APPROXIMATING CSISZÁR f-DIVERGENCE VIA TWO INTEGRAL IDENTITIES AND APPLICATIONS

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Abstract. Some approximations of the Csiszár $f-$divergence via the use of the integral identities obtained in [8] and [9] and applications are given.

1. Introduction

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [23], Kullback and Leibler [32], Rényi [42], Havrda and Charvat [21], Kapur [26], Sharma and Mittal [15], Burbea and Rao [5], Rao [41], Lin [34], Csiszár [14], Ali and Silvey [1], Vajda [51], Shioya and Da-te [45] and others (see for example [26] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [37], genetics [39], finance, economics, and political science [47], [48], [43], biology [39], the analysis of contingency tables [20], approximation of probability distributions [27], [24], signal processing [25], [3] and pattern recognition [10], [53]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set χ and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be $\Omega := \left\{p | p : \chi \to \mathbb{R},\ p(x) \geq 0,\ \int_\chi p(x)\, d\mu(x) = 1\right\}$. The Kullback-Leibler divergence [32] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\chi p(x) \log\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \Omega, \tag{1.1}$$

where log is to base 2.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [22], χ^2-*divergence* D_{χ^2}, $\alpha-$*divergence* D_α, *Bhattacharyya distance* D_B [4], *Harmonic distance* D_{Ha}, *Jeffreys distance* D_J [23], *triangular discrimination* D_Δ [49], etc... They are defined as follows:

$$D_v(p,q) := \int_\chi |p(x) - q(x)|\, d\mu(x), \quad p,q \in \Omega; \tag{1.2}$$

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$$D_H(p,q) := \int_\chi \left[\sqrt{p(x)} - \sqrt{q(x)}\right]^2 d\mu(x), \quad p,q \in \Omega; \tag{1.3}$$

$$D_{\chi^2}(p,q) := \int_\chi p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,q \in \Omega; \tag{1.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\chi [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x)\right], \quad p,q \in \Omega; \tag{1.5}$$

$$D_B(p,q) := \int_\chi \sqrt{p(x)\,q(x)}\,d\mu(x), \quad p,q \in \Omega; \tag{1.6}$$

$$D_{Ha}(p,q) := \int_\chi \frac{2p(x)\,q(x)}{p(x)+q(x)} d\mu(x), \quad p,q \in \Omega; \tag{1.7}$$

$$D_J(p,q) := \int_\chi [p(x) - q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \Omega; \tag{1.8}$$

$$D_\Delta(p,q) := \int_\chi \frac{[p(x) - q(x)]^2}{p(x)+q(x)} d\mu(x), \quad p,q \in \Omega. \tag{1.9}$$

For other divergence measures, see the paper [26] by Kapur or the book on line [46] by Taneja. For a comprehensive collection of preprints available on line, see the RGMIA web site `http://rgmia.vu.edu.au/papersinfth.html`

Csiszár $f-$divergence is defined as follows [14]

$$I_f(p,q) := \int_\chi q(x)\, f\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \Omega, \tag{1.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. All the above distances $(1.1) - (1.9)$, are particular instances of Csiszár $f-$divergence. There are also many others which are not in this class (see for example [26] or [46]). For the basic properties of Csiszár $f-$divergence see [12]-[16].

The main aim of this paper is to point out some representations of Csiszár $f-$divergence for the function which has the $(n-1)-$derivative $(n \geq 1)$ absolutely continuous by employing two recent integral identities from [8] and [9] involving interior point and end point identities. Estimates for the remainder are also provided.

2. Representation of Csiszár $f-$Divergence

In [8] (see also [6]), the authors proved the following integral identity generalising the mid-point rule.

Lemma 1. *Let $g:[a,b]\to\mathbb{R}$ be a function such that $g^{(n-1)}$ is absolutely continuous. Then for all $x\in[a,b]$, we have the identity:*

$$\int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(b-x)^{k+1}+(-1)^k(x-a)^{k+1}\right]g^{(k)}(x) + (-1)^n\int_a^b K_n(x,t)\,g^{(n)}(t)\,dt, \tag{2.1}$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t):=\begin{cases}\frac{(t-a)^n}{n!}, & a\le t\le x\le b\\ \frac{(t-b)^n}{n!}, & a\le x<t\le b.\end{cases} \tag{2.2}$$

In particular, if $x=\frac{a+b}{2}$, then

$$\int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{2^{k+1}}\left[\frac{1+(-1)^k}{(k+1)!}\right](b-a)^{k+1}g^{(k)}\left(\frac{a+b}{2}\right) + (-1)^n\int_a^b M_n(t)\,g^{(n)}(t)\,dt, \tag{2.3}$$

where

$$M_n(t):=\begin{cases}\frac{(t-a)^n}{n!}, & a\le t\le\frac{a+b}{2}\\ \frac{(t-b)^n}{n!}, & \frac{a+b}{2}<t\le b.\end{cases} \tag{2.4}$$

Another integral identity generalising the trapezoid rule is embodied in the following lemma (see [9] or [7]).

Lemma 2. *Let $g:[a,b]\to\mathbb{R}$ be as in Lemma 1. Then for all $x\in[a,b]$, we have the representation*

$$\int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}g^{(k)}(a)+(-1)^k(b-x)^{k+1}g^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n g^{(n)}(t)\,dt, \tag{2.5}$$

In particular, if $x=\frac{a+b}{2}$, then

$$\int_a^b g(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)!}(b-a)^{k+1}\left[g^{(k)}(a)+(-1)^k g^{(k)}(b)\right] + \frac{(-1)^n}{n!}\int_a^b\left(t-\frac{a+b}{2}\right)^n g^{(n)}(t)\,dt. \tag{2.6}$$

Let us consider $x=(1-\lambda)a+\lambda b$, $\lambda\in[0,1]$, then from (2.1) we obtain

$$\begin{aligned}\int_a^b g(t)\,dt &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(1-\lambda)^{k+1}+(-1)^k\lambda^{k+1}\right](b-a)^{k+1}g^{(k)}((1-\lambda)a+\lambda b)\\ &\quad +(-1)^n\int_a^b K_n((1-\lambda)a+\lambda b,t)\,g^{(n)}(t)\,dt,\end{aligned} \tag{2.7}$$

and from (2.5) we obtain

$$\begin{aligned}\int_a^b g(t)\,dt &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[\lambda^{k+1}g^{(k)}(a)+(-1)^k(1-\lambda)^{k+1}g^{(k)}(b)\right](b-a)^{k+1}\\ &\quad +\frac{1}{n!}\int_a^b[(1-\lambda)a+\lambda b-t]^n g^{(n)}(t)\,dt.\end{aligned} \tag{2.8}$$

We are now able to state and prove the following representation result for the Csiszár $f-$divergence.

Theorem 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous on any $[a,b]\subset\mathbb{R}$. If $p,q\in\Omega$, then*

$$\begin{aligned}I_f(p,q) &= f(1)+\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(1-\lambda)^{k+1}+(-1)^k\lambda^{k+1}\right]\\ &\quad\times I_{(\cdot-1)^{k+1}f^{(k+1)}[(1-\lambda)+\lambda\cdot]}(p,q)+(-1)^n\int_\Gamma q(x)\\ &\quad\times\left(\int_1^{\frac{p(x)}{q(x)}}K_n\left[\frac{(1-\lambda)q(x)+\lambda p(x)}{q(x)},t\right]f^{(n+1)}(t)\,dt\right)d\mu(x),\ \ \lambda\in[0,1]\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}I_f(p,q) &= f(1)+\sum_{k=0}^{n-1}\frac{\lambda^{k+1}}{(k+1)!}f^{(k+1)}(1)\,D_k(p,q)\\ &\quad+\sum_{k=0}^{n-1}\frac{(-1)^k(1-\lambda)^{k+1}}{(k+1)!}I_{(\cdot-1)^{k+1}f^{(k+1)}(\cdot)}(p,q)+\frac{1}{n!}\int_\Gamma[q(x)]^{-n+1}\\ &\quad\times\left(\int_1^{\frac{p(x)}{q(x)}}[\lambda p(x)+[(1-\lambda)-t]\,q(x)]^n f^{(n+1)}(t)\,dt\right)d\mu(x),\end{aligned} \tag{2.10}$$

where

$$D_k(p,q)=\int_\Gamma[p(x)-q(x)]^k[q(x)]^{-k+1}\,d\mu(x).$$

Proof. If we apply the identity (2.7) for f', we get

$$f(b) = f(a) + \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(1-\lambda)^{k+1} + (-1)^k \lambda^{k+1} \right] \times (b-a)^{k+1} f^{(k+1)}((1-\lambda)a + \lambda b) + (-1)^n \int_a^b K_n[(1-\lambda)a + \lambda b, t] f^{(n+1)}(t)\, dt. \tag{2.11}$$

If in (2.11) we choose $b = \frac{p(x)}{q(x)}$, $x \in \Gamma$ and $a = 1$, we get

$$f\left(\frac{p(x)}{q(x)}\right) = f(1) + \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(1-\lambda)^{k+1} + (-1)^k \lambda^{k+1} \right] \times \frac{(p(x) - q(x))^{k+1}}{[q(x)]^{k+1}} \cdot f^{(k+1)}\left[\frac{(1-\lambda)q(x) + \lambda p(x)}{q(x)}\right] + (-1)^n \int_1^{\frac{p(x)}{q(x)}} K_n\left[\frac{(1-\lambda)q(x) + \lambda p(x)}{q(x)}, t\right] f^{(n+1)}(t)\, dt \tag{2.12}$$

for all $x \in \Gamma$.

If we multiply (2.12) by $q(x) \geq 0$ $(x \in \Gamma)$, integrate on Γ and take into account that $\int_\Gamma q(x)\, d\mu(x) = 1$, then we get the representation (2.9).

If we apply the identity (2.8) for f', we get

$$f(b) = f(a) + \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\lambda^{k+1} f^{(k+1)}(a) + (-1)^k (1-\lambda)^{k+1} f^{(k+1)}(b) \right] (b-a)^{k+1} + \frac{1}{n!} \int_a^b [(1-\lambda)a + \lambda b - t]^n f^{(n+1)}(t)\, dt. \tag{2.13}$$

If in (2.13) we choose $b = \frac{p(x)}{q(x)}$, $x \in \Gamma$ and $a = 1$, we get

$$f\left(\frac{p(x)}{q(x)}\right) = f(1) + \sum_{k=0}^{n-1} \frac{\lambda^{k+1}}{(k+1)!} f^{(k+1)}(1) \left(\frac{p(x)}{q(x)} - 1\right)^{k+1} + \sum_{k=0}^{n-1} \frac{(-1)^k (1-\lambda)^{k+1}}{(k+1)!} f^{(k+1)}\left(\frac{p(x)}{q(x)}\right) \left(\frac{p(x)}{q(x)} - 1\right)^{k+1} + \frac{1}{n!} \int_1^{\frac{p(x)}{q(x)}} \left[\frac{(1-\lambda)q(x) + \lambda p(x)}{q(x)} - t\right]^n f^{(n+1)}(t)\, dt, \tag{2.14}$$

for all $x \in \Gamma$.

If we multiply (2.14) by $q(x) \geq 0$ $(x \in \Gamma)$, integrate on Γ and take into account that $\int_\Gamma q(x)\, d\mu(x) = 1$, we get the representation (2.10). ∎

Remark 1. *If in (2.9) we choose $\lambda = 0$ or, $\lambda = 1$ or, $\lambda = \frac{1}{2}$, we get, respectively*

$$I_f(p,q) = f(1) + \sum_{k=0}^{n-1} \frac{1}{(k+1)!} f^{(k+1)}(1) D_k(p,q) + (-1)^n \int_\Gamma q(x) \times \left(\int_1^{\frac{p(x)}{q(x)}} K_n(1,t) f^{(n+1)}(t)\, dt \right) d\mu(x), \tag{2.15}$$

$$\begin{aligned} & I_f(p,q) \\ = {} & f(1) + \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} I_{(\cdot-1)^{k+1} f^{(k+1)}(\cdot)}(p,q) \\ & + (-1)^n \int_\Gamma q(x) \left(\int_1^{\frac{p(x)}{q(x)}} K_n\left(\frac{p(x)}{q(x)}, t\right) f^{(n+1)}(t)\, dt \right) d\mu(x) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} & I_f(p,q) \\ = {} & f(1) + \sum_{k=0}^{n-1} \left[\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right] I_{(\cdot-1)^{k+1} f^{(k+1)}\left(\frac{1+\cdot}{2}\right)}(p,q) \\ & + (-1)^n \int_\Gamma q(x) \left(\int_1^{\frac{p(x)}{q(x)}} K_n\left(\frac{q(x)+p(x)}{2q(x)}, t\right) f^{(n+1)}(t)\, dt \right) d\mu(x). \end{aligned} \tag{2.17}$$

Remark 2. *If in (2.10) we choose $\lambda = 0$, or $\lambda = 1$ or, $\lambda = \frac{1}{2}$, we get, respectively*

$$\begin{aligned} I_f(p,q) = {} & f(1) + \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} I_{(\cdot-1)^{k+1} f^{(k+1)}(\cdot)}(p,q) \\ & + \frac{1}{n!} \int_\Gamma q(x) \left(\int_1^{\frac{p(x)}{q(x)}} (1-t)^n f^{(n+1)}(t)\, dt \right) d\mu(x), \end{aligned} \tag{2.18}$$

$$\begin{aligned} I_f(p,q) = {} & f(1) + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(1)}{(k+1)!} D_k(p,q) + \frac{1}{n!} \int_\Gamma [q(x)]^{-n+1} \\ & \times \left(\int_1^{\frac{p(x)}{q(x)}} (p(x) - tq(x))^n f^{(n+1)}(t)\, dt \right) d\mu(x) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} & I_f(p,q) \\ = {} & f(1) + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(1)}{2^{k+1}(k+1)!} D_k(p,q) \\ & + \sum_{k=0}^{n-1} \frac{(-1)^k}{2^{k+1}(k+1)!} I_{(\cdot-1)^{k+1} f^{(k+1)}(\cdot)}(p,q) + \frac{1}{n!} \int_\Gamma [q(x)]^{-n+1} \\ & \times \left(\int_1^{\frac{p(x)}{q(x)}} \left[\frac{1}{2} p(x) + \left(\frac{1}{2} - t \right) q(x) \right]^n f^{(n+1)}(t)\, dt \right) d\mu(x). \end{aligned} \tag{2.20}$$

3. Bounds for the Remainder

In this section we point out some bounds for the remainders in the representations (2.9) and (2.10), i.e.,

$$R_f(p,q) := (-1)^n \int_\Gamma q(x) \times \left(\int_1^{\frac{p(x)}{q(x)}} K_n\left[\frac{(1-\lambda)q(x)+\lambda p(x)}{q(x)}, t\right] \times f^{(n+1)}(t)\,dt \right) d\mu(x) \tag{3.1}$$

and

$$\tilde{R}_f(p,q) := \frac{1}{n!}\int_\Gamma [q(x)]^{-n+1} \times \left(\int_1^{\frac{p(x)}{q(x)}} [\lambda p(x) + [(1-\lambda)-t]\,q(x)]^n f^{(n+1)}(t)\,dt \right) d\mu(x), \tag{3.2}$$

where $p,q \in \Omega$, $\lambda \in [0,1]$ and $K_n(\cdot,\cdot)$ is the kernel defined in equation (2.2).

For $a,b \in \mathbb{R}$, let us denote

$$\|f\|_{[a,b],p} := \left| \int_a^b |f(t)|^p\,dt \right|^{\frac{1}{p}}, \quad p \geq 1$$

and

$$\|f\|_{[a,b],\infty} := ess \sup_{\substack{t\in[a,b] \\ (t\in[b,a])}} |f(t)|.$$

In order to obtain bounds on $R_f(p,q)$ as given in (3.1), we need to consider integrals of the form

$$I_1(z) := \int_1^z K_n[(1-\lambda)\cdot 1 + \lambda z, t]\, f^{(n+1)}(t)\,dt, \quad z \in (0,\infty).$$

Thus

$$\begin{aligned}
|I_1(z)| &\leq \left| \int_1^z |K_n[(1-\lambda)\cdot 1+\lambda z, t]| \left| f^{(n+1)}(t) \right| dt \right| \\
&\leq \left\| f^{(n+1)} \right\|_{[1,z],\infty} \left| \int_1^z |K_n((1-\lambda)\cdot 1+\lambda z, t)|\,dt \right| \\
&= \frac{1}{n!} \left\| f^{(n+1)} \right\|_{[1,z],\infty} \left| \int_1^{(1-\lambda)\cdot 1+\lambda z} |t-1|^n\,dt + \int_{(1-\lambda)\cdot 1+\lambda z}^z |t-z|^n\,dt \right| \\
&= \frac{1}{n!} \left[\frac{|(1-\lambda)+\lambda z-1|^{n+1} + |z-(1-\lambda)\cdot 1-\lambda z|^{n+1}}{n+1} \right] \left\| f^{(n+1)} \right\|_{[1,z],\infty} \\
&= \frac{1}{n!} \left[\frac{\lambda^{n+1}|z-1|^{n+1} + (1-\lambda)^{n+1}|z-1|^{n+1}}{n+1} \right] \left\| f^{(n+1)} \right\|_{[1,z],\infty} \\
&= \frac{|z-1|^{n+1}}{(n+1)!} \left[\lambda^{n+1} + (1-\lambda)^{n+1} \right] \left\| f^{(n+1)} \right\|_{[1,z],\infty}.
\end{aligned}$$

Using Hölder's inequality, we may write for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, that:

$$|I_1(z)| \leq \left\| f^{(n+1)} \right\|_{[1,z],\beta} \left| \int_1^z |K_n((1-\lambda)\cdot 1 + \lambda z, t)|^\alpha \, dt \right|^{\frac{1}{\alpha}}.$$

However,

$$\begin{aligned} & \left| \int_1^z |K_n((1-\lambda)\cdot 1 + \lambda z, t)|^\alpha \, dt \right|^{\frac{1}{\alpha}} \\ = & \frac{1}{n!} \left| \int_1^{(1-\lambda)\cdot 1+\lambda z} |t-1|^{\alpha n} \, dt + \int_{(1-\lambda)\cdot 1+\lambda z}^z |t-z|^{\alpha n} \, dt \right|^{\frac{1}{\alpha}} \\ = & \frac{1}{n!} \left[\frac{|(1-\lambda)+\lambda z - 1|^{\alpha n+1} + |z-(1-\lambda)\cdot 1 - \lambda z|^{\alpha n+1}}{\alpha n+1} \right]^{\frac{1}{\alpha}} \\ = & \frac{1}{n!} \left[\frac{\lambda^{\alpha n+1} |z-1|^{\alpha n+1} + (1-\lambda)^{\alpha n+1} |z-1|^{\alpha n+1}}{\alpha n+1} \right]^{\frac{1}{\alpha}} \\ = & \frac{|z-1|^{n+\frac{1}{\alpha}}}{n!(\alpha n+1)^{\frac{1}{\alpha}}} \left[\lambda^{\alpha n+1} + (1-\lambda)^{\alpha n+1} \right]^{\frac{1}{\alpha}} \end{aligned}$$

and then:

$$|I_1(z)| \leq \left\| f^{(n+1)} \right\|_{[1,z],\beta} \frac{|z-1|^{n+\frac{1}{\alpha}}}{n!(\alpha n+1)^{\frac{1}{\alpha}}} \left[\lambda^{\alpha n+1} + (1-\lambda)^{\alpha n+1} \right]^{\frac{1}{\alpha}}.$$

Finally, we observe that

$$\begin{aligned} & \sup_{t\in[1,z]} |K_n((1-\lambda)\cdot 1 + \lambda z, t)| \\ = & \frac{1}{n!} \max\left\{ ((1-\lambda)+\lambda z - 1)^n + (z-(1-\lambda)\cdot 1 - \lambda z)^n \right\} \\ = & \frac{1}{n!} (z-1)^n (\max\{\lambda, 1-\lambda\})^n \\ = & \frac{1}{n!} |z-1|^n \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right]^n \end{aligned}$$

and then

$$|I_1(z)| \leq \frac{1}{n!} |z-1|^n \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right]^n \left\| f^{(n+1)} \right\|_{[1,z],1}.$$

Using the above inequalities, we may state the following result

(3.3)

$$|I_1(z)| \leq \left\{ \begin{array}{c} \frac{|z-1|^{n+1}}{(n+1)!} \left[\lambda^{n+1} + (1-\lambda)^{n+1} \right] \left\| f^{(n+1)} \right\|_{[1,z],\infty} \\ \\ \frac{|z-1|^{n+\frac{1}{\alpha}}}{n!(\alpha n+1)^{\frac{1}{\alpha}}} \left[\lambda^{\alpha n+1} + (1-\lambda)^{\alpha n+1} \right]^{\frac{1}{\alpha}} \left\| f^{(n+1)} \right\|_{[1,z],\beta} \\ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \frac{1}{n!} |z-1|^n \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right]^n \left\| f^{(n+1)} \right\|_{[1,z],1} \end{array} \right\} =: \kappa(z,n)$$

for all $z > 0$, $n \in \mathbb{N}$.

We are now able to state the following theorem pertaining to the remainder $R_f(p,q)$.

Theorem 2. *Assume that the function f is as in Theorem 1. If $p,q\in\Omega$, then we have the inequality*

$$|R_f(p,q)| \tag{3.4}$$

$$\leq A := \begin{cases} \frac{1}{(n+1)!}\left[\lambda^{n+1}+(1-\lambda)^{n+1}\right] \\ \qquad \times\int_\Gamma\left\{[q(x)]^{-n}|p(x)-q(x)|^{n+1}\left\|f^{(n+1)}\right\|_{\left[1,\frac{p(x)}{q(x)}\right],\infty}\right\}d\mu(x) \\ \\ \frac{1}{n!(\alpha n+1)^{\frac{1}{\alpha}}}\left[\lambda^{\alpha n+1}+(1-\lambda)^{\alpha n+1}\right]^{\frac{1}{\alpha}}\int_\Gamma\left\{[q(x)]^{-n-\frac{1}{\alpha}+1}\right. \\ \qquad\left.\times|p(x)-q(x)|^{n+\frac{1}{\alpha}}\left\|f^{(n+1)}\right\|_{\left[1,\frac{p(x)}{q(x)}\right],\beta}\right\}d\mu(x) \\ \qquad\qquad \textit{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{n!}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n\int_\Gamma[q(x)]^{-n+1}|p(x)-q(x)|^n \\ \qquad\times\left\|f^{(n+1)}\right\|_{\left[1,\frac{p(x)}{q(x)}\right],1}d\mu(x). \end{cases}$$

Moreover, if we assume that $0\leq r\leq\frac{p(x)}{q(x)}\leq R<\infty$, $x\in\Gamma$, then the second term in (3.4) can be upper bounded by

$$B \tag{3.5}$$

$$: = \begin{cases} \frac{1}{(n+1)!}\left[\lambda^{n+1}+(1-\lambda)^{n+1}\right]\left\|f^{(n+1)}\right\|_{[r,R],\infty} \\ \qquad\times\int_\Gamma[q(x)]^{-n}|p(x)-q(x)|^{n+1}d\mu(x) \\ \\ \frac{1}{n!(\alpha n+1)^{\frac{1}{\alpha}}}\left[\lambda^{\alpha n+1}+(1-\lambda)^{\alpha n+1}\right]^{\frac{1}{\alpha}}\left\|f^{(n+1)}\right\|_{[r,R],\beta} \\ \qquad\times\int_\Gamma[q(x)]^{-n-\frac{1}{\alpha}+1}|p(x)-q(x)|^{n+\frac{1}{\alpha}}\ d\mu(x) \\ \qquad\qquad \textit{if}\quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{n!}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n\left\|f^{(n+1)}\right\|_{[r,R],1}\int_\Gamma[q(x)]^{-n+1}|p(x)-q(x)|^n d\mu(x) \end{cases}$$

$$\leq C := \begin{cases} \frac{1}{(n+1)!}\left[\lambda^{n+1}+(1-\lambda)^{n+1}\right]\left\|f^{(n+1)}\right\|_{[r,R],\infty}(R-r)^{n+1} \\ \\ \frac{1}{n!(\alpha n+1)^{\frac{1}{\alpha}}}\left[\lambda^{\alpha n+1}+(1-\lambda)^{\alpha n+1}\right]^{\frac{1}{\alpha}}\left\|f^{(n+1)}\right\|_{[r,R],\beta}(R-r)^{n+\frac{1}{\alpha}} \\ \qquad\qquad \textit{if}\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{n!}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n\left\|f^{(n+1)}\right\|_{[r,R],1}(R-r)^n. \end{cases}$$

The proof of (3.4) follows by the inequality (3.3) choosing $z=\frac{p(x)}{q(x)}$ and integrating.

The proof of (3.5) follows by the fact that $\left|\frac{p(x)}{q(x)}-1\right|\leq R-r$ for all $x\in\Gamma$.

We omit the details.

The following corollary may be useful in practical applications.

Corollary 1. *With the assumptions of Theorem 2, we have the inequality:*

$$
\begin{aligned}
&\left| I_f(p,q) - f(1) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(1)}{(k+1)!} D_k(p,q) \right| \\
&\leq \begin{cases}
\frac{1}{(n+1)!} \int_\Gamma [q(x)]^{-n} |p(x)-q(x)|^{n+1} \left\| f^{(n+1)} \right\|_{\left[1,\frac{p(x)}{q(x)}\right],\infty} d\mu(x) \\[2ex]
\frac{1}{n!(\alpha n+1)^{\frac{1}{\alpha}}} \int_\Gamma [q(x)]^{-n-\frac{1}{\alpha}+1} \\
\qquad \times |p(x)-q(x)|^{n+\frac{1}{\alpha}} \left\| f^{(n+1)} \right\|_{\left[1,\frac{p(x)}{q(x)}\right],\beta} d\mu(x) \\
\qquad \text{where } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\frac{1}{n!} \int_\Gamma [q(x)]^{-n+1} |p(x)-q(x)|^{n} \left\| f^{(n+1)} \right\|_{\left[1,\frac{p(x)}{q(x)}\right],1} d\mu(x)
\end{cases} \\
&=: M_1.
\end{aligned}
\tag{3.6}
$$

Furthermore, if we assume that $r \leq \frac{p(x)}{q(x)} \leq R < \infty$, $x \in \Gamma$, *then we have*

$$
\begin{aligned}
M_1 &\leq \begin{cases}
\frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{[r,R],\infty} \int_\Gamma [q(x)]^{-n} |p(x)-q(x)|^{n+1} d\mu(x) \\[2ex]
\frac{1}{n!(\alpha n+1)^{\frac{1}{\alpha}}} \left\| f^{(n+1)} \right\|_{[r,R],\beta} \int_\Gamma [q(x)]^{-n-\frac{1}{\alpha}+1} \\
\qquad \times |p(x)-q(x)|^{n+\frac{1}{\alpha}} d\mu(x) \\
\qquad \text{where } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\frac{1}{n!} \left\| f^{(n+1)} \right\|_{[r,R],1} \int_\Gamma [q(x)]^{-n+1} |p(x)-q(x)|^{n} d\mu(x)
\end{cases} \\
&=: M_2 \\
&\leq \left\{ \begin{array}{l}
\frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{[r,R],\infty} (R-r)^{n+1} \\[2ex]
\frac{1}{n!(\alpha n+1)^{\frac{1}{\alpha}}} \left\| f^{(n+1)} \right\|_{[r,R],\beta} (R-r)^{n+\frac{1}{\alpha}} \\
\qquad \text{where } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\frac{1}{n!} \left\| f^{(n+1)} \right\|_{[r,R],1} (R-r)^{n}
\end{array} \right\} =: M_3
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| I_f(p,q) - f(1) - \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} I_{(\cdot-1)^{k+1} f^{(k+1)}(\cdot)}(p,q) \right| \\
&\leq M_1 \leq M_2 \leq M_3
\end{aligned}
$$

and if $0 \le r \le \frac{p(x)}{q(x)} \le R < \infty,\ x \in \Gamma,$ *then*

$$\begin{aligned} & \left| I_f(p,q) - f(1) - \sum_{k=0}^{n-1} \left[\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right] I_{(\cdot-1)^{k+1} f^{(k+1)}\left(\frac{1+\cdot}{2}\right)}(p,q) \right| \\ \le\ & \frac{1}{2^n} M_1 \le \frac{1}{2^n} M_2 \le \frac{1}{2^n} M_3. \end{aligned}$$

Now, to obtain the bound on $\tilde{R}_f(p,q)$ as defined in (3.2), consider the integral

$$I_2(z) := \frac{1}{n!} \int_1^z ((1-\lambda)\cdot 1 + \lambda z - t)^n f^{(n+1)}(t)\, dt,$$

from which we have

$$\begin{aligned} |I_2(z)| &\le \left\| f^{(n+1)} \right\|_{[1,z],\infty} \frac{1}{n!} \left| \int_1^z |(1-\lambda)\cdot 1 + \lambda z - t|^n \right| dt \\ &= \frac{1}{n!} \left| \int_1^{(1-\lambda)\cdot 1+\lambda z} |(1-\lambda)\cdot 1 + \lambda z - t|^n\, dt \right. \\ &\quad \left. + \int_{(1-\lambda)\cdot 1+\lambda z}^z |(1-\lambda)\cdot 1 + \lambda z - t|^n\, dt \right| \left\| f^{(n+1)} \right\|_{[1,z],\infty} \\ &= \frac{1}{n!} \cdot \left[\frac{|(1-\lambda)\cdot 1 + \lambda z - 1|^{n+1} + |(1-\lambda)\cdot 1 + \lambda z - z|^{n+1}}{n+1} \right] \\ &\quad \times \left\| f^{(n+1)} \right\|_{[1,z],\infty} \\ &= \frac{(z-1)^{n+1}}{(n+1)!} \cdot \left[\lambda^{n+1} + (1-\lambda)^{n+1} \right] \cdot \left\| f^{(n+1)} \right\|_{[1,z],\infty}. \end{aligned}$$

Using Hölder's inequality, we may write, for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, that

$$\begin{aligned} |I_2(z)| &\le \frac{1}{n!} \left\| f^{(n+1)} \right\|_{[1,z],\beta} \left| \int_1^z |(1-\lambda)\cdot 1 + \lambda z - t|^{n\alpha}\, dt \right|^{\frac{1}{\alpha}} \\ &= \frac{1}{n!} \left\| f^{(n+1)} \right\|_{[1,z],\beta} \left| \int_1^{(1-\lambda)\cdot 1+\lambda z} |(1-\lambda)\cdot 1 + \lambda z - t|^{n\alpha}\, dt \right. \\ &\quad \left. + \int_{(1-\lambda)\cdot 1+\lambda z}^z |(1-\lambda)\cdot 1 + \lambda z - t|^{n\alpha}\, dt \right|^{\frac{1}{\alpha}} \\ &= \frac{1}{n!} \left\| f^{(n+1)} \right\|_{[1,z],\beta} \left[\frac{|(1-\lambda) + \lambda z - 1|^{n\alpha+1} + |z - (1-\lambda) - \lambda z|^{n\alpha+1}}{n\alpha+1} \right]^{\frac{1}{\alpha}} \\ &= \frac{1}{n!} \left\| f^{(n+1)} \right\|_{[1,z],\beta} \left[\frac{\lambda^{\alpha n+1} |z-1|^{\alpha n+1} + (1-\lambda)^{\alpha n+1} |z-1|^{\alpha n+1}}{\alpha n+1} \right]^{\frac{1}{\alpha}} \\ &= \frac{|z-1|^{n+\frac{1}{\alpha}}}{n!\,(\alpha n+1)^{\frac{1}{\alpha}}} \cdot \left[\lambda^{\alpha n+1} + (1-\lambda)^{\alpha n+1} \right]^{\frac{1}{\alpha}} \cdot \left\| f^{(n+1)} \right\|_{[1,z],\beta}. \end{aligned}$$

Finally, we observe that

$$\begin{aligned}
|I_2(z)| &\leq \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[1,z],1}\sup_{t\in[1,z]}|(1-\lambda)\cdot 1+\lambda z-t|^n \\
&= \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[1,z],1}\max\left\{|(1-\lambda)+\lambda z-1|^n+|z-(1-\lambda)\cdot 1-\lambda z|^n\right\} \\
&= \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[1,z],1}(z-1)^n\left(\max\{\lambda,1-\lambda\}\right)^n \\
&= \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[1,z],1}|z-1|^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n.
\end{aligned}$$

Using the above inequalities, we may state that

$$|I_2(z)|\leq\kappa(n,z), \tag{3.7}$$

where $\kappa(n,z)$ is defined in (3.3). That is, the bounds for $R_f(p,q)$ and $\tilde{R}_f(p,q)$ are the same.

We may now state the following theorem concerning a bound for the remainder $\tilde{R}_f(p,q)$.

Theorem 3. *Assume that the function f is as in Theorem 1. If $p,q\in\not\leq$, then we have the inequality:*

$$\left|\tilde{R}_f(p,q)\right|\leq A, \tag{3.8}$$

where A is given in (3.4).
Moreover, if we assume that $0\leq r\leq\frac{p(x)}{q(x)}\leq R<\infty$, $x\in\Gamma$, then

$$A\leq B\leq C, \tag{3.9}$$

with B and C being as defined in (3.5).

The following corollary may be useful in practical applications.

Corollary 2. *With the above assumptions, we have*

$$\begin{aligned}
&\left|I_f(p,q)-f(1)-\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}I_{(\cdot-1)^{k+1}f^{(k+1)}(\cdot)}(p,q)\right| \\
&\leq M_1\leq M_2\leq M_3,
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
&\left|I_f(p,q)-f(1)-\sum_{k=0}^{n-1}\frac{f^{(k+1)}(1)}{(k+1)!}D_k(p,q)\right| \\
&\leq M_1\leq M_2\leq M_3
\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}
&\left|I_f(p,q)-f(1)-\sum_{k=0}^{n-1}\frac{f^{(k+1)}(1)}{2^{k+1}(k+1)!}D_k(p,q)\right. \\
&\left.-\sum_{k=0}^{n-1}\frac{(-1)^k}{2^{k+1}(k+1)!}I_{(\cdot-1)^{k+1}f^{(k+1)}(\cdot)}(p,q)\right| \\
&\leq \frac{1}{2^n}M_1\leq\frac{1}{2^n}M_2\leq\frac{1}{2^n}M_3
\end{aligned}\tag{3.12}$$

for $r\leq\frac{p(x)}{q(x)}\leq R$, $x\in\Gamma$, where M_i $\left(i=\overline{1,3}\right)$ are as defined in Corollary 1.

Remark 3. *If in all the above results we choose f to be a particular function generating the divergences listed at (1.2) – (1.9), then we can obtain many interesting approximations for the above distances. We omit the details.*

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