & -\left[f\left(b,d\right)-f\left(a,c\right)\right]\cdot\left[u\left(b,d\right)-u\left(a,c\right)\right] \\ & \qquad -\left[M_1-f\left(a,d\right)+M_2-m\right]\cdot\left[u\left(b,d\right)-u\left(b,c\right)\right] \\ & \qquad -\left[M_2-f\left(b,c\right)+M_1-m\right]\cdot\left[u\left(b,d\right)-u\left(a,d\right)\right] \\ & \leq \mathcal{E}\left(f,u;Q\right) \\ & \leq \left[f\left(b,d\right)-f\left(a,c\right)\right]\cdot\left[u\left(b,d\right)-u\left(a,c\right)\right] \\ & \qquad +\left[m_1-f\left(a,d\right)+m_2-M\right]\cdot\left[u\left(b,c\right)-u\left(a,c\right)\right] \\ & \qquad +\left[m_2-f\left(b,c\right)+m_1-M\right]\cdot\left[u\left(a,d\right)-u\left(a,c\right)\right]. \end{aligned} \tag{4.4.3}$$

where,

$$M := \max\{M_1, M_2\}, \quad and \quad m := \min\{m_1, m_2\}.$$

Proof. From the condition $-\infty < m_1 \leq f(t,\cdot) \leq M_1 < \infty$, for all $t \in [a,b]$ and $-\infty < m_2 \leq f(\cdot,s) \leq M_2 < \infty$, for all $s \in [c,d]$. Setting

$$M := \max\{M_1, M_2\}, \quad and \quad m := \min\{m_1, m_2\},$$

then we have $m \leq f(t,s) \leq M$. Also, we may state that

$$\begin{aligned} m-f\left(t,c\right)-f\left(a,s\right)+f\left(a,c\right) &\leq f\left(t,s\right)-f\left(t,c\right)-f\left(a,s\right)+f\left(a,c\right) \\ &\leq M-f\left(t,c\right)-f\left(a,s\right)+f\left(a,c\right), \end{aligned} \tag{4.4.4}$$

$$\begin{aligned} f\left(t,d\right)-f\left(a,d\right)+f\left(a,s\right)-M &\leq f\left(t,d\right)-f\left(t,s\right)-f\left(a,d\right)+f\left(a,s\right) \\ &\leq f\left(t,d\right)-f\left(a,d\right)+f\left(a,s\right)-m, \end{aligned} \tag{4.4.5}$$

$$\begin{aligned} f\left(b,s\right)-f\left(b,c\right)+f\left(t,c\right)-M &\leq f\left(b,s\right)-f\left(b,c\right)-f\left(t,s\right)+f\left(t,c\right) \\ &\leq f\left(b,s\right)-f\left(b,c\right)+f\left(t,c\right)-m, \end{aligned} \tag{4.4.6}$$

and

$$\begin{aligned} f\left(b,d\right)-f\left(b,s\right)-f\left(t,d\right)+m &\leq f\left(b,d\right)-f\left(b,s\right)-f\left(t,d\right)+f\left(t,s\right) \\ &\leq f\left(b,d\right)-f\left(b,s\right)-f\left(t,d\right)+M \end{aligned} \tag{4.4.7}$$

Therefore, by assumptions the above inequalities become respectively; as follow:

$$\begin{aligned} m-M_1-M_2+f(a,c) &\leq f(t,s)-f(t,c)-f(a,s)+f(a,c) \\ &\leq M-m_1-m_2+f(a,c), \end{aligned} \tag{4.4.8}$$

$$\begin{aligned} m_1-f(a,d)+m_2-M &\leq f(t,d)-f(t,s)-f(a,d)+f(a,s) \\ &\leq M_1-f(a,d)+M_2-m, \end{aligned} \tag{4.4.9}$$

$$\begin{aligned} m_2-f(b,c)+m_1-M &\leq f(b,s)-f(b,c)-f(t,s)+f(t,c) \\ &\leq M_2-f(b,c)+M_1-m, \end{aligned} \tag{4.4.10}$$

and

$$\begin{aligned} f(b,d)-M_2-M_1+m &\leq f(b,d)-f(b,s)-f(t,d)+f(t,s) \\ &\leq f(b,d)-m_2-m_1+M \end{aligned} \tag{4.4.11}$$

If we multiply (4.4.8) by $(t-a)(s-c) \geq 0$, (4.4.9) by $(t-a)(s-d) \leq 0$, (4.4.10) by $(t-b)(s-c) \leq 0$, and (4.4.11) by $(t-b)(s-d) \geq 0$, we obtain

$$\begin{aligned} &(t-a)(s-c)[m-M_1-M_2+f(a,c)] \\ &\leq (t-a)(s-c)[f(t,s)-f(t,c)-f(a,s)+f(a,c)] \\ &\leq (t-a)(s-c)[M-m_1-m_2+f(a,c)], \end{aligned} \tag{4.4.12}$$

$$\begin{aligned} &(t-a)(s-d)[M_1-f(a,d)+M_2-m] \\ &\leq (t-a)(s-d)[f(t,d)-f(t,s)-f(a,d)+f(a,s)] \\ &\leq (t-a)(s-d)[m_1-f(a,d)+m_2-M], \end{aligned} \tag{4.4.13}$$

$$\begin{aligned} &(t-b)(s-c)[M_2-f(b,c)+M_1-m] \\ &\leq (t-b)(s-c)[f(b,s)-f(b,c)-f(t,s)+f(t,c)] \\ &\leq (t-b)(s-c)[m_2-f(b,c)+m_1-M], \end{aligned} \tag{4.4.14}$$

and

$$\begin{aligned}
&(t-b)(s-d)[f(b,d)-M_2-M_1+m] \\
&\leq (t-b)(s-d)[f(b,d)-f(b,s)-f(t,d)+f(t,s)] \\
&\leq (t-b)(s-d)[f(b,d)-m_2-m_1+M].
\end{aligned} \tag{4.4.15}$$

Summing the above inequalities (4.4.12)–(4.4.15), and then integrating over the bimonotonic nondecreasing function u we get,

$$\begin{aligned}
&[m-M_1-M_2+f(a,c)]\int_c^d\int_a^b(t-a)(s-c)d_td_su(t,s) \\
&\quad+[M_1-f(a,d)+M_2-m]\int_c^d\int_a^b(t-a)(s-d)d_td_su(t,s) \\
&\quad+[M_2-f(b,c)+M_1-m]\int_c^d\int_a^b(t-b)(s-c)d_td_su(t,s) \\
&\quad+[f(b,d)-M_2-M_1+m]\int_c^d\int_a^b(t-b)(s-d)d_td_su(t,s) \\
&\leq\int_c^d\int_a^b(t-a)(s-c)[f(t,s)-f(t,c)-f(a,s)+f(a,c)]d_td_su(t,s) \\
&\quad+\int_c^d\int_a^b(t-a)(s-d)[f(t,d)-f(t,s)-f(a,d)+f(a,s)]d_td_su(t,s) \\
&\quad+\int_c^d\int_a^b(t-b)(s-c)[f(b,s)-f(b,c)-f(t,s)+f(t,c)]d_td_su(t,s) \\
&\quad+\int_c^d\int_a^b(t-b)(s-d)[f(b,d)-f(b,s)-f(t,d)+f(t,s)]d_td_su(t,s) \\
&\leq[M-m_1-m_2+f(a,c)]\int_c^d\int_a^b(t-a)(s-c)d_td_su(t,s) \\
&\quad+[m_1-f(a,d)+m_2-M]\int_c^d\int_a^b(t-a)(s-d)d_td_su(t,s) \\
&\quad+[m_2-f(b,c)+m_1-M]\int_c^d\int_a^b(t-b)(s-c)d_td_su(t,s) \\
&\quad+[f(b,d)-m_2-m_1+M]\int_c^d\int_a^b(t-b)(s-d)d_td_su(t,s).
\end{aligned}$$

Dividing the above inequalities by $(b-a)(d-c)$, we deduce

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\Big\{[m-M_1-M_2+f(a,c)]\int_c^d\int_a^b (t-a)(s-c)d_td_su(t,s)\\
&\qquad+[M_1-f(a,d)+M_2-m]\int_c^d\int_a^b (t-a)(s-d)d_td_su(t,s)\\
&\qquad+[M_2-f(b,c)+M_1-m]\int_c^d\int_a^b (t-b)(s-c)d_td_su(t,s)\\
&\qquad+[f(b,d)-M_2-M_1+m]\int_c^d\int_a^b (t-b)(s-d)d_td_su(t,s)\Big\}\\
&\le \mathcal{E}(f,u;Q)\\
&\le \frac{1}{(b-a)(d-c)}\Big\{[M-m_1-m_2+f(a,c)]\int_c^d\int_a^b (t-a)(s-c)d_td_su(t,s)\\
&\qquad+[m_1-f(a,d)+m_2-M]\int_c^d\int_a^b (t-a)(s-d)d_td_su(t,s)\\
&\qquad+[m_2-f(b,c)+m_1-M]\int_c^d\int_a^b (t-b)(s-c)d_td_su(t,s)\\
&\qquad+[f(b,d)-m_2-m_1+M]\int_c^d\int_a^b (t-b)(s-d)d_td_su(t,s)\Big\}.
\end{aligned}$$

However,

$$\int_c^d\int_a^b (t-a)(s-c)\,d_td_su(t,s)=(b-a)(d-c)\,u(b,d)-\int_c^d\int_a^b u(t,s)\,dtds,$$

$$\int_c^d\int_a^b (t-a)(s-d)\,d_td_su(t,s)=(b-a)(d-c)\,u(b,c)-\int_c^d\int_a^b u(t,s)\,dtds,$$

$$\int_c^d\int_a^b (t-b)(s-c)\,d_td_su(t,s)=(b-a)(d-c)\,u(a,d)-\int_c^d\int_a^b u(t,s)\,dtds,$$

and

$$\int_c^d\int_a^b (t-b)(s-d)\,d_td_su(t,s)=(b-a)(d-c)\,u(a,c)-\int_c^d\int_a^b u(t,s)\,dtds.$$

Substituting these values in last inequality, we get

$$\begin{aligned}
&[m - M_1 - M_2 + f(a,c)] \cdot \left[u(b,d) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\quad + [M_1 - f(a,d) + M_2 - m] \cdot \left[u(b,c) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\quad + [M_2 - f(b,c) + M_1 - m] \cdot \left[u(a,d) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\quad + [f(b,d) - M_2 - M_1 + m] \cdot \left[u(a,c) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\le \mathcal{E}(f,u;Q) \\
&\le [M - m_1 - m_2 + f(a,c)] \cdot \left[u(b,d) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\quad + [m_1 - f(a,d) + m_2 - M] \cdot \left[u(b,c) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\quad + [m_2 - f(b,c) + m_1 - M] \cdot \left[u(a,d) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right] \\
&\quad + [f(b,d) - m_2 - m_1 + M] \cdot \left[u(a,c) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds\right].
\end{aligned}$$

Observe that, by the bimonotonicity of u,

$$u(a,c) \le \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b u(t,s)\,dtds \le u(b,d),$$

and then

$$\begin{aligned}
&[m - M_1 - M_2 + f(a,c)] \cdot [u(b,d) - u(a,c)] \\
&\quad + [M_1 - f(a,d) + M_2 - m] \cdot [u(b,c) - u(b,d)] \\
&\quad + [M_2 - f(b,c) + M_1 - m] \cdot [u(a,d) - u(b,d)] \\
&\quad - [f(b,d) - M_2 - M_1 + m] \cdot [u(b,d) - u(a,c)] \\
&\le \mathcal{E}(f,u;Q) \\
&\le -[M - m_1 - m_2 + f(a,c)] \cdot [u(b,d) - u(a,c)] \\
&\quad + [m_1 - f(a,d) + m_2 - M] \cdot [u(b,c) - u(a,c)] \\
&\quad + [m_2 - f(b,c) + m_1 - M] \cdot [u(a,d) - u(a,c)] \\
&\quad + [f(b,d) - m_2 - m_1 + M] \cdot [u(b,d) - u(a,c)].
\end{aligned}$$

which gives that,

$$
\begin{aligned}
& -\left[f(b,d)-f(a,c)\right]\cdot\left[u(b,d)-u(a,c)\right] \\
& \qquad -\left[M_1-f(a,d)+M_2-m\right]\cdot\left[u(b,d)-u(b,c)\right] \\
& \qquad -\left[M_2-f(b,c)+M_1-m\right]\cdot\left[u(b,d)-u(a,d)\right] \\
& \leq \mathcal{E}(f,u;Q) \\
& \leq \left[f(b,d)-f(a,c)\right]\cdot\left[u(b,d)-u(a,c)\right] \\
& \qquad +\left[m_1-f(a,d)+m_2-M\right]\cdot\left[u(b,c)-u(a,c)\right] \\
& \qquad +\left[m_2-f(b,c)+m_1-M\right]\cdot\left[u(a,d)-u(a,c)\right].
\end{aligned}
$$

as required. □

A result for $\mathcal{F}(f,u;Q)$ is incorporated in the following result:

Corollary 4.4.3. *Assume that $f,u:[a,b]\to\mathbb{R}$ are bounded on $[a,b]$ and the Riemann–Stieltjes (double) integral $\int_c^d\int_a^b f(t,s)\,d_t d_s u(t,s)$ exists. If $-\infty<n_1\leq u(t,\cdot)\leq N_1<\infty$, for all $t\in[a,b]$ and $-\infty<n_2\leq u(\cdot,s)\leq N_2<\infty$, for all $s\in[c,d]$, and f is bimonotonic nondecreasing on Q. Then, we have*

$$
\begin{aligned}
& -\left[u(b,d)-u(a,c)\right]\cdot\left[f(b,d)-f(a,c)\right] \\
& \qquad -\left[N_1-u(a,d)+N_2-n\right]\cdot\left[f(b,d)-f(b,c)\right] \\
& \qquad -\left[N_2-u(b,c)+N_1-n\right]\cdot\left[f(b,d)-f(a,d)\right] \\
& \leq \mathcal{F}(f,u;Q) \qquad\qquad (4.4.16) \\
& \leq \left[u(b,d)-u(a,c)\right]\cdot\left[f(b,d)-f(a,c)\right] \\
& \qquad +\left[n_1-u(a,d)+n_2-N\right]\cdot\left[f(b,c)-f(a,c)\right] \\
& \qquad +\left[n_2-u(b,c)+n_1-N\right]\cdot\left[f(a,d)-f(a,c)\right].
\end{aligned}
$$

where,

$$N:=\max\{N_1,N_2\},\quad \text{and}\quad n:=\min\{n_1,n_2\}.$$

Proof. The argument is similar to the proof of Theorem 4.4.2 and we shall omit the details □

In what follows, we establish some bounds for $\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)$ (respectively $\mathcal{F}\left(f,u;Q_{a,c}^{b,d}\right)$) in the situation where the function from the integrand satisfies the following conditions at the end points:

$$|f(t,s)-f(a,c)| \leq L_a (t-a)^{\alpha_1} + L_c (s-c)^{\alpha_2}, \quad \forall\ (t,s) \in Q. \tag{4.4.17}$$

$$|f(t,s)-f(a,d)| \leq L_a (t-a)^{\alpha_1} + L_d (d-s)^{\beta_2}, \quad \forall\ (t,s) \in Q. \tag{4.4.18}$$

$$|f(t,s)-f(b,c)| \leq L_b (b-t)^{\beta_1} + L_c (s-c)^{\alpha_2}, \quad \forall\ (t,s) \in Q. \tag{4.4.19}$$

$$|f(t,s)-f(b,d)| \leq L_b (b-t)^{\beta_1} + L_d (d-s)^{\beta_2}, \quad \forall\ (t,s) \in Q. \tag{4.4.20}$$

where, L_a, L_b, L_c, L_d and $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are given.

We notice that if the function f is of (r_1,r_2)-(H_1,H_2)–Hölder type, then obviously conditions (4.4.17)–(4.4.20) hold with $\alpha_1 = \beta_1 = r_1$, $\alpha_2 = \beta_2 = r_2$ and $L_a = L_b = H_1$, $L_c = L_d = H_2$. However, $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be greater than 1. Indeed, for instance, if we choose

1. $f(t,s) = (t-a)^{\alpha_1}(s-c)^{\alpha_2}$ with $\alpha_1, \alpha_2 > 0$ then f satisfies (4.4.17) with $L_a = (b-a)^{\alpha_1}$, $L_c = (d-c)^{\alpha_2}$.

2. $f(t,s) = (t-a)^{\alpha_1}(d-s)^{\beta_2}$ with $\alpha_1, \beta_2 > 0$ then f satisfies (4.4.18) with $L_a = (b-a)^{\alpha_1}$, $L_d = (d-c)^{\beta_2}$.

3. $f(t,s) = (b-t)^{\beta_1}(s-c)^{\alpha_2}$ with $\beta_1, \alpha_2 > 0$ then f satisfies (4.4.19) with $L_b = (b-a)^{\beta_1}$, $L_c = (d-c)^{\alpha_2}$.

4. $f(t,s) = (b-t)^{\beta_1}(d-s)^{\beta_2}$ with $\beta_1, \beta_2 > 0$ then f satisfies (4.4.20) with $L_b = (b-a)^{\beta_1}$, $L_d = (d-c)^{\beta_2}$.

Theorem 4.4.4. *If $f, u : Q \to \mathbb{R}$ be bounded on Q and such that the Riemann–Stieltjes double integral $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ exists. If f satisfies (4.4.17)–(4.4.20) and*

1. u *is of bounded bivariation, then*

$$\begin{aligned}&\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\leq 4\max\left\{\left[L_a\left(b-a\right)^{\alpha_1}+L_b\left(b-a\right)^{\beta_1}\right],\left[L_c\left(d-c\right)^{\alpha_2}+L_d\left(d-c\right)^{\beta_2}\right]\right\}\cdot\bigvee_Q\left(u\right).\end{aligned} \tag{4.4.21}$$

2. u *is bimonotonic nondecreasing, then*

$$\begin{aligned}&\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\leq\max\Big\{2L_a\left(b-a\right)^{\alpha_1}\left[u\left(b,d\right)-u\left(a,d\right)\right]-2L_b\left(b-a\right)^{\beta_1}\left[u\left(b,c\right)-u\left(a,c\right)\right],\\&2L_c\left(d-c\right)^{\alpha_2}\left[u\left(b,d\right)-u\left(a,d\right)\right]-2L_d\left(d-c\right)^{\beta_2}\left[u\left(b,c\right)-u\left(a,c\right)\right]\Big\}\end{aligned} \tag{4.4.22}$$

Proof. 1. If u is of bounded bivariation, then we have

$$\begin{aligned}\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|&\leq\frac{1}{\left(b-a\right)\left(d-c\right)}\\&\times\Bigg[\left|\int_c^d\int_a^b\left(t-a\right)\left(s-c\right)\left[f\left(t,s\right)-f\left(t,c\right)-f\left(a,s\right)+f\left(a,c\right)\right]d_td_su\left(t,s\right)\right|\\&+\left|\int_c^d\int_a^b\left(t-a\right)\left(s-d\right)\left[f\left(t,d\right)-f\left(t,s\right)-f\left(a,d\right)+f\left(a,s\right)\right]d_td_su\left(t,s\right)\right|\\&+\left|\int_c^d\int_a^b\left(t-b\right)\left(s-c\right)\left[f\left(b,s\right)-f\left(b,c\right)-f\left(t,s\right)+f\left(t,c\right)\right]d_td_su\left(t,s\right)\right|\\&+\left|\int_c^d\int_a^b\left(t-b\right)\left(s-d\right)\left[f\left(b,d\right)-f\left(b,s\right)-f\left(t,d\right)+f\left(t,s\right)\right]d_td_su\left(t,s\right)\right|\Bigg]\\&\leq\frac{\bigvee_Q\left(u\right)}{\left(b-a\right)\left(d-c\right)}\Bigg[\sup_{(t,s)\in Q}\left|\left(t-a\right)\left(s-c\right)\left[f\left(t,s\right)-f\left(t,c\right)-f\left(a,s\right)+f\left(a,c\right)\right]\right|\\&+\sup_{(t,s)\in Q}\left|\left(t-a\right)\left(s-d\right)\left[f\left(t,d\right)-f\left(t,s\right)-f\left(a,d\right)+f\left(a,s\right)\right]\right|\\&+\sup_{(t,s)\in Q}\left|\left(t-b\right)\left(s-c\right)\left[f\left(b,s\right)-f\left(b,c\right)-f\left(t,s\right)+f\left(t,c\right)\right]\right|\\&+\sup_{(t,s)\in Q}\left|\left(t-b\right)\left(s-d\right)\left[f\left(b,d\right)-f\left(b,s\right)-f\left(t,d\right)+f\left(t,s\right)\right]\right|\Bigg]\end{aligned}$$

$$\leq \frac{\bigvee_Q (u)}{(b-a)(d-c)} \left[2 \sup_{(t,s)\in Q} \left|(t-a)(s-c) L_c (s-c)^{\alpha_2}\right| \right.$$
$$+ 2 \sup_{(t,s)\in Q} \left|(t-a)(s-d) L_d (d-s)^{\beta_2}\right|$$
$$+ 2 \sup_{(t,s)\in Q} \left|(t-b)(s-c) L_c (s-c)^{\alpha_2}\right|$$
$$\left. + 2 \sup_{(t,s)\in Q} \left|(t-b)(s-d)(s-d) L_d (d-s)^{\beta_2}\right| \right]$$
$$\leq 4 \left[L_c (d-c)^{\alpha_2} + L_d (d-c)^{\beta_2} \right] \cdot \bigvee_Q (u)$$

Similarly, we may observe that

$$\left|\mathcal{E}\left(f, u; Q_{a,c}^{b,d}\right)\right| \leq 4 \left[L_a (b-a)^{\alpha_1} + L_b (b-a)^{\beta_1} \right] \cdot \bigvee_Q (u),$$

and therefore,

$$\left|\mathcal{E}\left(f, u; Q_{a,c}^{b,d}\right)\right|$$
$$\leq 4 \max \left\{ \left[L_a (b-a)^{\alpha_1} + L_b (b-a)^{\beta_1} \right], \left[L_c (d-c)^{\alpha_2} + L_d (d-c)^{\beta_2} \right] \right\} \cdot \bigvee_Q (u).$$

2. If u is bimonotonic nondecreasing, then we may state that

$$\left| \int_c^d \int_a^b (t-a)(s-c)\left[f(t,s) - f(t,c) - f(a,s) + f(a,c)\right] d_t d_s u(t,s) \right|$$
$$\leq \int_c^d \int_a^b (t-a)(s-c)\left|f(t,s) - f(t,c) - f(a,s) + f(a,c)\right| d_t d_s u(t,s)$$
$$\leq 2L_c \int_c^d \int_a^b (t-a)(s-c)^{\alpha_2+1} d_t d_s u(t,s)$$
$$\leq 2L_c (b-a)(d-c)^{\alpha_2+1} u(b,d) - 2(\alpha_2+1) L_c \int_c^d \int_a^b (s-c)^{\alpha_2} u(t,s)\, dt ds, \tag{4.4.23}$$

and

$$\left| \int_c^d \int_a^b (t-a)(s-d)\left[f(t,d) - f(t,s) - f(a,d) + f(a,s)\right] d_t d_s u(t,s) \right|$$
$$\leq \int_c^d \int_a^b (t-a)(d-s)\left|f(t,d) - f(t,s) - f(a,d) + f(a,s)\right| d_t d_s u(t,s)$$
$$\leq 2L_c \int_c^d \int_a^b (t-a)(d-s)^{\beta_2+1} d_t d_s u(t,s)$$
$$\leq -2L_d (b-a)(d-c)^{\beta_2+1} u(b,c) + 2(\beta_2+1) L_d \int_c^d \int_a^b (d-s)^{\beta_2} u(t,s)\, dt ds, \tag{4.4.24}$$

Similarly, we may observe that

$$\left|\int_c^d\int_a^b (t-b)(s-c)\left[f(b,s)-f(b,c)-f(t,s)+f(t,c)\right]d_td_su(t,s)\right|$$
$$\leq -2L_c(b-a)(d-c)^{\alpha_2+1}u(a,d)+2(\alpha_2+1)L_c\int_c^d\int_a^b (s-c)^{\alpha_2}u(t,s)\,dtds, \tag{4.4.25}$$

and

$$\left|\int_c^d\int_a^b (t-b)(s-d)\left[f(b,d)-f(b,s)-f(t,d)+f(t,s)\right]d_td_su(t,s)\right|$$
$$\leq 2L_d(b-a)(d-c)^{\beta_2+1}u(a,c)-2(\beta_2+1)L_d\int_c^d\int_a^b (d-s)^{\beta_2}u(t,s)\,dtds. \tag{4.4.26}$$

Adding the above inequalities (4.4.23)–(4.4.26), therefore we have

$$\begin{aligned}&\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\ &\qquad\leq 2L_c(d-c)^{\alpha_2}\left[u(b,d)-u(a,d)\right]-2L_d(d-c)^{\beta_2}\left[u(b,c)-u(a,c)\right].\end{aligned} \tag{4.4.27}$$

On the other hand, we may write the above inequalities as follows:

$$\begin{aligned}&\left|\int_c^d\int_a^b (t-a)(s-c)\left[f(t,s)-f(t,c)-f(a,s)+f(a,c)\right]d_td_su(t,s)\right|\\ &\leq \int_c^d\int_a^b (t-a)(s-c)\left|f(t,s)-f(t,c)-f(a,s)+f(a,c)\right|d_td_su(t,s)\\ &\leq 2L_a\int_c^d\int_a^b (t-a)^{\alpha_1+1}(s-c)\,d_td_su(t,s)\\ &\leq 2L_a(b-a)^{\alpha_1+1}(d-c)\,u(b,d)-2(\alpha_1+1)L_a\int_c^d\int_a^b (t-a)^{\alpha_2}u(t,s)\,dtds,\end{aligned} \tag{4.4.28}$$

and

$$\begin{aligned}&\left|\int_c^d\int_a^b (t-a)(s-d)\left[f(t,d)-f(t,s)-f(a,d)+f(a,s)\right]d_td_su(t,s)\right|\\ &\leq \int_c^d\int_a^b (t-a)(s-d)\left|f(t,d)-f(t,s)-f(a,d)+f(a,s)\right|d_td_su(t,s)\\ &\leq 2L_b\int_c^d\int_a^b (t-a)^{\beta_1+1}(d-s)\,d_td_su(t,s)\\ &\leq 2L_b(b-a)^{\beta_1+1}(d-c)\,u(b,c)-2(\beta_1+1)L_b\int_c^d\int_a^b (b-t)^{\beta_1}u(t,s)\,dtds,\end{aligned} \tag{4.4.29}$$

Similarly, we may observe that

$$\left|\int_c^d\int_a^b (t-b)(s-c)\left[f(b,s)-f(b,c)-f(t,s)+f(t,c)\right]d_td_su(t,s)\right|$$
$$\leq 2L_a(b-a)^{\alpha_1+1}(d-c)\,u(a,d)-2(\alpha_1+1)L_a\int_c^d\int_a^b (t-a)^{\alpha_2}u(t,s)\,dtds, \tag{4.4.30}$$

and

$$\left|\int_c^d\int_a^b (t-b)(s-d)\left[f(b,d)-f(b,s)-f(t,d)+f(t,s)\right]d_td_su(t,s)\right|$$
$$\leq 2L_b(b-a)^{\beta_1+1}(d-c)\,u(a,c)-2(\beta_1+1)L_b\int_c^d\int_a^b (b-t)^{\beta_1}u(t,s)\,dtds. \tag{4.4.31}$$

Adding the inequalities (4.4.28)–(4.4.31), therefore we have

$$\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|$$
$$\leq 2L_a(b-a)^{\alpha_1}\left[u(b,d)-u(a,d)\right]-2L_b(b-a)^{\beta_1}\left[u(b,c)-u(a,c)\right]. \tag{4.4.32}$$

Now, using ‘max’ property, from (4.4.27) and (4.4.32), we get

$$\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|$$
$$\leq \max\Big\{2L_a(b-a)^{\alpha_1}\left[u(b,d)-u(a,d)\right]-2L_b(b-a)^{\beta_1}\left[u(b,c)-u(a,c)\right],$$
$$2L_c(d-c)^{\alpha_2}\left[u(b,d)-u(a,d)\right]-2L_d(d-c)^{\beta_2}\left[u(b,c)-u(a,c)\right]\Big\} \tag{4.4.33}$$

□

Now, we may state the following result for $\mathcal{F}\left(f,u;Q_{a,c}^{b,d}\right)$:

Corollary 4.4.5. *With the assumptions of Theorem 4.3.4. If u satisfies (4.4.17)–(4.4.20), with $\alpha_1=\gamma_1$, $\alpha_2=\gamma_2$, $\beta_1=\delta_1$, $\beta_2=\delta_2$ and $L_a=H_a$, $L_b=H_b$, $L_c=H_ca$, $L_d=H_d$, then if*

1. f *is of bounded bivariation, then*

$$\begin{aligned}&\left|\mathcal{F}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\leq 4\max\left\{\left[H_a\left(b-a\right)^{\gamma_1}+H_b\left(b-a\right)^{\delta_1}\right],\left[H_c\left(d-c\right)^{\gamma_2}+H_d\left(d-c\right)^{\delta_2}\right]\right\}\cdot\bigvee_Q\left(f\right).\end{aligned} \tag{4.4.34}$$

2. f *is bimonotonic nondecreasing, then*

$$\begin{aligned}&\left|\mathcal{F}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\leq\max\Big\{2H_a\left(b-a\right)^{\gamma_1}\left[f\left(b,d\right)-f\left(a,d\right)\right]-2H_b\left(b-a\right)^{\delta_1}\left[f\left(b,c\right)-f\left(a,c\right)\right],\\&\qquad 2H_c\left(d-c\right)^{\gamma_2}\left[f\left(b,d\right)-f\left(a,d\right)\right]-2H_d\left(d-c\right)^{\delta_2}\left[f\left(b,c\right)-f\left(a,c\right)\right]\Big\}\end{aligned} \tag{4.4.35}$$

Theorem 4.4.6. *If $f,u:Q\to\mathbb{R}$ be bounded on Q and such that the Riemann–Stieltjes double integral $\int_c^d\int_a^b f\left(t,s\right)d_td_s\left(t,s\right)$ exists. If u is of bounded bivariation on Q and*

1. f *is of bounded bivariation on Q, then we have*

$$\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\leq\bigvee_Q\left(u\right)\cdot\bigvee_Q\left(f\right). \tag{4.4.36}$$

2. f *is bimonotonically nondecreasing on Q, the we have*

$$\begin{aligned}&\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\leq\left[f\left(b,d\right)-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(r_1,r_2\right)dr_1dr_2\right]\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}\left(u\right)\\&\quad-\left[f\left(b,c\right)-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(r_1,r_2\right)dr_1dr_2\right]\cdot\bigvee_{Q_{a,r_2}^{r_1,d}}\left(u\right)\\&\quad-\left[f\left(a,d\right)-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(r_1,r_2\right)dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,c}^{b,r_2}}\left(u\right)\\&\quad+\left[f\left(a,c\right)-\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(r_1,r_2\right)dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,r_2}^{b,d}}\left(u\right).\end{aligned} \tag{4.4.37}$$

Proof. Utilizing the equality between the first and the last terms in (4.3.7) we can write

$$\begin{aligned}
&\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\\
&=\frac{1}{(b-a)(d-c)}\left[\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right.\\
&\quad+\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_a^{r_2}(t-a)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\\
&\quad+\int_c^d\int_a^b\left(\int_c^{r_2}\int_{r_1}^{b}(t-b)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\\
&\quad\left.+\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_{r_1}^{b}(t-b)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right].
\end{aligned}$$

1. If f is of bounded bivariation, then

$$\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\qquad\le\sup_{(r_1,r_2)\in Q}\left|\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right|\cdot\bigvee_Q(f),
\end{aligned}$$

   also, since u is of bounded bivariation, then

$$\begin{aligned}
\left|\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right|&\le\sup_{(r_1,r_2)\in Q}(t-a)(s-c)\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u)\\
&=(b-a)(d-c)\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u),\qquad(4.4.38)
\end{aligned}$$

   which gives that

$$\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\qquad\le(b-a)(d-c)\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u)\cdot\bigvee_Q(f).
\end{aligned}$$

   Similarly, we may observe that

$$\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_a^{r_2}(t-a)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\qquad\le(b-a)(d-c)\cdot\bigvee_{Q_{a,r_2}^{r_1,d}}(u)\cdot\bigvee_Q(f),
\end{aligned}$$

$$\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_{r_1}^{b}(t-b)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\qquad\le(b-a)(d-c)\cdot\bigvee_{Q_{r_1,c}^{b,r_2}}(u)\cdot\bigvee_Q(f),
\end{aligned}$$

and

$$\left|\int_c^d\int_a^b\left(\int_{r_2}^d\int_{r_1}^b(t-b)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq(b-a)(d-c)\cdot\bigvee_{Q_{r_1,r_2}^{b,d}}(u)\cdot\bigvee_Q(f).$$

Therefore,

$$\begin{aligned}&\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\leq\left[\bigvee_{Q_{a,c}^{r_1,r_2}}(u)+\bigvee_{Q_{a,r_2}^{r_1,d}}(u)+\bigvee_{Q_{r_1,c}^{b,r_2}}(u)+\bigvee_{Q_{r_1,r_2}^{b,d}}(u)\right]\cdot\bigvee_Q(f)\\&=\bigvee_Q(u)\cdot\bigvee_Q(f)\end{aligned}$$

2. If f is bimonotonic nondecreasing, then

$$\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq\int_c^d\int_a^b\left|\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right|d_{r_1}d_{r_2}f(r_1,r_2)$$

and by (4.4.38), we have

$$\left|\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right|\leq(r_1-a)(r_2-c)\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u),$$

it follows that

$$\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq\int_c^d\int_a^b(r_1-a)(r_2-c)\,d_{r_1}d_{r_2}f(r_1,r_2)\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u).\quad(4.4.39)$$

Similarly, we may observe that

$$\left|\int_c^d\int_a^b\left(\int_{r_2}^d\int_a^{r_2}(t-a)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq\int_c^d\int_a^b(r_1-a)(d-r_2)\,d_{r_1}d_{r_2}f(r_1,r_2)\cdot\bigvee_{Q_{a,r_2}^{r_1,d}}(u),\quad(4.4.40)$$

$$\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_{r_1}^b(t-b)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq\int_c^d\int_a^b(b-r_1)(r_2-c)\,d_{r_1}d_{r_2}f(r_1,r_2)\cdot\bigvee_{Q_{r_1,c}^{b,r_2}}(u)\,, \quad (4.4.41)$$

and

$$\left|\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_{r_1}^b(t-b)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq\int_c^d\int_a^b(b-r_1)(d-r_2)\,d_{r_1}d_{r_2}f(r_1,r_2)\cdot\bigvee_{Q_{r_1,r_2}^{b,d}}(u)\,. \quad (4.4.42)$$

Now, using Riemann–Stieltjes integral, then by (4.4.39)–(4.4.42), we get

$$\int_c^d\int_a^b(r_1-a)(r_2-c)\,d_{r_1}d_{r_2}f(r_1,r_2)$$
$$=(b-a)(d-c)f(b,d)-\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2, \quad (4.4.43)$$

$$\int_c^d\int_a^b(r_1-a)(d-r_2)\,d_{r_1}d_{r_2}f(r_1,r_2)$$
$$=-(b-a)(d-c)f(b,c)+\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2, \quad (4.4.44)$$

$$\int_c^d\int_a^b(b-r_1)(r_2-c)\,d_{r_1}d_{r_2}f(r_1,r_2)$$
$$=-(b-a)(d-c)f(a,d)+\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2, \quad (4.4.45)$$

and

$$\int_c^d\int_a^b(b-r_1)(d-r_2)\,d_{r_1}d_{r_2}f(r_1,r_2)$$
$$=(b-a)(d-c)f(a,c)-\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2. \quad (4.4.46)$$

Substituting (4.4.43)–(4.4.46) in (4.4.39)–(4.4.42), respectively; we get

$$\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)f(r_1,r_2)\,dr_1dr_2\right|$$
$$\leq\left[(b-a)(d-c)f(b,d)-\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u)\,. \quad (4.4.47)$$

Similarly, we may observe that

$$\begin{aligned}&\left|\int_c^d\int_a^b\left(\int_{r_2}^d\int_a^{r_2}(t-a)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\&\quad\le\left[\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2-(b-a)(d-c)\,f(b,c)\right]\cdot\bigvee_{Q_{a,r_2}^{r_1,d}}(u),\qquad(4.4.48)\end{aligned}$$

$$\begin{aligned}&\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_{r_1}^b(t-b)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\&\quad\le\left[\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2-(b-a)(d-c)\,f(a,d)\right]\cdot\bigvee_{Q_{r_1,c}^{b,r_2}}(u),\qquad(4.4.49)\end{aligned}$$

and

$$\begin{aligned}&\left|\int_c^d\int_a^b\left(\int_{r_2}^d\int_{r_1}^b(t-b)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\&\quad\le\left[(b-a)(d-c)\,f(a,c)-\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,r_2}^{b,d}}(u).\qquad(4.4.50)\end{aligned}$$

Adding the inequalities to each other and then dividing by $(b-a)(d-c)$, we obtain

$$\begin{aligned}&\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right|\\&\le\left[f(b,d)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(u)\\&\qquad-\left[f(b,c)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{a,r_2}^{r_1,d}}(u)\\&\qquad-\left[f(a,d)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,c}^{b,r_2}}(u)\\&\qquad+\left[f(a,c)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,r_2}^{b,d}}(u)\end{aligned}$$

□

Corollary 4.4.7. *If $f,u:Q\to\mathbb{R}$ be bounded on Q and such that the Riemann–Stieltjes double integral $\int_c^d\int_a^b f(t,s)\,d_td_s(t,s)$ exists. If f is of bounded bivariation on Q and*

1. u *is of bounded bivariation on* Q*, then we have*

$$\left|\mathcal{F}\left(f,u;Q_{a,c}^{b,d}\right)\right| \le \bigvee_{Q}(f)\cdot\bigvee_{Q}(u)\,. \tag{4.4.51}$$

2. u *is bimonotonically nondecreasing on* Q*, the we have*

$$\begin{aligned}
&\left|\mathcal{F}\left(f,u;Q_{a,c}^{b,d}\right)\right| \qquad\qquad (4.4.52)\\
&\le\left[u(b,d)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{a,c}^{r_1,r_2}}(f)\\
&\quad-\left[u(b,c)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{a,r_2}^{r_1,d}}(f)\\
&\quad-\left[u(a,d)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,c}^{b,r_2}}(f)\\
&\quad+\left[u(a,c)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(r_1,r_2)\,dr_1dr_2\right]\cdot\bigvee_{Q_{r_1,r_2}^{b,d}}(f)\,.
\end{aligned}$$

Theorem 4.4.8. *If* $f,u:Q\to\mathbb{R}$ *be bounded on* Q *and such that the Riemann–Stieltjes double integral* $\int_c^d\int_a^b f(t,s)\,d_td_s(t,s)$ *exists. If* u *is bimonotonically nondecreasing on* Q *and*

1. f *is of bounded bivariation on* Q*, then we have*

$$\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right| \le [u(b,d)-u(b,c)-u(a,d)+u(a,c)]\cdot\bigvee_{Q}(f)\,. \tag{4.4.53}$$

2. f *is bimonotonically nondecreasing on* Q*, the we have*

$$\begin{aligned}
\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right| &\le \left[u(b,d)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(t,s)\,dtds\right]f(b,d)\\
&\quad-\left[u(b,c)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(t,s)\,dtds\right]f(b,c)\\
&\quad-\left[u(a,d)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(t,s)\,dtds\right]f(a,d)\\
&\quad+\left[u(a,c)-\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b u(t,s)\,dtds\right]f(a,c)
\end{aligned} \tag{4.4.54}$$

Proof. Utilizing the equality between the first and the last terms in (4.3.7) we can write

$$
\begin{aligned}
&\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\\
&=\frac{1}{(b-a)(d-c)}\left[\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right.\\
&\quad+\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_a^{r_1}(t-a)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\\
&\quad+\int_c^d\int_a^b\left(\int_c^{r_2}\int_{r_1}^{b}(t-b)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\\
&\quad\left.+\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_{r_1}^{b}(t-b)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right].
\end{aligned}
$$

1. If f is of bounded bivariation, then

$$
\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\qquad\le\sup_{(r_1,r_2)\in Q}\left|\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right|\cdot\bigvee_Q(f),
\end{aligned}
$$

also, since u is of bimonotonic non-decreasing, then

$$
\begin{aligned}
&\left|\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right|\\
&\le\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\\
&=(r_1-a)(r_2-c)\,u(r_1,r_2)-\int_c^{r_2}\int_a^{r_1}u(t,s)\,dtds, \qquad (4.4.55)
\end{aligned}
$$

which gives that

$$
\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_c^{r_2}\int_a^{r_1}(t-a)(s-c)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\le\sup_{(r_1,r_2)\in Q}\left[(r_1-a)(r_2-c)\,u(r_1,r_2)-\int_c^{r_2}\int_a^{r_1}u(t,s)\,dtds\right]\cdot\bigvee_Q(f)\\
&\qquad\le\left[(b-a)(d-c)\,u(b,d)-\int_c^{d}\int_a^{b}u(t,s)\,dtds\right]\cdot\bigvee_Q(f).
\end{aligned}
$$

Similarly, we may observe that

$$
\begin{aligned}
&\left|\int_c^d\int_a^b\left(\int_{r_2}^{d}\int_a^{r_1}(t-a)(s-d)\,d_td_su(t,s)\right)d_{r_1}d_{r_2}f(r_1,r_2)\right|\\
&\le\sup_{(r_1,r_2)\in Q}\left[\int_{r_2}^{d}\int_a^{r_1}u(t,s)\,dtds-(r_1-a)(d-r_2)\,u(r_1,r_2)\right]\cdot\bigvee_Q(f)\\
&\qquad\le\left[\int_c^{d}\int_a^{b}u(t,s)\,dtds-(b-a)(d-c)\,u(b,c)\right]\cdot\bigvee_Q(f),
\end{aligned}
$$

$$\left|\int_c^d \int_a^b \left(\int_c^{r_2} \int_{r_1}^b (t-b)(s-c)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \sup_{(r_1,r_2)\in Q} \left[\int_c^{r_2} \int_{r_1}^b u(t,s)\, dtds - (b-r_1)(r_2-c)\, u(r_1,r_2)\right] \cdot \bigvee_Q (f)$$
$$\leq \left[\int_c^d \int_a^b u(t,s)\, dtds - (b-a)(d-c)\, u(a,d)\right] \cdot \bigvee_Q (f),$$

and

$$\left|\int_c^d \int_a^b \left(\int_{r_2}^d \int_{r_1}^b (t-b)(s-d)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \sup_{(r_1,r_2)\in Q} \left[(b-r_1)(d-r_2)\, u(r_1,r_2) - \int_{r_2}^d \int_{r_1}^b u(t,s)\, dtds\right] \cdot \bigvee_Q (f)$$
$$\leq \left[(b-a)(d-c)\, u(a,c) - \int_c^d \int_a^b u(t,s)\, dtds\right] \cdot \bigvee_Q (f).$$

Therefore,

$$\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right| \leq [u(b,d) - u(b,c) - u(a,d) + u(a,c)] \cdot \bigvee_Q (f).$$

2. If f is bimonotonic nondecreasing, then

$$\left|\int_c^d \int_a^b \left(\int_c^{r_2} \int_a^{r_1} (t-a)(s-c)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \int_c^d \int_a^b \left|\int_c^{r_2} \int_a^{r_1} (t-a)(s-c)\, d_t d_s u(t,s)\right| d_{r_1} d_{r_2} f(r_1,r_2)$$

and by (4.4.55), we have

$$\begin{aligned} &\left|\int_c^{r_2} \int_a^{r_1} (t-a)(s-c)\, d_t d_s u(t,s)\right| \\ &\leq \int_c^{r_2} \int_a^{r_1} (t-a)(s-c)\, d_t d_s u(t,s) \\ &= (r_1-a)(r_2-c)\, u(r_1,r_2) - \int_c^{r_2} \int_a^{r_1} u(t,s)\, dtds, \end{aligned} \tag{4.4.56}$$

which gives that

$$\left|\int_c^d \int_a^b \left(\int_c^{r_2} \int_a^{r_1} (t-a)(s-c)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \int_c^d \int_a^b \left[(r_1-a)(r_2-c)\, u(r_1,r_2) - \int_c^{r_2} \int_a^{r_1} u(t,s)\, dtds\right] d_{r_1} d_{r_2} f(r_1,r_2). \tag{4.4.57}$$

Similarly, we may observe that

$$\left|\int_c^d \int_a^b \left(\int_{r_2}^d \int_a^{r_1} (t-a)(s-d)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \int_c^d \int_a^b \left[\int_{r_2}^d \int_a^{r_1} u(t,s)\, dtds - (r_1-a)(d-r_2)\, u(r_1,r_2)\right] d_{r_1} d_{r_2} f(r_1,r_2), \tag{4.4.58}$$

$$\left|\int_c^d \int_a^b \left(\int_c^{r_2} \int_{r_1}^b (t-b)(s-c)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \int_c^d \int_a^b \left[\int_c^{r_2} \int_{r_1}^b u(t,s)\, dtds - (b-r_1)(r_2-c)\, u(r_1,r_2)\right] d_{r_1} d_{r_2} f(r_1,r_2), \tag{4.4.59}$$

and

$$\left|\int_c^d \int_a^b \left(\int_{r_2}^d \int_{r_1}^b (t-b)(s-d)\, d_t d_s u(t,s)\right) d_{r_1} d_{r_2} f(r_1,r_2)\right|$$
$$\leq \int_c^d \int_a^b \left[(b-r_1)(d-r_2)\, u(r_1,r_2) - \int_{r_2}^d \int_{r_1}^b u(t,s)\, dtds\right] d_{r_1} d_{r_2} f(r_1,r_2). \tag{4.4.60}$$

Now, using Riemann–Stieltjes integral, then by (4.4.57), we get

$$\int_c^d \int_a^b \left[(r_1-a)(r_2-c)\, u(r_1,r_2) - \int_c^{r_2} \int_a^{r_1} u(t,s)\, dtds\right] d_{r_1} d_{r_2} f(r_1,r_2)$$
$$= \int_c^d \int_a^b (r_1-a)(r_2-c)\, u(r_1,r_2)\, d_{r_1} d_{r_2} f(r_1,r_2)$$
$$- \int_c^d \int_a^b \left[\int_c^{r_2} \int_a^{r_1} u(t,s)\, dtds\right] d_{r_1} d_{r_2} f(r_1,r_2), \tag{4.4.61}$$

therefore,

$$\int_c^d \int_a^b (r_1-a)(r_2-c)\, u(r_1,r_2)\, d_{r_1} d_{r_2} f(r_1,r_2)$$
$$= (b-a)(d-c)\, u(b,d)\, f(b,d) - \int_c^d \int_a^b f(r_1,r_2)\, d_{r_1} d_{r_2} u(r_1,r_2), \tag{4.4.62}$$

and

$$\int_c^d \int_a^b \left[\int_c^{r_2} \int_a^{r_1} u(t,s)\, dtds\right] d_{r_1} d_{r_2} f(r_1,r_2)$$
$$= \left[\int_c^d \int_a^b u(t,s)\, dtds\right] f(b,d) - \int_c^d \int_a^b f(r_1,r_2)\, d_{r_1} d_{r_2} u(r_1,r_2), \tag{4.4.63}$$

which gives by (4.4.61)–(4.4.63), we get

$$\int_c^d \int_a^b \left[(r_1-a)(r_2-c)\,u(r_1,r_2) - \int_c^{r_2}\int_a^{r_1} u(t,s)\,dtds\right] d_{r_1}d_{r_2}f(r_1,r_2)$$
$$= \left[(b-a)(d-c)\,u(b,d) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(b,d)\,. \quad (4.4.64)$$

Substituting (4.4.64) in (4.4.57)

$$\left|\int_c^d\int_a^b \left(\int_c^{r_2}\int_a^{r_1} (t-a)(s-c)\,d_td_su(t,s)\right) d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq \left[(b-a)(d-c)\,u(b,d) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(b,d)\,. \quad (4.4.65)$$

Similarly, we may observe that

$$\left|\int_c^d\int_a^b \left(\int_{r_2}^{d}\int_a^{r_1} (t-a)(s-d)\,d_td_su(t,s)\right) d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq \left[\int_c^d\int_a^b u(t,s)\,dtds - (b-a)(d-c)\,u(b,c)\right] f(b,c)\,, \quad (4.4.66)$$

$$\left|\int_c^d\int_a^b \left(\int_c^{r_2}\int_{r_1}^{b} (t-b)(s-c)\,d_td_su(t,s)\right) d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq \left[\int_c^d\int_a^b u(t,s)\,dtds - (b-a)(d-c)\,u(a,d)\right] f(a,d)\,, \quad (4.4.67)$$

and

$$\left|\int_c^d\int_a^b \left(\int_{r_2}^{d}\int_{r_1}^{b} (t-b)(s-d)\,d_td_su(t,s)\right) d_{r_1}d_{r_2}f(r_1,r_2)\right|$$
$$\leq \left[(b-a)(d-c)\,u(a,c) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(a,c)\,. \quad (4.4.68)$$

Adding the inequalities (4.4.65)–(4.4.68) to each other and then dividing by $(b-a)(d-c)$, we obtain

$$\begin{aligned}\left|\mathcal{E}\left(f,u;Q_{a,c}^{b,d}\right)\right| \leq &\left[(b-a)(d-c)\,u(b,d) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(b,d)\\ &- \left[(b-a)(d-c)\,u(b,c) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(b,c)\\ &- \left[(b-a)(d-c)\,u(a,d) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(a,d)\\ &+ \left[(b-a)(d-c)\,u(a,c) - \int_c^d\int_a^b u(t,s)\,dtds\right] f(a,c)\end{aligned}$$

$\square$

Corollary 4.4.9. *If $f, u : Q \to \mathbb{R}$ be bounded on Q and such that the Riemann–Stieltjes double integral $\int_c^d \int_a^b f(t,s)\, d_t d_s (t,s)$ exists. If f is bimonotonically nondecreasing on Q and*

1. *u is of bounded bivariation on Q, then we have*

$$\left|\mathcal{F}\left(f, u; Q_{a,c}^{b,d}\right)\right| \leq \left[f(b,d) - f(b,c) - f(a,d) + f(a,c)\right] \cdot \bigvee_Q (u). \quad (4.4.69)$$

2. *u is bimonotonically nondecreasing on Q, the we have*

$$\begin{aligned}\left|\mathcal{F}\left(f, u; Q_{a,c}^{b,d}\right)\right| \leq & \left[f(b,d) - \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(t,s)\, dtds\right] u(b,d) \\ & - \left[f(b,c) - \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(t,s)\, dtds\right] u(b,c) \\ & - \left[f(a,d) - \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(t,s)\, dtds\right] u(a,d) \\ & + \left[f(a,c) - \frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(t,s)\, dtds\right] u(a,c)\end{aligned} \quad (4.4.70)$$

4.5 A NUMERICAL QUADRATURE FORMULA FOR THE $\mathcal{RS}$–INTEGRAL

In this section, we apply some of the above obtained inequalities to give a sample of proposed quadrature rules for Riemann–Stieltjes integral. Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and $J_m : c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$, where $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, 1, \cdots, m-1)$ are intermediate points. Consider the Riemann sum

$$\begin{aligned}& A(f, I_n, J_m, \xi, \eta) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \frac{g(x_{i+1}, y_{j+1}) - g(x_{i+1}, y_j) - g(x_i, y_{j+1}) + g(x_i, y_j)}{(x_{i+1} - x_i)(y_{j+1} - y_j)} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)\, dtds\end{aligned} \quad (4.5.1)$$

Using Theorem 4.2.1, we can state the following theorem

Theorem 4.5.1. *Let f as in Theorem 4.2.1. Then we have*

$$\int_c^d \int_a^b f(t,s)\, d_t d_s g(t,s) = R(f, I_n, J_m, \xi, \eta) + E(f, I_n, J_m, \xi, \eta), \tag{4.5.2}$$

where $R(f, I_n, J_m, \xi, \eta)$ is the Riemann sum defined in (4.5.1) and the remainder the through the approximation $E(f, I_n, J_m, \xi, \eta)$ satisfies the bound

$$|E(f, I_n, J_m, \xi, \eta)| \leq \left[\frac{H_1 (b-a)^{\beta_1}}{2^{\beta_1+1}(\beta_1+1)} + \frac{H_2 (d-c)^{\beta_2}}{2^{\beta_2+1}(\beta_2+1)} \right] \cdot \bigvee_a^b \bigvee_c^d (g). \tag{4.5.3}$$

Proof. Applying Theorem 4.2.1 on the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get that

$$\begin{aligned}
&\Bigg| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)\, d_t d_s g(t,s) \\
&- \frac{g(x_{i+1}, y_{j+1}) - g(x_{i+1}, y_j) - g(x_i, y_{j+1}) + g(x_i, y_j)}{(x_{i+1}-x_i)(y_{j+1}-y_j)} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)\, dt ds \Bigg| \\
&\leq \left[\frac{H_1 (x_{i+1}-x_i)^{\beta_1}}{2^{\beta_1+1}(\beta_1+1)} + \frac{H_2 (y_{j+1}-y_j)^{\beta_2}}{2^{\beta_2+1}(\beta_2+1)} \right] \cdot \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (g).
\end{aligned}$$

Summing over i and j such that $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$ we get

$$\begin{aligned}
|E(f, I_n, J_m, \xi, \eta)| &\leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{H_1 (x_{i+1}-x_i)^{\beta_1}}{2^{\beta_1+1}(\beta_1+1)} + \frac{H_2 (y_{j+1}-y_j)^{\beta_2}}{2^{\beta_2+1}(\beta_2+1)} \right] \cdot \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (g) \\
&\leq \left[\frac{H_1}{2^{\beta_1+1}(\beta_1+1)} \sup_{1 \leq i \leq n-1} (x_{i+1}-x_i)^{\beta_1} + \frac{H_2}{2^{\beta_2+1}(\beta_2+1)} \sup_{1 \leq j \leq m-1} (y_{j+1}-y_j)^{\beta_2} \right] \\
&\qquad \times \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (g) \\
&= \left[\frac{H_1 (b-a)^{\beta_1}}{2^{\beta_1+1}(\beta_1+1)} + \frac{H_2 (d-c)^{\beta_2}}{2^{\beta_2+1}(\beta_2+1)} \right] \cdot \bigvee_a^b \bigvee_c^d (g),
\end{aligned}$$

which gives the result. □

Remark 4.5.2. *Similarly, we can give several estimations for the error $E(f, I_n, J_m, \xi, \eta)$ using the results the previous sections.*

CHAPTER V

SOME RELATED INEQUALITIES

5.1 INTRODUCTION

In this chapter, for mappings of two variables several inequalities of Trapezoid, Grüss and Ostrowski type are discussed. Namely, in the next two sections, by a Korkine type identity the Grüss type inequality for integrable functions holds. Inequalities for mappings of bounded variation, bounded bi-variation, Lipschitzian and bimonotonic are also provided. In the section after, approximating real functions of two variables which possess n-th partial derivatives of bounded variation, Lipschitzian and absolutely continuous are proved. In the section 5.5, Trapezoid-type rules for $\mathcal{RS}$-Double integrals are proved, and therefore, the classical Hermite–Hadamard inequality for mappings of two variables is hold. As applications quadrature rules for $\mathcal{RS}$–double integral are deduced.

5.2 GRÜSS TYPE INEQUALITIES

We start with the following lemma:

Lemma 5.2.1. *Let $F_1, F_2, G_1, G_2 : Q \to \mathbb{R}$ be a Riemann-integrable mappings on Q.*

Then the following identity holds:

$$\begin{aligned}
&\int_c^d\int_a^b F_1(x,y)\,F_2(x,y)\,dxdy\int_c^d\int_a^b G_1(x,y)\,G_2(x,y)\,dxdy\\
&\qquad-\int_c^d\int_a^b F_1(x,y)\,G_2(x,y)\,dxdy\int_c^d\int_a^b F_2(x,y)\,G_1(x,y)\,dxdy\\
&=\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b\{(F_1(x_1,y_1)\,G_1(x_2,y_2)-F_1(x_2,y_2)\,G_1(x_1,y_1))\\
&\qquad\times(F_2(x_1,y_1)\,G_2(x_2,y_2)-F_2(x_2,y_2)\,G_2(x_1,y_1))\}\,dx_1dx_2dy_1dy_2,
\end{aligned}\tag{5.2.1}$$

provided that the above integrals are exist.

Proof. Simple calculations yield that

$$\begin{aligned}
&\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b\{(F_1(x_1,y_1)\,G_1(x_2,y_2)-F_1(x_2,y_2)\,G_1(x_1,y_1))\\
&\qquad\times(F_2(x_1,y_1)\,G_2(x_2,y_2)-F_2(x_2,y_2)\,G_2(x_1,y_1))\}\,dx_1dx_2dy_1dy_2\\
&=\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b(F_1(x_1,y_1)\,G_1(x_2,y_2)\,F_2(x_1,y_1)\,G_2(x_2,y_2))\,dx_1dx_2dy_1dy_2\\
&\qquad-\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b(F_1(x_1,y_1)\,G_1(x_2,y_2)\,F_2(x_2,y_2)\,G_2(x_1,y_1))\,dx_1dx_2dy_1dy_2\\
&\qquad-\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b(F_1(x_2,y_2)\,G_1(x_1,y_1)\,F_2(x_1,y_1)\,G_2(x_2,y_2))\,dx_1dx_2dy_1dy_2\\
&\qquad+\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b(F_1(x_2,y_2)\,G_1(x_1,y_1)\,F_2(x_2,y_2)\,G_2(x_1,y_1))\,dx_1dx_2dy_1dy_2\\
&=\frac{1}{2}\int_c^d\int_a^b F_1(x_1,y_1)\,F_2(x_1,y_1)\,dx_1dy_1\int_c^d\int_a^b G_1(x_2,y_2)\,G_2(x_2,y_2)\,dx_2dy_2\\
&\qquad-\frac{1}{2}\int_c^d\int_a^b F_1(x_1,y_1)\,G_2(x_1,y_1)\,dx_1dy_1\int_c^d\int_a^b F_2(x_2,y_2)\,G_1(x_2,y_2)\,dx_2dy_2\\
&\qquad-\frac{1}{2}\int_c^d\int_a^b F_2(x_1,y_1)\,G_1(x_1,y_1)\,dx_1dy_1\int_c^d\int_a^b F_1(x_2,y_2)\,G_2(x_2,y_2)\,dx_2dy_2\\
&\qquad+\frac{1}{2}\int_c^d\int_a^b G_1(x_1,y_1)\,G_2(x_1,y_1)\,dx_1dy_1\int_c^d\int_a^b F_1(x_2,y_2)\,F_2(x_2,y_2)\,dx_2dy_2\\
&=\int_c^d\int_a^b F_1(x,y)\,F_2(x,y)\,dxdy\int_c^d\int_a^b G_1(x,y)\,G_2(x,y)\,dxdy\\
&\qquad-\int_c^d\int_a^b F_1(x,y)\,G_2(x,y)\,dxdy\int_c^d\int_a^b F_2(x,y)\,G_1(x,y)\,dxdy,
\end{aligned}$$

where, the last equality holds by changing of variables and thus the required result holds.

□

Therefore, we may deduce the following Korkine type identity for mappings of two variables.

Corollary 5.2.2. *Let F_1, F_2, G_1, G_2 as in Lemma 5.2.1. Then the following identity holds:*

$$\Delta \int_c^d \int_a^b f(x,y)\, g(x,y)\, dxdy - \left(\int_c^d \int_a^b f(x,y)\, dxdy \right) \left(\int_c^d \int_a^b g(x,y)\, dxdy \right)$$
$$= \frac{1}{2} \int_c^d \int_c^d \int_a^b \int_a^b \left(f(x_1,y_1) - f(x_2,y_2) \right) \left(g(x_1,y_1) - g(x_2,y_2) \right) dx_1 dx_2 dy_1 dy_2, \tag{5.2.2}$$

where, $\Delta = (b-a)(d-c)$.

Proof. In Lemma 5.2.1, choose $G_1(x,y) = G_2(x,y) = 1$, $F_1(x,y) = f(x,y)$, $F_2(x,y) = g(x,y)$, then the required result holds. □

For two measurable functions $f, g : Q \to \mathbb{R}$, define the functional, which is known in the literature as Čebyčsev's functional, by

$$\mathcal{T}(f,g) = \mathcal{M}(fg) - \mathcal{M}(f)\, \mathcal{M}(g), \tag{5.2.3}$$

where the integral mean is given by

$$\mathcal{M}(f) = \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\, dxdy. \tag{5.2.4}$$

The integrals in (5.2.3) are assumed to exist.

Further, the weighted Čebyčsev functional is defined by

$$\Im(f,g;p) = \Re(f,g;p) - \Re(f;p)\, \Re(g;p), \tag{5.2.5}$$

where the weighted integral mean is given by

$$\Re(f;p) = \frac{\int_c^d \int_a^b p(x,y)\, f(x,y)\, dxdy}{\int_c^d \int_a^b p(x,y)\, dxdy}. \tag{5.2.6}$$

Here, we note that,

$$\Im (f, g; 1) \equiv \mathcal{T} (f, g) \quad and \;\; \Re (f; 1) \equiv \mathcal{M} (f) \, .$$

Let $S\Re (f)$ be an operator defined by

$$S (f) (x, y) := f (x, y) - \mathcal{M} (f) \, , \tag{5.2.7}$$

which shifts a function by its integral mean, then the following identity holds. Namely,

$$\mathcal{T} (f, g) = \mathcal{T} (S (f) \, , g) = \mathcal{T} (f, S (g)) = \mathcal{T} (S (f) \, , S (g)) \tag{5.2.8}$$

and so

$$\mathcal{T} (f, g) = \mathcal{M} (S (f) \, g) = \mathcal{M} (f S (g)) = \mathcal{M} (S (f) \, S (g)) \, , \tag{5.2.9}$$

since $\mathcal{M} (S (f)) = \mathcal{M} (S (g)) = 0$.

For the last term in (5.2.8) (or 5.2.9) only one of the functions needs to be shifted by its integral mean. If the other were to be shifted by any other quantity, the identities would still hold.

Using the above Korkine type identity (5.2.2), we give another proof for the well-known Grüss inequality:

Theorem 5.2.3. *Let $f, g : Q \to \mathbb{R}$ be integrable on Q and satisfy*

$$\phi \leq f (x, y) \leq \Phi, \;\; \gamma \leq g (x, y) \leq \Gamma, \;\; \forall (x, y) \in Q$$

Then we have the inequality

$$|\mathcal{T} (f, g)| \leq \frac{1}{4} (\Phi - \phi) (\Gamma - \gamma) \tag{5.2.10}$$

where,

$$\mathcal{T} (f, g)$$
$$:= \frac{1}{\Delta} \int_c^d \int_a^b f (x, y) \, g (x, y) \, dx dy - \left(\frac{1}{\Delta} \int_c^d \int_a^b f (x, y) \, dx dy \right) \left(\frac{1}{\Delta} \int_c^d \int_a^b g (x, y) \, dx dy \right)$$

is the Čebyčsev's functional, and $\Delta := (b - a) (d - c)$.

Proof. First of all, we note that

$$\begin{aligned}
&\mathcal{T}\left(f,g\right) \\
&:= \frac{1}{2\Delta^2}\int_c^d\int_c^d\int_a^b\int_a^b \left(f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right)\left(g\left(x_1,y_1\right)-g\left(x_2,y_2\right)\right)dx_1dx_2dy_1dy_2
\end{aligned}$$

Applying the Cauchy-Bunyakovsky-Schwarz integral inequality on the Korkine identity (5.2.2), we get

$$\begin{aligned}
&\left|\int_c^d\int_c^d\int_a^b\int_a^b \left(f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right)\left(g\left(x_1,y_1\right)-g\left(x_2,y_2\right)\right)dx_1dx_2dy_1dy_2\right| \\
&\qquad \le \left[\int_c^d\int_c^d\int_a^b\int_a^b \left(f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right)^2 dx_1dx_2dy_1dy_2\right]^{1/2} \\
&\qquad \times \left[\int_c^d\int_c^d\int_a^b\int_a^b \left(g\left(x_1,y_1\right)-g\left(x_2,y_2\right)\right)^2 dx_1dx_2dy_1dy_2\right]^{1/2}. \qquad (5.2.11)
\end{aligned}$$

Now, observe from (5.2.2) that we have

$$\begin{aligned}
&\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b \left(f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right)^2 dx_1dx_2dy_1dy_2 \\
&\qquad = \Delta\int_c^d\int_a^b f^2\left(x,y\right)dxdy - \left(\int_c^d\int_a^b f\left(x,y\right)dxdy\right)^2, \qquad (5.2.12)
\end{aligned}$$

and a similar identity for g, i.e.,

$$\begin{aligned}
&\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b \left(g\left(x_1,y_1\right)-g\left(x_2,y_2\right)\right)^2 dx_1dx_2dy_1dy_2 \\
&\qquad = \Delta\int_c^d\int_a^b g^2\left(x,y\right)dxdy - \left(\int_c^d\int_a^b g\left(x,y\right)dxdy\right)^2. \qquad (5.2.13)
\end{aligned}$$

A simple calculation yields that

$$\begin{aligned}
&\frac{1}{\Delta}\int_c^d\int_a^b f^2\left(x,y\right)dxdy - \left(\frac{1}{\Delta}\int_c^d\int_a^b f\left(x,y\right)dxdy\right)^2 \\
&= \left(\Phi - \frac{1}{\Delta}\int_c^d\int_a^b f\left(x,y\right)dxdy\right)\left(\frac{1}{\Delta}\int_c^d\int_a^b f\left(x,y\right)dxdy - \phi\right) \\
&\quad - \frac{1}{\Delta}\int_c^d\int_a^b \left(f\left(x,y\right)-\phi\right)\left(\Phi - f\left(x,y\right)\right)dxdy. \qquad (5.2.14)
\end{aligned}$$

and a similar identity for g

$$\begin{aligned}
&\frac{1}{\Delta}\int_c^d\int_a^b g^2\left(x,y\right)dxdy - \left(\frac{1}{\Delta}\int_c^d\int_a^b g\left(x,y\right)dxdy\right)^2 \\
&= \left(\Gamma - \frac{1}{\Delta}\int_c^d\int_a^b g\left(x,y\right)dxdy\right)\left(\frac{1}{\Delta}\int_c^d\int_a^b g\left(x,y\right)dxdy - \gamma\right) \\
&\quad - \frac{1}{\Delta}\int_c^d\int_a^b \left(g\left(x,y\right)-\gamma\right)\left(\Gamma - g\left(x,y\right)\right)dxdy. \qquad (5.2.15)
\end{aligned}$$

By the assumption we have $(f(x,y)-\phi)(\Phi-f(x,y))\geq 0$ and $(g(x,y)-\gamma)(\Gamma-g(x,y))\geq 0$ for all $x,y\in Q$, and so

$$\int_c^d\int_a^b (f(x,y)-\phi)(\Phi-f(x,y))\,dxdy\geq 0$$

$$\int_c^d\int_a^b (g(x,y)-\gamma)(\Gamma-g(x,y))\,dxdy\geq 0$$

which implies that from (5.2.14)

$$\begin{aligned}
&\frac{1}{\Delta}\int_c^d\int_a^b f^2(x,y)\,dxdy-\left(\frac{1}{\Delta}\int_c^d\int_a^b f(x,y)\,dxdy\right)^2\\
&=\left(\Phi-\frac{1}{\Delta}\int_c^d\int_a^b f(x,y)\,dxdy\right)\left(\frac{1}{\Delta}\int_c^d\int_a^b f(x,y)\,dxdy-\phi\right)\\
&\leq\left[\left(\Phi-\frac{1}{\Delta}\int_c^d\int_a^b f(x,y)\,dxdy\right)+\left(\frac{1}{\Delta}\int_c^d\int_a^b f(x,y)\,dxdy-\phi\right)\right]^2\\
&=\frac{1}{4}(\Phi-\phi)^2
\end{aligned}\tag{5.2.16}$$

where we have used the fact that $AB\leq\left(\frac{A+B}{2}\right)^2$. A similar argument for g, gives

$$\begin{aligned}
&\frac{1}{\Delta}\int_c^d\int_a^b g^2(x,y)\,dxdy-\left(\frac{1}{\Delta}\int_c^d\int_a^b g(x,y)\,dxdy\right)^2\\
&\leq\frac{1}{4}(\Gamma-\gamma)^2.
\end{aligned}\tag{5.2.17}$$

Using the inequality (5.2.11) via (5.2.12), (5.2.13) and the estimations (5.2.16) and (5.2.17), we get

$$\begin{aligned}
&\left|\frac{1}{2}\int_c^d\int_c^d\int_a^b\int_a^b (f(x_1,y_1)-f(x_2,y_2))(g(x_1,y_1)-g(x_2,y_2))\,dx_1dx_2dy_1dy_2\right|\\
&\qquad\qquad\leq\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)(b-a)(d-c),
\end{aligned}$$

and then, by (5.2.2), we deduce the desired inequality (5.2.10). □

Theorem 5.2.4. *Let $f,g:Q\to\mathbb{R}$ be L_1, L_2–Lipschitzian mappings on Q, so that*

$$|f(x_1,y_1)-f(x_2,y_2)|\leq L_1\|(x_1,y_1)-(x_2,y_2)\|$$

and

$$|g(x_1,y_1)-g(x_2,y_2)|\leq L_2\|(x_1,y_1)-(x_2,y_2)\|$$

for all $(x_1, y_1), (x_2, y_2) \in Q$*. We then have the inequality*

$$|\mathcal{T}(f,g)| \leq \frac{L_1 L_2}{12}\left[(b-a)^2 + (d-c)^2\right], \tag{5.2.18}$$

where, $\|\cdot\|$ *is the usual Euclidean norm, i.e.,* $\|(x,y)\| = \sqrt{x^2+y^2}$*.*

Proof. We have the Korkine identity

$$\begin{aligned}
&\Delta \int_c^d \int_a^b f(x,y)\, g(x,y)\, dxdy - \left(\int_c^d \int_a^b f(x,y)\, dxdy\right)\left(\int_c^d \int_a^b g(x,y)\, dxdy\right) \\
&= \frac{1}{2}\int_c^d \int_c^d \int_a^b \int_a^b (f(x_1,y_1) - f(x_2,y_2))(g(x_1,y_1) - g(x_2,y_2))\, dx_1 dx_2 dy_1 dy_2,
\end{aligned} \tag{5.2.19}$$

where, $\Delta = (b-a)(d-c)$.

By assumptions we have

$$\begin{aligned}
&|(f(x_1,y_1) - f(x_2,y_2))| \cdot |(g(x_1,y_1) - g(x_2,y_2))| \\
&\leq [L_1 |(x_1,y_1) - (x_2,y_2)|] \cdot [L_2 |(x_1,y_1) - (x_2,y_2)|] \\
&= L_1 L_2 \left(\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\right)^2 \\
&= L_1 L_2 \left[(x_1-x_2)^2 + (y_1-y_2)^2\right]
\end{aligned} \tag{5.2.20}$$

for all $(x_1, y_1), (x_2, y_2) \in Q$.

Integrating (5.2.20) on Q^2, we get

$$\begin{aligned}
&\int_c^d \int_c^d \int_a^b \int_a^b (f(x_1,y_1) - f(x_2,y_2))(g(x_1,y_1) - g(x_2,y_2))\, dx_1 dx_2 dy_1 dy_2 \\
&= L_1 L_2 \int_c^d \int_c^d \int_a^b \int_a^b \left[(x_1-x_2)^2 + (y_1-y_2)^2\right] dx_1 dx_2 dy_1 dy_2 \\
&= L_1 L_2 \left[(d-c)^2 \int_a^b \int_a^b (x_1-x_2)^2\, dx_1 dx_2 + (b-a)^2 \int_c^d \int_c^d (y_1-y_2)^2\, dy_1 dy_2\right] \\
&= L_1 L_2 (d-c)^2 (b-a)^2 \left[\frac{(b-a)^2}{6} + \frac{(d-c)^2}{6}\right]
\end{aligned}$$

Using (5.2.19), we get (5.2.18). □

Theorem 5.2.5. *Let $f, g : Q \to \mathbb{R}$ be satisfy*

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L_1 |x_1 - x_2| |y_1 - y_2|$$

and

$$|g(x_1, y_1) - g(x_2, y_2)| \leq L_2 |x_1 - x_2| |y_1 - y_2|$$

for all $(x_1, y_1), (x_2, y_2) \in Q$. We then have the inequality

$$|\mathcal{T}(f, g)| \leq \frac{L_1 L_2}{36} (d-c)^4 (b-a)^4, \tag{5.2.21}$$

where, $\|\cdot\|$ is the usual Euclidean norm, i.e., $\|(x, y)\| = \sqrt{x^2 + y^2}$.

Proof. We have the Korkine identity

$$\begin{aligned}
&\Delta \int_c^d \int_a^b f(x, y) g(x, y) \, dxdy - \left(\int_c^d \int_a^b f(x, y) \, dxdy \right) \left(\int_c^d \int_a^b g(x, y) \, dxdy \right) \\
&= \frac{1}{2} \int_c^d \int_c^d \int_a^b \int_a^b (f(x_1, y_1) - f(x_2, y_2)) (g(x_1, y_1) - g(x_2, y_2)) \, dx_1 dx_2 dy_1 dy_2,
\end{aligned} \tag{5.2.22}$$

where, $\Delta = (b-a)(d-c)$.

By assumptions we have

$$\begin{aligned}
&|(f(x_1, y_1) - f(x_2, y_2))| \cdot |(g(x_1, y_1) - g(x_2, y_2))| \\
&\leq [L_1 |x_1 - x_2| |y_1 - y_2|] \cdot [L_2 |x_1 - x_2| |y_1 - y_2|] \\
&= L_1 L_2 (x_1 - x_2)^2 (y_1 - y_2)^2
\end{aligned} \tag{5.2.23}$$

for all $(x_1, y_1), (x_2, y_2) \in Q$.

Integrating (5.2.23) on Q^2, we get

$$\begin{aligned}
&\int_c^d\int_c^d\int_a^b\int_a^b \left(f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right)\left(g\left(x_1,y_1\right)-g\left(x_2,y_2\right)\right)dx_1dx_2dy_1dy_2 \\
&= L_1L_2\int_c^d\int_c^d\int_a^b\int_a^b \left[\left(x_1-x_2\right)^2\cdot\left(y_1-y_2\right)^2\right]dx_1dx_2dy_1dy_2 \\
&= L_1L_2\left[(d-c)^2\int_a^b\int_a^b\left(x_1-x_2\right)^2dx_1dx_2\cdot(b-a)^2\int_c^d\int_c^d\left(y_1-y_2\right)^2dy_1dy_2\right] \\
&= L_1L_2\left[\frac{(d-c)^2(b-a)^4}{6}\cdot\frac{(b-a)^2(d-c)^4}{6}\right] \\
&= \frac{1}{36}L_1L_2(d-c)^6(b-a)^6\,.
\end{aligned}$$

Using (5.2.22), we get (5.2.21). □

The following result presents an identity for the Čebyčsev functional that involves a Riemann-Stieltjes integral and provides a Peano kernel representation:

Lemma 5.2.6. *Let $f,g:Q\to\mathbb{R}$ where f is of bounded variation and g is continuous on Q, then*

$$\mathcal{T}(f,g)=\frac{1}{\Delta^2}\int_c^d\int_a^b\psi(t,s)\,d_td_sf(t,s), \tag{5.2.24}$$

where,

$$\begin{aligned}
\psi(t,s):&=(s-c)(t-a)A(t,b;s,d)-(s-c)(b-t)A(a,t;s,d) \\
&\quad-(d-s)(t-a)A(t,b;c,s)+(d-s)(b-t)A(a,t;c,s)
\end{aligned} \tag{5.2.25}$$

with

$$A(a,b;c,d):=\int_c^d\int_a^b g(x,y)\,dxdy \tag{5.2.26}$$

Proof. From (5.2.24), integrating the Riemann–Stieltjes integral by parts produces

$$\begin{aligned}
&\frac{1}{\Delta^2}\int_c^d\int_a^b\psi(t,s)\,d_td_sf(t,s) \\
&\quad=\frac{1}{\Delta^2}\left[\psi(b,d)f(b,d)-\psi(b,c)f(b,c)-\psi(a,d)f(a,d)+\psi(a,c)f(a,c)\right] \\
&\qquad\qquad-\frac{1}{\Delta^2}\int_c^d\int_a^b f(t,s)\frac{\partial^2\psi}{\partial t\partial s}(t,s)\,dtds
\end{aligned}$$

since $\psi(t,s)$ differentiable. Thus, from (5.2.25) we have $\psi(b,d) = \psi(b,c) = \psi(a,d) = \psi(a,c) = 0$, and so

$$\begin{aligned}
&\frac{1}{\Delta^2}\int_c^d\int_a^b \psi(t,s)\, d_t d_s f(t,s)\\
&= \frac{1}{\Delta^2}\int_c^d\int_a^b \left[(b-a)(d-c)\, g(t,s) - A(a,b;c,d)\right] f(t,s)\, dtds\\
&= \frac{1}{\Delta}\int_c^d\int_a^b \left[g(t,s) - \mathcal{M}(g)\right] f(t,s)\, dtds\\
&= \mathcal{M}(fS(g)) = \mathcal{T}(f,g)\,.
\end{aligned}$$

from which the result (5.2.25) is obtained on noting identity (5.2.24). $\square$

Remark 5.2.7. *We remark that $\psi(t,s)$ attain its maximum at $\left(\frac{a+b}{2},\frac{c+d}{2}\right)$ and therefore, $\sup\limits_{(t,s)\in Q}\psi(t,s) = \psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)$.*

Theorem 5.2.8. *Let $f,g : Q \to \mathbb{R}$ where f is of bounded variation and g is continuous on Q, then*

$$\Delta^2\left|\mathcal{T}(f,g)\right| \le \begin{cases} \sup\limits_{(t,s)\in Q}\psi(t,s)\cdot\bigvee_c^d\bigvee_a^b(f) & \\ L\int_c^d\int_a^b|\psi(t,s)|\,dtds, & for\ f\ L-Lipschitzian\\ \int_c^d\int_a^b|\psi(t,s)|\,d_t d_s f(t,s), & for\ f\ bimonotonic\ nondecreasing\end{cases} \tag{5.2.27}$$

Proof. The first part may be done by Lemma, the second by Lemma and the last part by Lemma. We shall omit the details. $\square$

The following result gives an identity for the weighted Čebyčsev functional that involves a Riemann-Stieltjes double integral.

Theorem 5.2.9. *Let $f,g,p : Q \to \mathbb{R}$ where f is of bounded variation and g,p are continuous on Q. Further $P(b,d) = \int_c^d\int_a^b p(t,s)\,dtds > 0$, then*

$$\Im(f,g;p) = \frac{1}{P^2(b,d)}\int_c^d\int_a^b \psi(t,s)\, d_t d_s f(t,s) \tag{5.2.28}$$

$\Im(f,g;p)$ is given in (...),

$$\Psi(t,s) = P(t,s)\,G^*(t,s) - P^*(t,s)\,G(t,s) \tag{5.2.29}$$

with

$$P(t,s) = \int_c^s \int_a^t p(x,y)\,dxdy,$$
$$P^*(t,s) = P(b,d) - P(b,s) - P(t,d) + P(t,s) \quad (5.2.30)$$

and

$$G(t,s) = \int_c^s \int_a^t p(x,y)\,g(x,y)\,dxdy,$$
$$G^*(t,s) = G(b,d) - G(b,s) - g(t,d) + G(t,s)\,. \quad (5.2.31)$$

Proof. The proof may be very closely to proof of Lemma 5.2.6. We shall omit the details. □

Theorem 5.2.10. *Under the assumptions of Theorem 5.2.9, we have*

$$P^2(b,d)\,|\Im(f,g;p)| \le \begin{cases} \sup\limits_{(t,s)\in Q} \Psi(t,s)\cdot \bigvee_c^d \bigvee_a^b (f) \\ L\int_c^d \int_a^b |\Psi(t,s)|\,dtds, & for\ f\ L-Lipschitzian \\ \int_c^d \int_a^b |\Psi(t,s)|\,d_t d_s f(t,s), & for\ f\ bimonotonic\ nondecreasing \end{cases} \quad (5.2.32)$$

Proof. The proof uses results 5.2.6 through 5.2.8 and follows closely the proof in procuring the bounds in (5.2.32). □

In the following, we derive a new inequality of Grüuss' type for Riemann–Stieltjes double integral. In Chapter 4, we have discussed several properties of the functional

$$\aleph(f,g,Q) := \int_c^d \int_a^b f(x,y)\,d_x d_y g(x,y) - \left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right]\cdot \int_c^d \int_a^b f(t,s)\,dtds \quad (5.2.33)$$

which is of Ostrowski's type for Riemann–Stieltjes double integral.

The following result holds:

Theorem 5.2.11. *Let $f, g : Q \to \mathbb{R}$, be such that g is L–Lipschitz on Q and f is Riemann-integrable on Q and there exist real numbers m, M such that $m \le f(x,y) \le M$, for all $(x,y) \in Q$. Then we have the following inequality:*

$$|\aleph(f,g,Q)| \le \frac{1}{2}(M-m)(b-a)(d-c). \tag{5.2.34}$$

Proof. Using Lemma 3.2.5, we have

$$\begin{aligned}
|\aleph(f,g,Q)| &:= \left|\int_c^d \int_a^b f(x,y)\, d_x d_y g(x,y)\right.\\
&\qquad \left. - \left[\frac{g(b,d)-g(a,d)-g(b,c)+g(a,c)}{(b-a)(d-c)}\right] \cdot \int_c^d \int_a^b f(t,s)\, dtds\right|
\end{aligned}$$

$$\begin{aligned}
&= \left|\int_c^d \int_a^b \left[f(x,y) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right] d_x d_y g(x,y)\right|\\
&\le L \int_c^d \int_a^b \left|f(x,y) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right| dxdy.
\end{aligned} \tag{5.2.35}$$

Now, define

$$I := \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \left(f(x,y) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right)^2 dxdy,$$

then, we have

$$\begin{aligned}
I &:= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \left[f^2(x,y) - 2f(x,y)\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right.\\
&\qquad \left. + \left(\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right)^2\right] dxdy\\
&= \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f^2(x,y) dxdy - \left(\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right)^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
I &= \left(M - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds\right)\\
&\qquad \times \left(\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds - m\right)\\
&\qquad - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b [M - f(t,s)] \cdot [f(t,s) - m]\, dtds.
\end{aligned}$$

As $m \le f(x,y) \le M$, for all $(x,y) \in Q$, then

$$\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b [M - f(t,s)] \cdot [f(t,s) - m]\, dtds \ge 0$$

which implies that

$$I \leq \left(M - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds \right) \times \left(\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds - m \right).$$

Using the elementary inequality

$$(M-k)(k-m) \leq \frac{1}{4}(M-m)^2$$

which holds for all $m, k, M \in \mathbb{R}$, we get

$$I \leq \frac{1}{4}(M-m)^2. \tag{5.2.36}$$

Using Cauchy-Buniakowski-Schwarz's integral inequality we have

$$I \geq \left[\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds \right| dxdy \right]^2$$

Now, by (5.2.36), we get

$$\int_c^d \int_a^b \left| f(x,y) - \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(t,s)\, dtds \right| dxdy \leq \frac{1}{2}(M-m)(b-a)(d-c)$$

and then by (5.2.35) we obtain the desired inequality (5.2.34). □

5.3 APPROXIMATING REAL FUNCTIONS OF TWO VARIABLES WHICH POSSESS N-TH DERIVATIVES OF BOUNDED VARIATION

Theorem 5.3.1. *Let $Q := I \times J$ be a closed rectangle on $\mathbb{R}^2$, let $a, b \in I$ with $a < b$, $c, d \in J$ with $c < d$ and let n be a nonnegative integer. If $f : Q \to \mathbb{R}$ is such that the n-th partial derivatives $D^n f$ is of bounded variation on Q, then, for any $(x, y) \in Q$ we* *have the representation*

$$f\left(x,y\right)=\frac{1}{(b-a)(d-c)}\left[(b-x)(d-y)f(a,c)+(b-x)(y-c)f(a,d)\right.$$
$$\left.+(x-a)(d-y)f(b,c)+(x-a)(y-c)f(b,d)\right]+\frac{(y-c)(d-y)}{(b-a)(d-c)}$$
$$\times\sum_{j=1}^{n}\frac{1}{j!}\begin{pmatrix} n \\ j \end{pmatrix}\Big\{(b-x)\left(x-a\right)^{n-j}\Big[\left(y-c\right)^{j-1}D^{n}f\left(a,c\right)+\left(-1\right)^{j}\left(d-y\right)^{j-1}D^{n}f\left(a,d\right)\Big]$$
$$+\left(x-a\right)\left(b-x\right)^{n-j}\Big[\left(-1\right)^{j}\left(y-c\right)^{j-1}D^{n}f\left(b,c\right)+\left(d-y\right)^{j-1}D^{n}f\left(b,d\right)\Big]\Big\}$$
$$+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{c}^{d}\int_{a}^{b}S_{n}\left(x,t;y,s\right)d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right) \quad (5.3.1)$$

where, $D^{n}f\left(t,s\right)=\frac{\partial^{n}f}{\partial t^{n-j}\partial s^{j}}\left(t,s\right)$ *and*

$$S_{n}\left(x,t;y,s\right)=\frac{1}{n!}\begin{cases}\left(x-t\right)^{n}\left(b-x\right)\left(y-s\right)^{n}\left(d-y\right), & a\leq t\leq x,\quad c\leq s\leq y\\ \left(-1\right)^{n}\left(t-x\right)^{n}\left(x-a\right)\left(y-s\right)^{n}\left(d-y\right), & x<t\leq b,\quad c\leq s\leq y\\ \left(-1\right)^{n}\left(x-t\right)^{n}\left(b-x\right)\left(s-y\right)^{n}\left(y-c\right), & a\leq t\leq x,\quad y<s\leq d\\ \left(t-x\right)^{n}\left(x-a\right)\left(s-y\right)^{n}\left(y-c\right), & x<t\leq b,\quad y<s\leq d\end{cases}$$

Proof. We utilize the following Taylor's representation formula for functions $f:Q\subset\mathbb{R}^{2}\rightarrow\mathbb{R}$ such that the n-th partial derivatives $D^{n}f$ are of locally bounded variation on Q,

$$f\left(x,y\right)=P_{n}\left(x,y\right)+R_{n}\left(x,y\right) \quad (5.3.2)$$

such that,

$$P_{n}\left(x,y\right)=\sum_{j=0}^{n}\frac{1}{j!}\begin{pmatrix} n \\ j \end{pmatrix}\left(x-x_{0}\right)^{n-j}\left(y-y_{0}\right)^{j}D^{n}f\left(x_{0},y_{0}\right) \quad (5.3.3)$$

and

$$R_{n}\left(x,y\right)=\frac{1}{n!}\int_{y_{0}}^{y}\int_{x_{0}}^{x}\left(x-t\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right) \quad (5.3.4)$$

where, $D^{n}f\left(t,s\right)=\frac{\partial^{n}f}{\partial t^{n-j}\partial s^{j}}\left(t,s\right)$, and (x,y), (x_{0},y_{0}) are in Q and the double integral in the remainder is taken in the Riemann-Stieltjes sense.

Choosing $x_0 = a$, $y_0 = c$ and then $x_0 = b$, $y_0 = d$ in (5.3.1) we can write that

$$f\left(x,y\right)=\sum_{j=0}^{n}\frac{1}{j!}\left(\begin{array}{c}n\\ j\end{array}\right)\left(x-a\right)^{n-j}\left(y-c\right)^{j}D^{n}f\left(a,c\right)$$
$$+\frac{1}{n!}\int_{c}^{y}\int_{a}^{x}\left(x-t\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right),\quad(5.3.5)$$

$$f\left(x,y\right)=\sum_{j=0}^{n}\frac{(-1)^{j}}{j!}\left(\begin{array}{c}n\\ j\end{array}\right)\left(x-a\right)^{n-j}\left(d-y\right)^{j}D^{n}f\left(a,d\right)$$
$$+\frac{(-1)^{n+1}}{n!}\int_{y}^{d}\int_{a}^{x}\left(x-t\right)^{n}\left(s-y\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right),\quad(5.3.6)$$

$$f\left(x,y\right)=\sum_{j=0}^{n}\frac{(-1)^{j}}{j!}\left(\begin{array}{c}n\\ j\end{array}\right)\left(b-x\right)^{n-j}\left(y-c\right)^{j}D^{n}f\left(b,c\right)$$
$$+\frac{(-1)^{n+1}}{n!}\int_{c}^{y}\int_{x}^{b}\left(t-x\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right)\quad(5.3.7)$$

$$f\left(x,y\right)=\sum_{j=0}^{n}\frac{1}{j!}\left(\begin{array}{c}n\\ j\end{array}\right)\left(b-x\right)^{n-j}\left(d-y\right)^{j}D^{n}f\left(b,d\right)$$
$$+\frac{1}{n!}\int_{y}^{d}\int_{x}^{b}\left(t-x\right)^{n}\left(s-y\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right)\quad(5.3.8)$$

for ny $(x,y)\in Q$.

Now, by multiplying (5.3.5) with $(b-x)(d-y)$, (5.3.6) with $(b-x)(y-c)$, (5.3.7) with $(x-a)(d-y)$, (5.3.8) with $(x-a)(y-c)$, we get

$$(b-x)(d-y)f\left(x,y\right)=(b-x)(d-y)f(a,c)$$
$$+(b-x)(d-y)\sum_{j=1}^{n}\frac{1}{j!}\left(\begin{array}{c}n\\ j\end{array}\right)\left(x-a\right)^{n-j}\left(y-c\right)^{j}D^{n}f\left(a,c\right)$$
$$+\frac{1}{n!}(b-x)(d-y)\int_{c}^{y}\int_{a}^{x}\left(x-t\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right),\quad(5.3.9)$$

$$(b-x)(y-c)f\left(x,y\right)=(b-x)(y-c)f(a,d)$$
$$+(b-x)(y-c)\sum_{j=0}^{n}\frac{(-1)^{j}}{j!}\left(\begin{array}{c}n\\ j\end{array}\right)\left(x-a\right)^{n-j}\left(d-y\right)^{j}D^{n}f\left(a,d\right)$$
$$+\frac{(-1)^{n+1}}{n!}(b-x)(y-c)\int_{y}^{d}\int_{a}^{x}\left(x-t\right)^{n}\left(s-y\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right),\quad(5.3.10)$$

$$
\begin{aligned}
(x-a)(d-y)f\left(x,y\right) &= (x-a)(d-y)f(b,c) \\
&+ (x-a)(d-y)\sum_{j=0}^{n}\frac{(-1)^{j}}{j!}\begin{pmatrix} n \\ j \end{pmatrix}(b-x)^{n-j}\left(y-c\right)^{j}D^{n}f\left(b,c\right) \\
&+ \frac{(-1)^{n+1}}{n!}(x-a)(d-y)\int_{c}^{y}\int_{x}^{b}\left(t-x\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right)
\end{aligned} \tag{5.3.11}
$$

$$
\begin{aligned}
(x-a)(y-c)f\left(x,y\right) &= (x-a)(y-c)f(b,d) \\
&+ (x-a)(y-c)\sum_{j=0}^{n}\frac{1}{j!}\begin{pmatrix} n \\ j \end{pmatrix}(b-x)^{n-j}\left(d-y\right)^{j}D^{n}f\left(b,d\right) \\
&+ \frac{1}{n!}(x-a)(y-c)\int_{y}^{d}\int_{x}^{b}\left(t-x\right)^{n}\left(s-y\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right)
\end{aligned} \tag{5.3.12}
$$

respectively, for ny $(x,y)\in Q$.

Finally, by adding the equalities (5.3.9)–(5.3.12) and dividing the sum with $(b-a)\,(d-c)$, we obtain

$$
\begin{aligned}
f\left(x,y\right) = \frac{1}{(b-a)(d-c)}&\left[(b-x)(d-y)f(a,c)+(b-x)(y-c)f(a,d)\right. \\
&\left.+(x-a)(d-y)f(b,c)+(x-a)(y-c)f(b,d)\right] \\
&+\frac{(y-c)(d-y)}{(b-a)(d-c)}\sum_{j=1}^{n}\frac{1}{j!}\begin{pmatrix} n \\ j \end{pmatrix}\left\{(b-x)\left(x-a\right)^{n-j}\left[\left(y-c\right)^{j-1}D^{n}f\left(a,c\right)\right.\right. \\
&\left.+\left(-1\right)^{j}\left(d-y\right)^{j-1}D^{n}f\left(a,d\right)\right]+(x-a)\left(b-x\right)^{n-j}\left[\left(-1\right)^{j}\left(y-c\right)^{j-1}D^{n}f\left(b,c\right)\right. \\
&\left.\left.+\left(d-y\right)^{j-1}D^{n}f\left(b,d\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{n!}\frac{(b-x)(d-y)}{(b-a)\,(d-c)}\int_{c}^{y}\int_{a}^{x}\left(x-t\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right) \\
&+\frac{(-1)^{n+1}}{n!}\frac{(b-x)(y-c)}{(b-a)\,(d-c)}\int_{y}^{d}\int_{a}^{x}\left(x-t\right)^{n}\left(s-y\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right), \\
&+\frac{(-1)^{n+1}}{n!}\frac{(x-a)(d-y)}{(b-a)\,(d-c)}\int_{c}^{y}\int_{x}^{b}\left(t-x\right)^{n}\left(y-s\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right) \\
&+\frac{1}{n!}\frac{(x-a)(y-c)}{(b-a)\,(d-c)}\int_{y}^{d}\int_{x}^{b}\left(t-x\right)^{n}\left(s-y\right)^{n}d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right)
\end{aligned}
$$

which gives the desired representation (5.3.1). $\square$

Remark 5.3.2. *The case $n=0$ provides the representation*

$$f\left(x,y\right)=\frac{1}{(b-a)(d-c)}\left[(b-x)(d-y)f(a,c)+(b-x)(y-c)f(a,d)\right.$$
$$\left.+(x-a)(d-y)f(b,c)+(x-a)(y-c)f(b,d)\right]$$
$$+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{c}^{d}\int_{a}^{b}S_{0}\left(x,t;y,s\right)d_{t}d_{s}\left(f\left(t,s\right)\right)\quad(5.3.13)$$

for any $x\in Q$, where

$$S_{0}\left(x,t;y,s\right)=\begin{cases}\left(b-x\right)\left(d-y\right), & a\le t\le x,\quad c\le s\le y\\ \left(x-a\right)\left(d-y\right), & x<t\le b,\quad c\le s\le y\\ \left(b-x\right)\left(y-c\right), & a\le t\le x,\quad y<s\le d\\ \left(x-a\right)\left(y-c\right), & x<t\le b,\quad y<s\le d\end{cases}$$

and f is of bounded variation on Q.

The above representation provides, as a natural consequence, the possibility to compare the value of a function at the mid point $(\frac{a+b}{2},\frac{c+d}{2})$ with the values of the function and its derivatives at the end points (the corners of the rectangle generated by the end points). Therefore, we can state the following corollary:

Corollary 5.3.3. *With the assumptions of Theorem 5.3.1 for f and Q, we have the identity*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$
$$+\frac{1}{2^{n+2}}\sum_{j=1}^{n}\frac{1}{j!}\begin{pmatrix}n\\ j\end{pmatrix}\left(b-a\right)^{n-j}\left(d-c\right)^{j}\left\{D^{n}f\left(a,c\right)+(-1)^{j}D^{n}f\left(a,d\right)\right.$$
$$\left.+\left(-1\right)^{j}D^{n}f\left(b,c\right)+D^{n}f\left(b,d\right)\right\}$$
$$+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{c}^{d}\int_{a}^{b}M_{n}\left(t,s\right)d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right)\quad(5.3.14)$$

where, $D^{n}f\left(t,s\right)=\frac{\partial^{n}f}{\partial t^{n-j}\partial s^{j}}\left(t,s\right)$ *and*

$$M_{n}\left(t,s\right)=\frac{\left(b-a\right)\left(d-c\right)}{4n!}\begin{cases}\left(\frac{a+b}{2}-t\right)^{n}\left(\frac{c+d}{2}-s\right)^{n}, & a\le t\le\frac{a+b}{2},\quad c\le s\le\frac{c+d}{2}\\ (-1)^{n}\left(t-\frac{a+b}{2}\right)^{n}\left(\frac{c+d}{2}-s\right)^{n}, & \frac{a+b}{2}<t\le b,\quad c\le s\le\frac{c+d}{2}\\ (-1)^{n}\left(\frac{a+b}{2}-t\right)^{n}\left(s-\frac{c+d}{2}\right)^{n}, & a\le t\le\frac{a+b}{2},\quad \frac{c+d}{2}<s\le d\\ \left(t-\frac{a+b}{2}\right)^{n}\left(s-\frac{c+d}{2}\right)^{n}, & \frac{a+b}{2}<t\le b,\quad \frac{c+d}{2}<s\le d\end{cases}$$

On utilizing the following notations

$$\mathcal{A}\left(f,Q\right)=\frac{1}{(b-a)(d-c)}\left[(b-x)(d-y)f(a,c)+(b-x)(y-c)f(a,d)\right.$$
$$\left.+(x-a)(d-y)f(b,c)+(x-a)(y-c)f(b,d)\right]+\frac{(y-c)(d-y)}{(b-a)(d-c)}$$
$$\times\sum_{j=1}^{n}\frac{1}{j!}\begin{pmatrix} n \\ j \end{pmatrix}\Big\{(b-x)\left(x-a\right)^{n-j}\Big[\left(y-c\right)^{j-1}D^{n}f\left(a,c\right)+\left(-1\right)^{j}\left(d-y\right)^{j-1}D^{n}f\left(a,d\right)\Big]$$
$$+\left(x-a\right)\left(b-x\right)^{n-j}\Big[\left(-1\right)^{j}\left(y-c\right)^{j-1}D^{n}f\left(b,c\right)+\left(d-y\right)^{j-1}D^{n}f\left(b,d\right)\Big]\Big\} \tag{5.3.15}$$

and

$$\mathcal{B}_{n}\left(f,Q\right):=\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{c}^{d}\int_{a}^{b}S_{n}\left(x,t;y,s\right)d_{t}d_{s}\left(D^{n}f\left(t,s\right)\right) \tag{5.3.16}$$

under the assumptions of Theorem 5.3.1, we can approximate the function f utilizing the polynomials $\mathcal{A}_{n}\left(f,Q\right)$ with the error $\mathcal{B}_{n}\left(f,Q\right)$. In other words, we have

$$f(x,y)=\mathcal{A}_{n}\left(f,Q\right)+\mathcal{B}_{n}\left(f,Q\right)$$

for any $(x,y)\in Q$.

It is then natural to ask for a priori error bounds provided that f belongs to different classes of functions for which the RiemannStieltjes integral defining the expression in (5.3.16) exists and can be bounded in absolute value.

Theorem 5.3.4. *With the assumptions of Theorem 5.3.1 for f and Q, we have*

$$\begin{aligned}
&\left|\mathcal{B}_{n}\left(f,Q\right)\right| \\
&\leq\frac{\left(x-a\right)\left(b-x\right)\left(y-c\right)\left(d-y\right)}{n!\left(b-a\right)\left(d-c\right)} \\
&\qquad\times\Bigg[\left(x-a\right)^{n-1}\left(y-c\right)^{n-1}\cdot\bigvee_{c}^{y}\bigvee_{a}^{x}\left(D^{n}f\right)\left(x-a\right)^{n-1}\left(d-y\right)^{n-1}\cdot\bigvee_{y}^{d}\bigvee_{a}^{x}\left(D^{n}f\right) \\
&\qquad+\left(b-x\right)^{n-1}\left(y-c\right)^{n-1}\cdot\bigvee_{c}^{y}\bigvee_{x}^{b}\left(D^{n}f\right)+\left(b-x\right)^{n-1}\left(d-y\right)^{n-1}\cdot\bigvee_{y}^{d}\bigvee_{x}^{b}\left(D^{n}f\right)\Bigg]
\end{aligned} \tag{5.3.17}$$

$$\leq \frac{(x-a)(b-x)(y-c)(d-y)}{n!(b-a)(d-c)}$$

$$\times \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]^{n-1}\cdot\bigvee_c^d\bigvee_a^b(D^n f); \\ \\ \left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{1/p}\cdot\left[(y-c)^{p(n-1)}+(d-y)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(\bigvee_c^y\bigvee_a^x(f)\right)^q+\left(\bigvee_y^d\bigvee_a^x(f)\right)^q+\left(\bigvee_c^y\bigvee_x^b(f)\right)^q+\left(\bigvee_y^d\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \hfill p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max\left\{\bigvee_c^y\bigvee_a^x(D^n f),\bigvee_y^d\bigvee_a^x(D^n f),\bigvee_c^y\bigvee_x^b(D^n f),\bigvee_y^d\bigvee_x^b(D^n f)\right\} \\ \times\left[(x-a)^{n-1}+(b-x)^{n-1}\right]\cdot\left[(y-c)^{n-1}+(d-y)^{n-1}\right]. \end{cases} \tag{5.3.18}$$

$$\leq \frac{(b-a)(d-c)}{16n!}$$

$$\times \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]^{n-1}\cdot\bigvee_c^d\bigvee_a^b(D^n f); \\ \\ \left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{1/p}\cdot\left[(y-c)^{p(n-1)}+(d-y)^{p(n-1)}\right]^{1/p} \\ \times\left[\left(\bigvee_c^y\bigvee_a^x(f)\right)^q+\left(\bigvee_y^d\bigvee_a^x(f)\right)^q+\left(\bigvee_c^y\bigvee_x^b(f)\right)^q+\left(\bigvee_y^d\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \hfill p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \max\left\{\bigvee_c^y\bigvee_a^x(D^n f),\bigvee_y^d\bigvee_a^x(D^n f),\bigvee_c^y\bigvee_x^b(D^n f),\bigvee_y^d\bigvee_x^b(D^n f)\right\} \\ \times\left[(x-a)^{n-1}+(b-x)^{n-1}\right]\cdot\left[(y-c)^{n-1}+(d-y)^{n-1}\right]. \end{cases} \tag{5.3.19}$$

Proof. Using the inequality for the Riemann–Stieltjes integral of continuous integrands and bounded variation integrators, we have

$$
\begin{aligned}
\left|\mathcal{B}_n(f,Q)\right| &= \left|\frac{1}{n!\,(b-a)\,(d-c)}\left[\int_c^y\int_a^x (x-t)^n(b-x)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right.\right.\\
&\quad+\int_y^d\int_a^x (-1)^{n+1}(x-t)^n(b-x)(s-y)^n(y-c)d_td_s\left(D^nf(t,s)\right)\\
&\quad+\int_c^y\int_x^b (-1)^{n+1}(t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\\
&\quad\left.\left.+\int_y^d\int_x^b (t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right]\right|\\
&\le \frac{1}{n!\,(b-a)\,(d-c)}\left[\left|\int_c^y\int_a^x (x-t)^n(b-x)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right|\right.\\
&\quad+\left|\int_y^d\int_a^x (-1)^{n+1}(x-t)^n(b-x)(s-y)^n(y-c)d_td_s\left(D^nf(t,s)\right)\right|\\
&\quad+\left|\int_c^y\int_x^b (-1)^{n+1}(t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right|\\
&\quad\left.+\left|\int_y^d\int_x^b (t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right|\right]\\
&\le \frac{1}{n!\,(b-a)\,(d-c)}\left[\max_{\substack{t\in[a,x]\\ s\in[c,y]}}\left\{(x-t)^n(b-x)(y-s)^n(d-y)\right\}\cdot\bigvee_c^y\bigvee_a^x\left(D^nf\right)\right.\\
&\quad+\max_{\substack{t\in[a,x]\\ s\in[y,d]}}\left\{(x-t)^n(b-x)(s-y)^n(y-c)\right\}\cdot\bigvee_y^d\bigvee_a^x\left(D^nf\right)\\
&\quad+\max_{\substack{t\in[x,b]\\ s\in[c,y]}}\left\{(t-x)^n(x-a)(y-s)^n(d-y)\right\}\cdot\bigvee_c^y\bigvee_x^b\left(D^nf\right)\\
&\quad\left.+\max_{\substack{t\in[x,b]\\ s\in[y,d]}}\left\{(t-x)^n(x-a)(s-y)^n(y-c)\right\}\cdot\bigvee_y^d\bigvee_x^b\left(D^nf\right)\right]\\
&\le \frac{1}{n!\,(b-a)\,(d-c)}\left[(x-a)^n(b-x)(y-c)^n(d-y)\cdot\bigvee_c^y\bigvee_a^x\left(D^nf\right)\right.\\
&\quad+(x-a)^n(b-x)(d-y)^n(y-c)\cdot\bigvee_y^d\bigvee_a^x\left(D^nf\right)\\
&\quad+(b-x)^n(x-a)(y-c)^n(d-y)\cdot\bigvee_c^y\bigvee_x^b\left(D^nf\right)\\
&\quad\left.+(b-x)^n(x-a)(d-y)^n(y-c)\cdot\bigvee_y^d\bigvee_x^b\left(D^nf\right)\right]
\end{aligned}
$$

$$= \frac{(x-a)(b-x)(y-c)(d-y)}{n!\,(b-a)(d-c)} \Bigg[(x-a)^{n-1}(y-c)^{n-1} \cdot \bigvee_c^y \bigvee_a^x (D^n f)$$
$$+ (x-a)^{n-1}(d-y)^{n-1} \cdot \bigvee_y^d \bigvee_a^x (D^n f) + (b-x)^{n-1}(y-c)^{n-1} \cdot \bigvee_c^y \bigvee_x^b (D^n f)$$
$$+ (b-x)^{n-1}(d-y)^{n-1} \cdot \bigvee_y^d \bigvee_x^b (D^n f) \Bigg]$$

and the first inequality in (5.3.17) is proved.

However, by Hölder's discrete inequality we also have

$$(x-a)^{n-1}(y-c)^{n-1} \cdot \bigvee_c^y \bigvee_a^x (D^n f) + (x-a)^{n-1}(d-y)^{n-1} \cdot \bigvee_y^d \bigvee_a^x (D^n f)$$
$$+ (b-x)^{n-1}(y-c)^{n-1} \cdot \bigvee_c^y \bigvee_x^b (D^n f) + (b-x)^{n-1}(d-y)^{n-1} \cdot \bigvee_y^d \bigvee_x^b (D^n f)$$

$$\leq \begin{cases} \max\left\{(x-a)^{n-1}(y-c)^{n-1}, (x-a)^{n-1}(d-y)^{n-1}, (b-x)^{n-1}(y-c)^{n-1},\right. \\ \left.(b-x)^{n-1}(d-y)^{n-1}\right\} \times \left[\bigvee_c^y \bigvee_a^x (D^n f) + \bigvee_y^d \bigvee_a^x (D^n f) + \bigvee_c^y \bigvee_x^b (D^n f)\right. \\ \left. + \bigvee_y^d \bigvee_x^b (D^n f)\right]; \\ \\ \left[(x-a)^{p(n-1)}(y-c)^{p(n-1)} + (x-a)^{p(n-1)}(d-y)^{p(n-1)}\right. \\ \left. + (b-x)^{p(n-1)}(y-c)^{p(n-1)} + (b-x)^{p(n-1)}(d-y)^{p(n-1)}\right]^{1/p} \\ \times \left[(\bigvee_c^y \bigvee_a^x (f))^q + \left(\bigvee_y^d \bigvee_a^x (f)\right)^q + \left(\bigvee_c^y \bigvee_x^b (f)\right)^q + \left(\bigvee_y^d \bigvee_x^b (f)\right)^q\right]^{1/q}; \\ \qquad\qquad\qquad\qquad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \max\left\{\bigvee_c^y \bigvee_a^x (D^n f), \bigvee_y^d \bigvee_a^x (D^n f), \bigvee_c^y \bigvee_x^b (D^n f), \bigvee_y^d \bigvee_x^b (D^n f)\right\} \\ \times \left[(x-a)^{n-1}(y-c)^{n-1} + (x-a)^{n-1}(d-y)^{n-1} + (b-x)^{n-1}(y-c)^{n-1}\right. \\ \left. + (b-x)^{n-1}(d-y)^{n-1}\right]; \end{cases}$$

$$= \begin{cases} \max\left\{(x-a)^{n-1},(b-x)^{n-1}\right\}\cdot\max\left\{(y-c)^{n-1},(d-y)^{n-1}\right\}\cdot\bigvee_c^d\bigvee_a^b(D^nf)\,; \\ \\ \left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{1/p}\cdot\left[(y-c)^{p(n-1)}+(d-y)^{p(n-1)}\right]^{1/p} \\ \times\left[(\bigvee_c^y\bigvee_a^x(f))^q+\left(\bigvee_y^d\bigvee_a^x(f)\right)^q+\left(\bigvee_c^y\bigvee_x^b(f)\right)^q+\left(\bigvee_y^d\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \hfill p,q>1,\frac{1}{p}+\frac{1}{q}=1; \\ \\ \max\left\{\bigvee_c^y\bigvee_a^x(D^nf),\bigvee_y^d\bigvee_a^x(D^nf),\bigvee_c^y\bigvee_x^b(D^nf),\bigvee_y^d\bigvee_x^b(D^nf)\right\} \\ \times\left[(x-a)^{n-1}+(b-x)^{n-1}\right]\cdot\left[(y-c)^{n-1}+(d-y)^{n-1}\right]. \end{cases}$$

$$= \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]^{n-1}\cdot\bigvee_c^d\bigvee_a^b(D^nf)\,; \\ \\ \left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{1/p}\cdot\left[(y-c)^{p(n-1)}+(d-y)^{p(n-1)}\right]^{1/p} \\ \times\left[(\bigvee_c^y\bigvee_a^x(f))^q+\left(\bigvee_y^d\bigvee_a^x(f)\right)^q+\left(\bigvee_c^y\bigvee_x^b(f)\right)^q+\left(\bigvee_y^d\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \hfill p,q>1,\frac{1}{p}+\frac{1}{q}=1; \\ \\ \max\left\{\bigvee_c^y\bigvee_a^x(D^nf),\bigvee_y^d\bigvee_a^x(D^nf),\bigvee_c^y\bigvee_x^b(D^nf),\bigvee_y^d\bigvee_x^b(D^nf)\right\} \\ \times\left[(x-a)^{n-1}+(b-x)^{n-1}\right]\cdot\left[(y-c)^{n-1}+(d-y)^{n-1}\right]. \end{cases}$$

which proves the second inequality in (5.3.18).

The last part is obvious by the elementary inequalities

$$(x-a)(b-x)\le\frac{1}{4}(b-a)^2,\forall x\in[a,b]$$

and

$$(y-c)(d-y)\le\frac{1}{4}(d-c)^2,\forall y\in[c,d].$$

The proof is complete. □

Remark 5.3.5. *Under the assumptions of Theorem 5.3.4 for f and Q, with the case $n=0$ provides the following inequality:*

$$|\mathcal{B}_n(f,Q)|\le\frac{1}{(b-a)(d-c)}\left[(b-x)(d-y)\cdot\bigvee_c^y\bigvee_a^x(f)+(b-x)(y-c)\cdot\bigvee_y^d\bigvee_a^x(f)\right.$$
$$\left.+(x-a)(d-y)\cdot\bigvee_c^y\bigvee_x^b(f)+(x-a)(y-c)\cdot\bigvee_y^d\bigvee_x^b(f)\right] \quad (5.3.20)$$

Now, if we denote

$$\mathcal{E}_M(f;Q) = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$
$$+\frac{1}{2^{n+2}}\sum_{j=1}^{n}\frac{1}{j!}\left(\begin{array}{c} n \\ j \end{array}\right)(b-a)^{n-j}(d-c)^j\left\{D^n f(a,c)+(-1)^j D^n f(a,d)\right.$$
$$\left. + (-1)^j D^n f(b,c) + D^n f(b,d)\right\} \quad (5.3.21)$$

and

$$\mathcal{F}_M(f;Q) = \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b M_n(t,s)\, d_t d_s\left(D^n f(t,s)\right) \quad (5.3.22)$$

where,

$$M_n(t,s) = \frac{(b-a)(d-c)}{4n!}$$
$$\times\begin{cases} \left(\frac{a+b}{2}-t\right)^n\left(\frac{c+d}{2}-s\right)^n, & a\le t\le\frac{a+b}{2},\quad c\le s\le\frac{c+d}{2} \\ (-1)^{n+1}\left(t-\frac{a+b}{2}\right)^n\left(\frac{c+d}{2}-s\right)^n, & \frac{a+b}{2}<t\le b,\quad c\le s\le\frac{c+d}{2} \\ (-1)^{n+1}\left(\frac{a+b}{2}-t\right)^n\left(s-\frac{c+d}{2}\right)^n, & a\le t\le\frac{a+b}{2},\quad \frac{c+d}{2}<s\le d \\ \left(t-\frac{a+b}{2}\right)^n\left(s-\frac{c+d}{2}\right)^n, & \frac{a+b}{2}<t\le b,\quad \frac{c+d}{2}<s\le d \end{cases}$$

then we can approximate the value of the function at the midpoint in terms of the values of the function and its partial derivatives taken at the end points with the error $\mathcal{F}_M(f;Q)$. Namely, we have the representation formula

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) = \mathcal{E}_M(f;Q)+\mathcal{F}_M(f;Q)$$

The absolute value of the error can be bounded as follows:

Corollary 5.3.6. *With the assumptions of Theorem 5.3.4 for f, Q and n, we have the inequality*

$$|\mathcal{F}_M(f;Q)| \le \frac{(b-a)^n(d-c)^n}{2^{2n+2}n!}\cdot\bigvee_c^d\bigvee_a^b(f) \quad (5.3.23)$$

The following result concerning Lipschitz mappings may be stated as well:

Theorem 5.3.7. *Let $Q := I\times J$ be a closed rectangle on $\mathbb{R}^2$, let $a,b\in I$ with $a<b$, $c,d\in J$ with $c<d$ and let n be a nonnegative integer. If $f:Q\to\mathbb{R}$ is such that the n-th*

partial derivatives $D^n f$ is L_1–Lipschitz on $[a,x]\times[c,y]$, L_2–Lipschitz on $[a,x]\times[y,d]$, L_3–Lipschitz on $[x,b]\times[c,y]$ and L_4–Lipschitz on $[x,b]\times[y,d]$, then we have

$$\begin{aligned}
\left|\mathcal{B}_n(f,Q)\right| &\leq \frac{(b-x)(x-a)(d-y)(y-c)}{n!\,(n+1)^2\,(b-a)(d-c)}\left\{(x-a)^n\cdot\left[L_1(y-c)^n+L_2(d-y)^n\right]\right.\\
&\qquad\left.+(b-x)^n\cdot\left[L_3(y-c)^n+L_4(d-y)^n\right]\right\}\\
&\leq \frac{(b-a)(d-c)}{n!16\,(n+1)^2}\left\{(x-a)^n\cdot\left[L_1(y-c)^n+L_2(d-y)^n\right]\right.\\
&\qquad\left.+(b-x)^n\cdot\left[L_3(y-c)^n+L_4(d-y)^n\right]\right\}
\end{aligned}\tag{5.3.24}$$

Proof. Since $D^n f$ is L–Lipschitz on Q, then by 3.2.5, we have

$$\begin{aligned}
\left|\mathcal{B}_n(f,Q)\right| &= \left|\frac{1}{n!\,(b-a)(d-c)}\left[\int_c^y\int_a^x (x-t)^n(b-x)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right.\right.\\
&\quad+\int_y^d\int_a^x(-1)^{n+1}(x-t)^n(b-x)(s-y)^n(y-c)d_td_s\left(D^nf(t,s)\right)\\
&\quad+\int_c^y\int_x^b(-1)^{n+1}(t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\\
&\quad\left.\left.+\int_y^d\int_x^b(t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right]\right|\\
&\leq \frac{1}{n!\,(b-a)(d-c)}\left[\left|\int_c^y\int_a^x (x-t)^n(b-x)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right|\right.\\
&\quad+\left|\int_y^d\int_a^x(-1)^{n+1}(x-t)^n(b-x)(s-y)^n(y-c)d_td_s\left(D^nf(t,s)\right)\right|\\
&\quad+\left|\int_c^y\int_x^b(-1)^{n+1}(t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right|\\
&\quad\left.+\left|\int_y^d\int_x^b(t-x)^n(x-a)(y-s)^n(d-y)d_td_s\left(D^nf(t,s)\right)\right|\right]\\
&\leq \frac{1}{n!\,(b-a)(d-c)}\left[L_1\int_c^y\int_a^x\left|(x-t)^n(b-x)(y-s)^n(d-y)\right|dtds\right.\\
&\quad+L_2\int_y^d\int_a^x\left|(-1)^{n+1}(x-t)^n(b-x)(s-y)^n(y-c)\right|dtds\\
&\quad+L_3\int_c^y\int_x^b\left|(-1)^{n+1}(t-x)^n(x-a)(y-s)^n(d-y)\right|dtds\\
&\quad\left.+L_4\int_y^d\int_x^b\left|(t-x)^n(x-a)(y-s)^n(d-y)\right|dtds\right]
\end{aligned}$$

$$= \frac{1}{n!\,(b-a)\,(d-c)} \Big[L_1 \,(b-x)\,(d-y) \int_c^y (y-s)^n \,ds \int_a^x (x-t)^n \,dt$$

$$+ L_2 \,(b-x)\,(y-c) \int_y^d (s-y)^n \,ds \int_a^x (x-t)^n \,dt$$

$$+ L_3 \,(x-a)\,(d-y) \int_c^y (y-s)^n \,ds \int_x^b (t-x)^n \,dt$$

$$+ L_4 \,(x-a)\,(d-y) \int_y^d (y-s)^n \,ds \int_x^b (t-x)^n \,dt \Big]$$

$$= \frac{(b-x)\,(x-a)\,(d-y)\,(y-c)}{n!\,(n+1)^2\,(b-a)\,(d-c)} \left\{ (x-a)^n \cdot \left[L_1 \,(y-c)^n + L_2 \,(d-y)^n \right] \right.$$

$$\left. + (b-x)^n \cdot \left[L_3 \,(y-c)^n + L_4 \,(d-y)^n \right] \right\}$$

which proves the first inequality in (5.3.24). The last part follows by elementary inequalities

$$(x-a)\,(b-x) \le \frac{1}{4}\,(b-a)^2, \forall x \in [a,b],$$

and

$$(y-c)\,(d-y) \le \frac{1}{4}\,(d-c)^2, \forall y \in [c,d],$$

the details are omitted. The proof is complete. □

Remark 5.3.8. *If the function $D^n f$ is L–Lipschitzian on the whole bidimentional interval $[a,b] \times [c,d]$, which, in fact, is a more natural assumption, then we get from (5.3.24) that*

$$|\mathcal{B}_n\,(f,Q)| \le L \frac{(b-x)\,(x-a)\,(d-y)\,(y-c)}{n!\,(n+1)^2\,(b-a)\,(d-c)} \cdot \left[(x-a)^n + (b-x)^n \right]$$

$$\cdot \left[(y-c)^n + (d-y)^n \right]$$

$$\le L \frac{(b-a)\,(d-c)}{n!16\,(n+1)^2} \times \left[(x-a)^n + (b-x)^n \right] \cdot \left[(y-c)^n + (d-y)^n \right] \quad (5.3.25)$$

Corollary 5.3.9. *Let $Q := I \times J$ be a closed rectangle on $\mathbb{R}^2$, let $a,b \in I$ with $a<b$, $c,d \in J$ with $c<d$ and let n be a nonnegative integer. If $f : Q \to \mathbb{R}$ is such that the n-th partial derivatives $D^n f$ is L_1–Lipschitz on $[a, \frac{a+b}{2}] \times [c, \frac{c+d}{2}]$, L_2–Lipschitz on $[a, \frac{a+b}{2}] \times [\frac{c+d}{2}, d]$, L_3–Lipschitz on $[\frac{a+b}{2}, b] \times [c, \frac{c+d}{2}]$ and L_4–Lipschitz on $[\frac{a+b}{2}, b] \times [\frac{c+d}{2}, d]$, then we have*

$$|\mathcal{F}_M\,(f,Q)| \le \frac{(d-c)^{n+1}\,(b-a)^{n+1}}{n! 2^{2n+4}\,(n+1)^2} \cdot \left[L_1 + L_2 + L_3 + L_4 \right] \quad (5.3.26)$$

In particular, if $D^n f$ is L–Lipschitzian on Q, then

$$\left|\mathcal{F}_M(f,Q)\right| \leq L\frac{(d-c)^{n+1}(b-a)^{n+1}}{n!2^{2n+2}(n+1)^2} \tag{5.3.27}$$

Remark 5.3.10. *In Section 4.3, we have considered an integral representation of error for mappings of two variables. Similarly, one may consider another more accurate error representation using Taylor representation formula with $n=0$ by defining $\psi_f(t,s)$ such as for a function $f: Q_{a,c}^{b,d} \to \mathbb{R}$, we define $\phi_f, \psi_f : Q_{a,c}^{b,d} \to \mathbb{R}$ by*

$$\begin{aligned}\phi_f(t,s) :&= [(b-t)(d-s)f(a,c)+(b-t)(s-c)f(a,d)\\ &\quad +(t-a)(d-s)f(b,c)+(t-a)(s-c)f(b,d)]\end{aligned}$$

and

$$\psi_f(t,s) := f(t,s) - \frac{\phi_f(t,s)}{(b-a)(d-c)}.$$

A generalization of this error may be extended to be for $n=k$, provided that the k-th partial derivatives $D^k f$ exist, as we shown above.

Finally, the case when $D^n f$ is absolutely continuous on Q produces the following estimates for the remainder:

Theorem 5.3.11. *Let $Q := I \times J$ be a closed rectangle on $\mathbb{R}^2$, let $a,b \in I$ with $a<b$, $c,d \in J$ with $c<d$ and let n be a nonnegative integer. If $f: Q \to \mathbb{R}$ is such that the n-th partial derivatives $D^n f$ is absolutely continuous on Q, then for any $(x,y) \in Q$ we have*

$$\begin{aligned}\left|\mathcal{B}_n(f,Q)\right| \leq \frac{1}{n!(b-a)(d-c)} \Bigg[&(b-x)(d-y)\int_c^y \int_a^x (x-t)^n (y-s)^n \left|D^{n+1} f(t,s)\right| dtds\\ &+ (y-c)(b-x)\int_y^d \int_a^x (x-t)^n (s-y)^n \left|D^{n+1} f(t,s)\right| dtds\\ &+ (d-y)(x-a)\int_c^y \int_x^b (t-x)^n (y-s)^n \left|D^{n+1} f(t,s)\right| dtds\\ &+ (d-y)(x-a)\int_y^d \int_x^b (t-x)^n (y-s)^n \left|D^{n+1} f(t,s)\right| dtds \Bigg]\end{aligned}$$

$$
\leq \frac{(b-x)(d-y)}{n!\,(b-a)(d-c)} \times \begin{cases} \frac{(x-a)^{n+1}(y-c)^{n+1}}{(n+1)^2} \cdot \left\|D^{n+1}f\right\|_{[a,x]\times[c,y],\infty}, & D^{n+1}f \in L_\infty(Q_{a,c}^{x,y}); \\ \frac{(x-a)^{n+1/q}(y-c)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\|D^{n+1}f\right\|_{[a,x]\times[c,y],p}, & D^{n+1}f \in L_p(Q_{a,c}^{x,y}), \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)^n (y-c)^n \cdot \left\|D^{n+1}f\right\|_{[a,x]\times[c,y],1}, & D^{n+1}f \in L_1(Q_{a,c}^{x,y}); \end{cases}
$$

$$
+\frac{(b-x)(y-c)}{n!\,(b-a)(d-c)} \times \begin{cases} \frac{(x-a)^{n+1}(d-y)^{n+1}}{(n+1)^2} \cdot \left\|D^{n+1}f\right\|_{[a,x]\times[y,d],\infty}, & D^{n+1}f \in L_\infty(Q_{a,y}^{x,d}); \\ \frac{(x-a)^{n+1/q}(d-y)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\|D^{n+1}f\right\|_{[a,x]\times[y,d],p}, & D^{n+1}f \in L_p(Q_{a,y}^{x,d}), \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)^n (d-y)^n \cdot \left\|D^{n+1}f\right\|_{[a,x]\times[c,y],1}, & D^{n+1}f \in L_1(Q_{a,y}^{x,d}); \end{cases}
$$

$$
+\frac{(x-a)(d-y)}{n!\,(b-a)(d-c)} \times \begin{cases} \frac{(b-x)^{n+1}(y-c)^{n+1}}{(n+1)^2} \cdot \left\|D^{n+1}f\right\|_{[x,b]\times[c,y],\infty}, & D^{n+1}f \in L_\infty(Q_{x,c}^{b,y}); \\ \frac{(b-x)^{n+1/q}(y-c)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\|D^{n+1}f\right\|_{[x,b]\times[c,y],p}, & D^{n+1}f \in L_p(Q_{x,c}^{b,y}), \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (b-x)^n (y-c)^n \cdot \left\|D^{n+1}f\right\|_{[x,b]\times[c,y],1}, & D^{n+1}f \in L_1(Q_{x,c}^{b,y}); \end{cases}
$$

$$
+\frac{(x-a)(y-c)}{n!\,(b-a)(d-c)} \times \begin{cases} \frac{(b-x)^{n+1}(d-y)^{n+1}}{(n+1)^2} \cdot \left\|D^{n+1}f\right\|_{[x,b]\times[y,d],\infty}, & D^{n+1}f \in L_\infty(Q_{x,y}^{b,d}); \\ \frac{(b-x)^{n+1/q}(d-y)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\|D^{n+1}f\right\|_{[x,b]\times[y,d],p}, & D^{n+1}f \in L_p(Q_{x,y}^{b,d}), \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (b-x)^n (d-y)^n \cdot \left\|D^{n+1}f\right\|_{[x,b]\times[y,d],1}, & D^{n+1}f \in L_1(Q_{x,y}^{b,d}); \end{cases}
\tag{5.3.28}
$$

Proof. Since $D^n f$ is absolutely continuous on Q then for any $(x,y) \in Q$ we have the representation

$$f(x,y) = \frac{1}{(b-a)(d-c)}\left[(b-x)(d-y)f(a,c) + (b-x)(y-c)f(a,d)\right.$$
$$\left. +(x-a)(d-y)f(b,c) + (x-a)(y-c)f(b,d)\right] + \frac{(y-c)(d-y)}{(b-a)(d-c)}$$
$$\times \sum_{j=1}^{n} \frac{1}{j!}\begin{pmatrix} n \\ j \end{pmatrix}\left\{(b-x)(x-a)^{n-j}\left[(y-c)^{j-1}D^{n}f(a,c) + (-1)^{j}(d-y)^{j-1}D^{n}f(a,d)\right]\right.$$
$$\left. + (x-a)(b-x)^{n-j}\left[(-1)^{j}(y-c)^{j-1}D^{n}f(b,c) + (d-y)^{j-1}D^{n}f(b,d)\right]\right\}$$
$$+ \frac{1}{(b-a)(d-c)}\int_{c}^{d}\int_{a}^{b} S_{n}(x,t;y,s)\, D^{n+1}f(t,s)\, dtds \quad (5.3.29)$$

where the integral is considered in the Lebesgue sense and the kernel $S_n(x,t;y,s)$ is given in Theorem 5.3.1.

Utilizing the properties of the Stieltjes integral, we have

$$|\mathcal{B}_n(f;Q)| = \frac{1}{(b-a)(d-c)}\left|\int_{c}^{d}\int_{a}^{b} S_{n}(x,t;y,s)\, D^{n+1}f(t,s)\, dtds\right|$$
$$= \left|\frac{1}{n!(b-a)(d-c)}\left[\int_{c}^{y}\int_{a}^{x}(x-t)^{n}(b-x)(y-s)^{n}(d-y)D^{n+1}f(t,s)\,dtds\right.\right.$$
$$+ \int_{y}^{d}\int_{a}^{x}(-1)^{n+1}(x-t)^{n}(b-x)(s-y)^{n}(y-c)D^{n+1}f(t,s)\,dtds$$
$$+ \int_{c}^{y}\int_{x}^{b}(-1)^{n+1}(t-x)^{n}(x-a)(y-s)^{n}(d-y)D^{n+1}f(t,s)\,dtds$$
$$\left.\left. + \int_{y}^{d}\int_{x}^{b}(t-x)^{n}(x-a)(y-s)^{n}(d-y)D^{n+1}f(t,s)\,dtds\right]\right|$$
$$\leq \frac{1}{n!(b-a)(d-c)}\left[\left|\int_{c}^{y}\int_{a}^{x}(x-t)^{n}(b-x)(y-s)^{n}(d-y)D^{n+1}f(t,s)\,dtds\right|\right.$$
$$+ \left|\int_{y}^{d}\int_{a}^{x}(-1)^{n+1}(x-t)^{n}(b-x)(s-y)^{n}(y-c)D^{n+1}f(t,s)\,dtds\right|$$
$$+ \left|\int_{c}^{y}\int_{x}^{b}(-1)^{n+1}(t-x)^{n}(x-a)(y-s)^{n}(d-y)D^{n+1}f(t,s)\,dtds\right|$$
$$\left. + \left|\int_{y}^{d}\int_{x}^{b}(t-x)^{n}(x-a)(y-s)^{n}(d-y)D^{n+1}f(t,s)\,dtds\right|\right]$$

$$\leq \frac{1}{n!\,(b-a)\,(d-c)} \left[(b-x)\,(d-y) \int_c^y \int_a^x (x-t)^n\,(y-s)^n \left| D^{n+1} f\,(t,s) \right| dtds \right.$$
$$+ (y-c)\,(b-x) \int_y^d \int_a^x (x-t)^n\,(s-y)^n \left| D^{n+1} f\,(t,s) \right| dtds$$
$$+ (d-y)\,(x-a) \int_c^y \int_x^b (t-x)^n\,(y-s)^n \left| D^{n+1} f\,(t,s) \right| dtds$$
$$\left. + (d-y)\,(x-a) \int_y^d \int_x^b (t-x)^n\,(y-s)^n \left| D^{n+1} f\,(t,s) \right| dtds \right]$$

and the first part of the inequality (5.3.28) is proved.

Utilizing Hölder integral inequality for the Lebesgue integral we have

$$\int_c^y \int_a^x (x-t)^n\,(y-s)^n \left| D^{n+1} f\,(t,s) \right| dtds$$
$$= \begin{cases} \frac{(x-a)^{n+1}(y-c)^{n+1}}{(n+1)^2} \cdot \left\| D^{n+1} f \right\|_{[a,x]\times[c,y],\infty}, & D^{n+1} f \in L_\infty\,[a,x]\times[c,y]\,; \\ \frac{(x-a)^{n+1/q}(y-c)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\| D^{n+1} f \right\|_{[a,x]\times[c,y],p}, & D^{n+1} f \in L_p\,[a,x]\times[c,y]\,, \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)^n\,(y-c)^n \cdot \left\| D^{n+1} f \right\|_{[a,x]\times[c,y],1}, & D^{n+1} f \in L_1\,[a,x]\times[c,y]\,; \end{cases} \tag{5.3.30}$$

$$\int_y^d \int_a^x (x-t)^n\,(s-y)^n \left| D^{n+1} f\,(t,s) \right| dtds$$
$$= \begin{cases} \frac{(x-a)^{n+1}(d-y)^{n+1}}{(n+1)^2} \cdot \left\| D^{n+1} f \right\|_{[a,x]\times[y,d],\infty}, & D^{n+1} f \in L_\infty\,[a,x]\times[y,d]\,; \\ \frac{(x-a)^{n+1/q}(d-y)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\| D^{n+1} f \right\|_{[a,x]\times[y,d],p}, & D^{n+1} f \in L_p\,[a,x]\times[y,d]\,, \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)^n\,(d-y)^n \cdot \left\| D^{n+1} f \right\|_{[a,x]\times[y,d],1}, & D^{n+1} f \in L_1\,[a,x]\times[y,d] \end{cases} \tag{5.3.31}$$

$$\int_c^y \int_x^b (t-x)^n (y-s)^n \left|D^{n+1} f(t,s)\right| dtds$$
$$= \begin{cases} \frac{(b-x)^{n+1}(y-c)^{n+1}}{(n+1)^2} \cdot \left\|D^{n+1} f\right\|_{[x,b]\times[c,y],\infty}, & D^{n+1} f \in L_\infty [x,b]\times[c,y]; \\ \frac{(b-x)^{n+1/q}(y-c)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\|D^{n+1} f\right\|_{[x,b]\times[c,y],p}, & D^{n+1} f \in L_p [x,b]\times[c,y], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (b-x)^n (y-c)^n \cdot \left\|D^{n+1} f\right\|_{[x,b]\times[c,y],1}, & D^{n+1} f \in L_1 [x,b]\times[c,y]; \end{cases} \tag{5.3.32}$$

$$\int_y^d \int_x^b (t-x)^n (y-s)^n \left|D^{n+1} f(t,s)\right| dtds$$
$$= \begin{cases} \frac{(b-x)^{n+1}(d-y)^{n+1}}{(n+1)^2} \cdot \left\|D^{n+1} f\right\|_{[x,b]\times[y,d],\infty}, & D^{n+1} f \in L_\infty [x,b]\times[y,d]; \\ \frac{(b-x)^{n+1/q}(d-y)^{n+1/q}}{(nq+1)^{1/q}} \cdot \left\|D^{n+1} f\right\|_{[x,b]\times[y,d],p}, & D^{n+1} f \in L_p [x,b]\times[y,d], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (b-x)^n (d-y)^n \cdot \left\|D^{n+1} f\right\|_{[x,b]\times[y,d],1}, & D^{n+1} f \in L_1 [x,b]\times[y,d] \end{cases} \tag{5.3.33}$$

On making use of (5.3.30)–(5.3.33) we deduce the second part of (5.3.28). □

5.4 TRAPEZOID-TYPE RULES FOR $\mathcal{RS}$-DOUBLE INTEGRALS

Let $f, u : Q \to \mathbb{R}$ be such that $\int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s)$ exists. Define

$$T(f,u;Q)$$
$$:= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\left[u(b,d)-u(b,c)-u(a,d)+u(a,c)\right]$$
$$- \int_c^d \int_a^b f(t,s)\, d_t d_s u(t,s).$$

For integrators of bounded variation, the following result holds:

Theorem 5.4.1. *Let $f, u : Q \to \mathbb{R}$ be such that f is (α_1, α_2)-(H_1, H_2)–Hölder type mapping, where $H_1, H_2 > 0$ and $\alpha_1, \alpha_2 > 0$ are given, and u is a mapping of bounded variation on Q. Then we have the inequality*

$$|T(f,u;Q)| \le \left[H_1 \left(\frac{b-a}{2}\right)^{\alpha_1} + H_2 \left(\frac{d-c}{2}\right)^{\alpha_2}\right] \cdot \bigvee_Q (u). \tag{5.4.1}$$

Proof. Using the inequality for the Riemann–Stieltjes integral of continuous integrands and bounded variation integrators, we have

$$\begin{aligned}|T(f,u;Q)| &= \left|\int_c^d \int_a^b \left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right] d_t d_s u(t,s)\right| \\ &\le \sup\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right| \cdot \bigvee_Q (u).\end{aligned} \tag{5.4.2}$$

As f is of (α_1,α_2)-(H_1,H_2)–Hölder type mapping, then we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right| \\ &\le \frac{1}{4}\left[|f(a,c)-f(t,s)|+|f(a,d)-f(t,s)|+|f(b,c)-f(t,s)|+|f(b,d)-f(t,s)|\right] \\ &\le \frac{1}{4}\left\{\left[H_1(t-a)^{\alpha_1}+H_2(s-c)^{\alpha_2}\right]+\left[H_1(t-a)^{\alpha_1}+H_2(d-s)^{\alpha_2}\right]\right. \\ &\qquad \left.+\left[H_1(b-t)^{\alpha_1}+H_2(s-c)^{\alpha_2}\right]+\left[H_1(b-t)^{\alpha_1}+H_2(d-s)^{\alpha_2}\right]\right\} \\ &= \frac{H_1}{2}\left[(t-a)^{\alpha_1}+(b-t)^{\alpha_1}\right]+\frac{H_2}{2}\left[(s-c)^{\alpha_2}+(d-s)^{\alpha_2}\right],\end{aligned} \tag{5.4.3}$$

for any $t\in[a,b]$ and $s\in[c,d]$.

Now, consider the mapping $\gamma_1:[a,b]\to\mathbb{R}$, given by $\gamma_1(t):=(t-a)^{\alpha_1}+(b-t)^{\alpha_1}$, $t\in[a,b]$, $\alpha_1\in(0,1]$. Then, $\gamma_1'(t):=\alpha_1(t-a)^{\alpha_1-1}-\alpha_1(b-t)^{\alpha_1-1}$ iff $t=\frac{a+b}{2}$, $\gamma_1'(t)>0$ on $(a,\frac{a+b}{2})$ and $\gamma_1'(t)<0$ on $(\frac{a+b}{2},b)$, which shows that its maximum is realized at $t=\frac{a+b}{2}$ and

$$\max_{t\in[a,b]}\{\gamma_1(t)\}=\gamma_1\left(\frac{a+b}{2}\right)=2^{1-\alpha_1}(b-a)^{\alpha_1}.$$

Similarly, if we consider the mapping $\gamma_2:[a,b]\to\mathbb{R}$, given by $\gamma_2(s):=(s-c)^{\alpha_2}+(d-s)^{\alpha_2}$, $s\in[c,d]$, $\alpha_2\in(0,1]$. Therefore, its maximum is realized at $s=\frac{c+d}{2}$ and

$$\max_{s\in[c,d]}\{\gamma_2(s)\}=\gamma_2\left(\frac{c+d}{2}\right)=2^{1-\alpha_2}(d-c)^{\alpha_2}.$$

Consequently, by (5.4.3), we have

$$\begin{aligned}&\sup\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right| \\ &\qquad\le H_1\left(\frac{b-a}{2}\right)^{\alpha_1}+H_2\left(\frac{d-c}{2}\right)^{\alpha_2}.\end{aligned} \tag{5.4.4}$$

Using (5.4.2) we obtain the desired inequality (5.4.1). □

Remark 5.4.2. *We notice that, if f is (L_1, L_2)–Lipschitzian on Q, then (5.4.1) becomes*

$$|T(f,u;Q)| \leq \frac{1}{2}\left[L_1(b-a)+L_2(d-c)\right]\cdot \bigvee_Q(u). \tag{5.4.5}$$

Corollary 5.4.3. *If we assume that $g : Q \to \mathbb{R}$, is Lebesgue integrable on Q, then $u(x,y) := \int_c^y \int_a^x g(t,s)\,dtds$ is differentiable almost everywhere, $u(b,d) = \int_c^d \int_a^b g(t,s)\,dtds$, $u(b,c) = u(a,d) = u(a,c) = 0$ and $\bigvee_Q(u) = \int_c^d \int_a^b |g(t,s)|\,dtds$. Consequently, by (5.4.1) we obtain*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\cdot\int_c^d\int_a^b g(t,s)\,dtds - \int_c^d\int_a^b f(t,s)\,g(t,s)\,dtds\right| \leq \left[H_1\left(\frac{b-a}{2}\right)^{\alpha_1} + H_2\left(\frac{d-c}{2}\right)^{\alpha_2}\right]\cdot\int_c^d\int_a^b |g(t,s)|\,dtds. \tag{5.4.6}$$

From (5.4.6) we get a weighted version of the trapezoid inequality,

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{\int_c^d\int_a^b f(t,s)\,g(t,s)\,dtds}{\int_c^d\int_a^b g(t,s)\,dtds}\right| \leq \left[H_1\left(\frac{b-a}{2}\right)^{\alpha_1} + H_2\left(\frac{d-c}{2}\right)^{\alpha_2}\right], \tag{5.4.7}$$

provided that $g(t,s) > 0$, for almost every $(t,s) \in Q$ and $\int_c^d \int_a^b g(t,s)\,dtds \neq 0$

Remark 5.4.4. *We notice that, if f is (L_1, L_2)–Lipschitzian on Q, then (5.4.7) becomes*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{\int_c^d\int_a^b f(t,s)\,g(t,s)\,dtds}{\int_c^d\int_a^b g(t,s)\,dtds}\right| \leq \frac{1}{2}\left[L_1(b-a)+L_2(d-c)\right]. \tag{5.4.8}$$

For a bimonotonic non-decreasing integrators, the following result holds:

Theorem 5.4.5. *Let $f, u : Q \to \mathbb{R}$ be such that f is (α_1, α_2)-(H_1, H_2)–Hölder type mapping, where $H_1, H_2 > 0$ and $\alpha_1, \alpha_2 > 0$ are given, and u is bimonotonic non-decreasing on Q. Then we have the inequality*

$$\begin{aligned}
&|T(f,u;Q)| \\
&\leq \left[\frac{H_1}{2}(b-a)^{\alpha_1} + \frac{H_2}{2}(d-c)^{\alpha_2}\right]\cdot\left[u(b,d)-u(b,c)-u(a,d)+u(a,c)\right] \\
&\quad + \left[\frac{H_1}{2}(b-a)^{\alpha_1}(d-c) + \frac{H_2}{2}(b-a)(d-c)^{\alpha_2}\right]\left[u(b,d)-u(a,c)\right].
\end{aligned} \tag{5.4.9}$$

Proof. Using the inequality for the Riemann–Stieltjes integral of continuous integrands and a bimonotonic non-decreasing integrators, we have

$$|T(f,u;Q)| = \left|\int_c^d \int_a^b \left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right] d_t d_s u(t,s)\right|$$
$$\leq \int_c^d \int_a^b \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right| d_t d_s u(t,s). \tag{5.4.10}$$

As f is of (α_1,α_2)-(H_1,H_2)–Hölder type mapping, then by (5.4.3) we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - f(t,s)\right|$$
$$\leq \frac{H_1}{2}\left[(t-a)^{\alpha_1}+(b-t)^{\alpha_1}\right] + \frac{H_2}{2}\left[(s-c)^{\alpha_2}+(d-s)^{\alpha_2}\right], \tag{5.4.11}$$

for any $t \in [a,b]$ and $s \in [c,d]$. Consequently, by (5.4.10) and then using integration by parts, we have

$$\begin{aligned}
&|T(f,u;Q)| \\
&\leq \int_c^d \int_a^b \left[\frac{H_1}{2}\left[(t-a)^{\alpha_1}+(b-t)^{\alpha_1}\right] + \frac{H_2}{2}\left[(s-c)^{\alpha_2}+(d-s)^{\alpha_2}\right]\right] d_t d_s u(t,s) \\
&= \left[\frac{H_1}{2}(b-a)^{\alpha_1} + \frac{H_2}{2}(d-c)^{\alpha_2}\right] \cdot \left[u(b,d)-u(b,c)-u(a,d)+u(a,c)\right] \\
&\quad - \frac{H_1}{2}\alpha_1 \int_c^d \int_a^b \left[(t-a)^{\alpha_1-1}-(b-t)^{\alpha_1-1}\right] u(t,s)\,dtds \\
&\quad - \frac{H_2}{2}\alpha_2 \int_c^d \int_a^b \left[(s-c)^{\alpha_2-1}-(d-s)^{\alpha_2-1}\right] u(t,s)\,dtds.
\end{aligned} \tag{5.4.12}$$

Now, on utilizing the bimonotonicity property of u on Q, we have

$$\begin{aligned}
\int_c^d \int_a^b (t-a)^{\alpha_1-1} u(t,s)\,dtds &\geq u(a,c) \int_c^d \int_a^b (t-a)^{\alpha_1-1}\,dtds \\
&= \frac{1}{\alpha_1} u(a,c)(b-a)^{\alpha_1}(d-c),
\end{aligned} \tag{5.4.13}$$

$$\begin{aligned}
\int_c^d \int_a^b (b-t)^{\alpha_1-1} u(t,s)\,dtds &\leq u(b,d) \int_c^d \int_a^b (b-t)^{\alpha_1-1}\,dtds \\
&= \frac{1}{\alpha_1} u(b,d)(b-a)^{\alpha_1}(d-c),
\end{aligned} \tag{5.4.14}$$

$$\begin{aligned}
\int_c^d \int_a^b (s-c)^{\alpha_2-1} u(t,s)\,dtds &\geq u(a,c) \int_c^d \int_a^b (s-a)^{\alpha_2-1}\,dtds \\
&= \frac{1}{\alpha_2} u(a,c)(b-a)(d-c)^{\alpha_2},
\end{aligned} \tag{5.4.15}$$

and

$$\int_c^d \int_a^b (d-s)^{\alpha_2 - 1} u(t,s)\, dtds \leq u(b,d) \int_c^d \int_a^b (d-s)^{\alpha_2 - 1}\, dtds$$
$$= \frac{1}{\alpha_2} u(b,d)(b-a)(d-c)^{\alpha_2}. \qquad (5.4.16)$$

Substituting (5.4.13)–(5.4.16), in (5.4.12) we get

$$\begin{aligned} &|T(f,u;Q)| \\ &\leq \left[\frac{H_1}{2}(b-a)^{\alpha_1} + \frac{H_2}{2}(d-c)^{\alpha_2}\right] \cdot [u(b,d) - u(b,c) - u(a,d) + u(a,c)] \\ &\quad + \left[\frac{H_1}{2}(b-a)^{\alpha_1}(d-c) + \frac{H_2}{2}(b-a)(d-c)^{\alpha_2}\right][u(b,d) - u(a,c)]. \end{aligned}$$

$\square$

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