SOME INEQUALITIES FOR THE CSISZÁR $\phi-$DIVERGENCE WHEN ϕ IS AN L- LIPSCHITZIAN FUNCTION AND APPLICATIONS

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Abstract. Some inequalities of Jessen's type for vector valued Lipschitzian functions and applications for the discrete Csiszár $\phi-$divergence are given.

1. Introduction

Given a convex function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, the Φ- *divergence functional*

$$I_{\Phi}(p,q) := \sum_{i=1}^{n} q_i \Phi\left(\frac{p_i}{q_i}\right), \tag{1.1}$$

was introduced in Csiszár [1], [2] as a generalized measure of information, a "distance function" on the set of probability distributions $\mathbb{P}^n$. The restriction here to discrete distribution is only for convenience, similar results hold for general distributions.

As in Csiszár [2], we interpret undefined expressions by

$$\begin{aligned} \Phi(0) &= \lim_{t\to 0+} \Phi(t), \quad 0\Phi\left(\frac{0}{0}\right) = 0, \\ 0\Phi\left(\frac{a}{0}\right) &= \lim_{\varepsilon\to 0+} \Phi\left(\frac{a}{\varepsilon}\right) = a \lim_{t\to\infty} \frac{\Phi(t)}{t}, \ a > 0. \end{aligned}$$

The following results were essentially given by Csiszár and Körner [3].

Theorem 1. *If $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is convex, then $I_{\Phi}(p,q)$ is jointly convex in p and q.*

The following lower bound for the $\Phi-$divergence functional also holds.

Theorem 2. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be convex. Then for every $p, q \in \mathbb{R}_+^n$, we have the inequality:*

$$I_{\Phi}(p,q) \geq \sum_{i=1}^{n} q_i \Phi\left(\frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i}\right). \tag{1.2}$$

If Φ is strictly convex, equality holds in (1.2) iff

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \ldots = \frac{p_n}{q_n}. \tag{1.3}$$

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Corollary 1. *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be convex and normalized, i.e.,*

$$\Phi(1) = 0. \tag{1.4}$$

Then for any $p, q \in \mathbb{R}_+^n$ with

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i \tag{1.5}$$

we have the inequality

$$I_\Phi(p, q) \geq 0. \tag{1.6}$$

If Φ is strictly convex, the equality holds in (1.6) iff $p_i = q_i$ for all $i \in \{1, ..., n\}$.

In particular, if p, q are probability vectors, then (1.5) is assured. Corollary 1 then shows, for strictly convex and normalized $\Phi : \mathbb{R}_+ \to \mathbb{R}$,

$$I_\Phi(p, q) \geq 0 \text{ for all } p, q \in \mathbb{P}^n. \tag{1.7}$$

The equality holds in (1.7) iff $p = q$.

These are "distance properties". However, I_Φ is not a metric: It violates the triangle inequality, and is **asymmetric**, i.e, for general $p, q \in \mathbb{R}_+^n$, $I_\Phi(p, q) \neq I_\Phi(q, p)$.

In the examples below we obtain, for suitable choices of the kernel Φ, some of the best known distance functions I_Φ used in mathematical statistics [4]-[5], information theory [6]-[8] and signal processing [9]-[10].

Example 1. ***(Kullback-Leibler)*** *For*

$$\Phi(t) := t \log t,\ t > 0 \tag{1.8}$$

the $\Phi-$divergence is

$$I_\Phi(p, q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right), \tag{1.9}$$

*the **Kullback-Leibler distance*** [11]-[12].

Example 2. ***(Hellinger)*** *Let*

$$\Phi(t) = \left(1 - \sqrt{t}\right)^2,\ t > 0. \tag{1.10}$$

*Then I_Φ gives the **Hellinger distance*** [13]

$$I_\Phi(p, q) = \sum_{i=1}^{n} \left(\sqrt{p_i} - \sqrt{q_i}\right)^2, \tag{1.11}$$

which is symmetric.

Example 3. ***(Renyi)*** *For $\alpha > 1$, let*

$$\Phi(t) = t^\alpha,\ t > 0. \tag{1.12}$$

Then

$$I_\Phi(p, q) = \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}, \tag{1.13}$$

*which is the $\alpha-$**order** entropy* [14].

Example 4. *(χ^2-**distance**) Let*

$$\Phi(t) = (t-1)^2, \ t > 0. \tag{1.14}$$

Then

$$I_\Phi(p,q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} \tag{1.15}$$

*is the χ^2-**distance** between p and q.*

Finally, we have

Example 5. *(**Variational distance**). Let $\Phi(t) = |t-1|, \ t > 0$. The corresponding divergence, called the **variational distance**, is symmetric,*

$$I_\Phi(p,q) = \sum_{i=1}^{n} |p_i - q_i|.$$

For other examples of divergence measures, see the paper [22] by J.N. Kapur, where further references are given.

2. Some Inequalities for Lipschitzian Mappings. The Case of $l_\infty-$Norm

2.1. **Some Inequalities.** We start with the following result.

Theorem 3. *Let X,Y be two normed linear spaces with the norms $\|\cdot\|$ and $|\cdot|$ respectively. If $F : X \to Y$ is L - Lipschitzian, that is,*

$$|F(x) - F(y)| \le L\|x-y\| \quad \text{for all } x,y \in X, \tag{2.1}$$

then for all $x_i \in X$, $p_i \ge 0$ with $\sum_{i=1}^{n} p_i = 1$ $(i = 1,...,n)$, we have the inequality

$$\left|F\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i F(x_i)\right| \le L \max_{k=1,...,n-1} \|\Delta x_k\| \sum_{i,j=1}^{n} p_i p_j |i-j|. \tag{2.2}$$

Proof. As F is $L-$Lipschitzian, we can choose $x = \sum_{i=1}^{n} p_i x_i$ and $y = x_j$ $(j = 1,...,n)$ in (2.1) to get

$$\begin{aligned} \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| &\le L\left\|\sum_{i=1}^{n} p_i x_i - x_j\right\| \\ &= L\left\|\sum_{i=1}^{n} p_i (x_i - x_j)\right\| \le L\sum_{i=1}^{n} p_i \|x_i - x_j\|. \end{aligned} \tag{2.3}$$

Multiplying (2.3) by $p_j \ge 0$ and summing over j from 1 to n, we deduce

$$\sum_{j=1}^{n} p_j \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \le L \sum_{i,j=1}^{n} p_i p_j \|x_j - x_i\|. \tag{2.4}$$

By the generalised triangle inequality we have

$$\sum_{j=1}^{n} p_j \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \ge \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j F(x_j)\right|. \tag{2.5}$$

Also, it is obvious that

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| = 2 \sum_{1 \le i < j \le n} p_i p_j \|x_j - x_i\| . \tag{2.6}$$

Now, observe that, for $i < j$, we have

$$x_j - x_i = \sum_{k=i}^{j-1} \Delta x_k,$$

where $\Delta x_k := x_{k+1} - x_k$ is the forward difference.

Therefore, by the generalised triangle inequality we have

$$\begin{aligned} (2.7) \quad & \sum_{1 \le i < j \le n} p_i p_j \|x_j - x_i\| \\ = & \sum_{1 \le i < j \le n} p_i p_j \left\| \sum_{k=i}^{j-1} \Delta x_k \right\| \le \sum_{1 \le i < j \le n} p_i p_j \sum_{k=i}^{j-1} \|\Delta x_k\| \\ \le & \sum_{1 \le i < j \le n} p_i p_j (j - i) \max_{k=1,\dots,n-1} \|\Delta x_k\| \\ = & \max_{k=1,\dots,n-1} \|\Delta x_k\| \sum_{1 \le i < j \le n} p_i p_j (j - i) \\ = & \frac{1}{2} \max_{k=1,\dots,n-1} \|\Delta x_k\| \sum_{i,j=1}^{n} p_i p_j |j - i| . \end{aligned}$$

Using (2.4) - (2.7) we deduce the desired inequality (2.2). ∎

Corollary 2. *With the above assumptions for F and x_i $(i = 1, ..., n)$, we have the inequality*

$$\left| F\left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) - \frac{1}{n} \sum_{i=1}^{n} F(x_i) \right| \le L \cdot \frac{n^2 - 1}{3n} \max_{k=1,\dots,n-1} \|\Delta x_k\| . \tag{2.8}$$

Proof. We choose $p_i = \frac{1}{n}$ $(i = 1, ..., n)$ in (2.2) and have to compute

$$I := \sum_{i,j=1}^{n} |i - j| .$$

We observe that

$$\begin{aligned} \sum_{j=1}^{n} |i - j| &= \sum_{j=1}^{i} |i - j| + \sum_{j=i+1}^{n} |i - j| = \sum_{j=1}^{i} (i - j) + \sum_{j=i+1}^{n} (j - i) \\ &= i^2 - \frac{i(i+1)}{2} + \sum_{j=1}^{n} j - i(n - i) \\ &= i^2 - (n+1)i + \frac{n(n+1)}{2} = \frac{n^2 - 1}{4} + \left(i - \frac{n+1}{2} \right)^2 . \end{aligned}$$

Then

$$\begin{aligned} I &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}|i-j|\right)=\sum_{i=1}^{n}\left[i^{2}-(n+1)\,i+\frac{n\,(n+1)}{2}\right] \\ &= \frac{(n-1)\,n\,(n+1)}{3}, \end{aligned}$$

and the inequality (2.8) is proved. ∎

The following corollary provides a counterpart of the generalised triangle inequality in normed spaces.

Corollary 3. *Let $(X,\|\cdot\|)$ be a normed space and $x_i \in X$, $p_i \geq 0$ $(i=1,...,n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\| - \left\|\sum_{i=1}^{n} p_i x_i\right\| \leq \max_{k=1,\ldots,n-1} \|\Delta x_k\| \sum_{i,j=1}^{n} p_i p_j |j-i| \tag{2.9}$$

and in particular

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \frac{n^2-1}{3} \max_{k=1,\ldots,n-1} \|\Delta x_k\|. \tag{2.10}$$

The proof is obvious by Theorem 3 on choosing $F : X \to \mathbb{R}$, $F(X) = \|x\|$ which is $L-$Lipschitzian with $L = 1$ as

$$|\|x\| - \|y\|| \leq \|x-y\| \quad \text{for all } x, y \in X.$$

2.2. **Applications for Csiszár $\phi-$Divergence.** The following theorem holds.

Theorem 4. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be $L-$Lipschitzian on $\mathbb{R}_+$. Then for all $p, q \in \mathbb{R}_+^n$, we have the inequality*

$$\left|I_\phi(p,q) - Q_n \phi\left(\frac{P_n}{Q_n}\right)\right| \leq L \max_{k=1,\ldots,n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| \frac{1}{Q_n} \sum_{i,j=1}^{n} q_i q_j |i-j|. \tag{2.11}$$

Proof. We apply inequality (2.2) for $F = \phi$ and $p_i = \frac{q_i}{Q_n}$, $x_i = \frac{p_i}{q_i}$ to get

$$\begin{aligned} &\left|\phi\left(\frac{\sum_{i=1}^{n} p_i}{Q_n}\right) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right)\right| \\ &\leq L \max_{k=1,\ldots,n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| \frac{1}{Q_n^2} \sum_{i,j=1}^{n} q_i q_j |i-j|. \end{aligned}$$

from where we obtain (2.11). ∎

Corollary 4. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be $L-$Lipschitzian and normalised. Then for any $p, q \in \mathbb{R}_+^n$ with $P_n = Q_n$, we have the inequality*

$$0 \leq |I_\phi(p,q)| \leq L \max_{k=1,\ldots,n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| \frac{1}{Q_n} \sum_{i,j=1}^{n} q_i q_j |i-j|. \tag{2.12}$$

Corollary 5. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex, with a bounded derivative, that is, $\|\phi'\|_\infty := \sup_{t\geq 0} |\phi'(t)| < \infty$. Then we have the inequality:*

$$(2.13) \qquad \begin{aligned} 0 &\leq I_\phi(p,q) - Q_n \phi\left(\frac{P_n}{Q_n}\right) \\ &\leq \|\phi'\|_\infty \max_{k=1,\dots,n-1} \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| \frac{1}{Q_n} \sum_{i,j=1}^{n} q_i q_j |i-j| . \end{aligned}$$

Moreover, if ϕ is normalised and $P_n = Q_n$, then

$$(2.14) \qquad 0 \leq I_\phi(p,q) \leq \|\phi'\|_\infty \max_{k=1,\dots,n-1} \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| \frac{1}{Q_n} \sum_{i,j=1}^{n} q_i q_j |i-j| .$$

Recall the *Kullback-Leibler distance* given by (1.9)

$$(2.15) \qquad KL(p,q) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

We observe that, for the convex mapping $\phi(t) := -\log(t)$, $t > 0$,

$$(2.16) \qquad I_\phi(p,q) = \sum_{i=1}^{n} q_i \left[-\log\left(\frac{p_i}{q_i}\right)\right] = \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right) = KL(q,p).$$

The following proposition for the Kullback-Leibler distance holds.

Proposition 1. *Let $p, q \in \mathbb{R}_+^n$ satisfy the condition*

$$(2.17) \qquad 0 < m \leq r_k := \frac{p_k}{q_k} \quad \text{for all } k = 1, \dots, n.$$

Then, we have the inequality

$$(2.18) \qquad \begin{aligned} 0 &\leq KL(q,p) - Q_n \log\left(\frac{Q_n}{P_n}\right) \\ &\leq \frac{1}{m} \max_{k=1,\dots,n-1} |r_{k+1} - r_k| \frac{1}{Q_n} \sum_{i,j=1}^{n} q_i q_j |i-j| . \end{aligned}$$

Proof. As $\phi(t) = -\log(t)$, then $\phi'(t) = -\frac{1}{t}$, $t > 0$. The restriction of ϕ' in the interval $[m, \infty)$ is bounded and $\|\phi'\|_\infty = \sup_{t\in[m,\infty)} \left|\frac{1}{t}\right| = \frac{1}{m} < \infty$. Applying the inequality (2.13), we deduce (2.18). ∎

Remark 1. *If we assume that $P_n = Q_n$, then $m \leq 1$ and (2.18) becomes*

$$(2.19) \qquad 0 \leq KL(q,p) \leq \frac{1}{m} \max_{k=1,\dots,n-1} |r_{k+1} - r_k| \frac{1}{Q_n} \sum_{i,j=1}^{n} q_i q_j |i-j| .$$

We also know that for $\phi(t) = t\log t$, $t > 0$, the Csiszár $\phi-$divergence is (see (1.9)), $\phi(p,q) = KL(p,q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right)$.

The following proposition also holds.

Proposition 2. *Let* $p,q\in\mathbb{R}_+^n$ *satisfy the condition*

$$0<m\le r_k\le M<\infty \quad \text{for all } k=1,\dots,n. \tag{2.20}$$

Then we have the inequality

$$\begin{aligned} 0 &\le KL(q,p)-P_n\log\left(\frac{P_n}{Q_n}\right) \\ &\le \max\{|\log(Ml)|,|\log(ml)|\} \\ &\quad\times\max_{k=1,\dots,n-1}|r_{k+1}-r_k|\frac{1}{Q_n}\sum_{i,j=1}^{n}q_iq_j|i-j|. \end{aligned} \tag{2.21}$$

Proof. For the mapping $\phi(t)=t\log t$, $t>0$, we have $\phi'(t)=\log(t)+1$. On the interval $[m,M]$ we have

$$\log(m)+1\le\phi'(M)\le\log(M)+1,\quad t\in[m,M].$$

Applying the inequality (2.13), we deduce (2.21). ∎

Remark 2. *If we assume that* $P_n=Q_n$*, then* $m\le1\le M$ *and (2.21) becomes*

$$\begin{aligned} 0 &\le KL(q,p) \\ &\le \max\{|\log(Ml)|,|\log(ml)|\} \\ &\quad\times\max_{k=1,\dots,n-1}|r_{k+1}-r_k|\frac{1}{Q_n}\sum_{i,j=1}^{n}q_iq_j|i-j|. \end{aligned} \tag{2.22}$$

Let $\phi(t)=\left(\sqrt{t}-1\right)^2$, $t>0$. Then I_ϕ gives the *Hellinger distance*

$$H_e(p,q)=\sum_{i=1}^{n}\left(\sqrt{p_i}-\sqrt{q_i}\right)^2.$$

Using Corollary 5, we can state the following proposition.

Proposition 3. *Let* $p,q\in\mathbb{R}_+^n$ *satisfy the condition (2.20). Then we have the inequality*

$$\begin{aligned} 0 &\le H_e(p,q)-\left(\sqrt{P_n}-\sqrt{Q_n}\right)^2 \\ &\le \max\left\{\frac{|\sqrt{m}-1|}{\sqrt{m}},\frac{\left|\sqrt{M}-1\right|}{\sqrt{M}}\right\} \\ &\quad\times\max_{k=1,\dots,n-1}|r_{k+1}-r_k|\frac{1}{Q_n}\sum_{i,j=1}^{n}q_iq_j|i-j|. \end{aligned} \tag{2.23}$$

Proof. As $\phi(t)=\left(\sqrt{t}-1\right)^2$, $t>0$, then $\phi'(t)=1-\frac{1}{\sqrt{t}}$. If we consider the mapping ϕ' restricted to the interval $[m,M]\subset(0,\infty)$, then we observe that

$$\frac{|\sqrt{m}-1|}{\sqrt{m}}\le\phi'(t)\le\frac{\left|\sqrt{M}-1\right|}{\sqrt{M}},\quad t\in[m,M]$$

and thus

$$\|\phi'\|_\infty\le\max\left\{\frac{|\sqrt{m}-1|}{\sqrt{m}},\frac{\left|\sqrt{M}-1\right|}{\sqrt{M}}\right\}.$$

$\blacksquare$

Remark 3. *If we assume that $P_n = Q_n$, then $m \leq 1 \leq M$ and (2.23) becomes*

$$
\begin{aligned}
0 &\leq H_e(p,q) \\
&\leq \max\left\{\frac{|1-\sqrt{m}|}{\sqrt{m}}, \frac{\left|\sqrt{M}-1\right|}{\sqrt{M}}\right\} \\
&\quad \times \max_{k=1,\dots,n-1} |r_{k+1}-r_k| \frac{1}{Q_n}\sum_{i,j=1}^{n} q_i q_j |i-j|.
\end{aligned} \tag{2.24}
$$

Consider now the mapping $\phi(t) = t^\alpha$, $\alpha > 1$, $t > 0$ and the $\alpha-$*order entropy by Rényi* $\operatorname{Re}_\alpha(p,q) := \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}$.

We can state the following proposition.

Proposition 4. *Let $p,q \in \mathbb{R}_+^n$ be such that*

$$
0 < r_k \leq M < \infty \quad \textit{for all } k = 1,\dots,n. \tag{2.25}
$$

Then we have the inequality

$$
\begin{aligned}
0 &\leq \operatorname{Re}_\alpha(p,q) - P_n^\alpha Q_n^{1-\alpha} \\
&\leq \alpha M^{\alpha-1} \max_{k=1,\dots,n-1} |r_{k+1}-r_k| \frac{1}{Q_n}\sum_{i,j=1}^{n} q_i q_j |i-j|.
\end{aligned} \tag{2.26}
$$

In particular, if $P_n = Q_n$, then $M \geq 1$ and (2.26) becomes

$$
\begin{aligned}
0 &\leq \operatorname{Re}_\alpha(p,q) - Q_n \\
&\leq \alpha M^{\alpha-1} \max_{k=1,\dots,n-1} |r_{k+1}-r_k| \frac{1}{Q_n}\sum_{i,j=1}^{n} q_i q_j |i-j|.
\end{aligned} \tag{2.27}
$$

The proof is obvious by Corollary 5 applied for $\phi(t) = t^\alpha$, and we omit the details.

Finally, if we consider the χ^2-*distance* (see (1.15))

$$
D_{\chi^2}(p,q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}
$$

obtained from the Csiszár $\phi-$divergence for the choice $\phi(t) = (t-1)^2$, $t > 0$, then we can state the following proposition as well.

Proposition 5. *Let $p,q \in \mathbb{R}_+^n$ fulfill the properties of (2.20). Then we have the converse inequality*

$$
\begin{aligned}
0 &\leq D_{\chi^2}(p,q) - \frac{1}{Q_n}(P_n - Q_n)^2 \\
&\leq 2\max\{|m-1|, |M-1|\} \\
&\quad \times \max_{k=1,\dots,n-1} |r_{k+1}-r_k| \frac{1}{Q_n}\sum_{i,j=1}^{n} q_i q_j |i-j|.
\end{aligned} \tag{2.28}
$$

In particular, if $P_n = Q_n$, then $m \leq 1 \leq M$ and (2.28) becomes

$$0 \leq D_{\chi^2}(p,q) \leq 2\max\{1-m, M-1\} \times \max_{k=1,\ldots,n-1} |r_{k+1}-r_k| \frac{1}{Q_n}\sum_{i,j=1}^{n} q_i q_j |i-j| . \tag{2.29}$$

3. The Case of l_1-Norm

3.1. **Some Inequalities.** We start with the following result.

Theorem 5. *Let X,Y be two normed linear spaces with the norms $\|\cdot\|$ and $|\cdot|$ respectively. If $F : X \to Y$ is L - Lipschitzian, that is,*

$$|F(x) - F(y)| \leq L\|x-y\| \quad \text{for all } x, y \in X, \tag{3.1}$$

then for all $x_i \in X$, $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ $(i = 1, ..., n)$, we have the inequality

$$\left| F\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i F(x_i)\right| \leq L\sum_{i=1}^{n} p_i (1-p_i) \sum_{k=1}^{n-1} \|\Delta x_k\| . \tag{3.2}$$

Proof. As F is $L-$Lipschitzian, we can state (see the proof of Theorem 3) that

$$\left| F\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j F(x_j)\right| \leq 2L \sum_{1\leq i<j\leq n} p_i p_j \|x_j - x_i\| . \tag{3.3}$$

in (3.1). Now, observe that, for $i < j$, we have

$$x_j - x_i = \sum_{k=i}^{j-1} \Delta x_k,$$

where $\Delta x_k := x_{k+1} - x_k$ is the forward difference.

Therefore, as in the proof of Theorem 3, we have

$$\begin{aligned} &\sum_{1\leq i<j\leq n} p_i p_j \|x_j - x_i\| \\ &= \sum_{1\leq i<j\leq n} p_i p_j \left\|\sum_{k=i}^{j-1} \Delta x_k\right\| \leq \sum_{1\leq i<j\leq n} p_i p_j \sum_{k=i}^{j-1} \|\Delta x_k\| \\ &\leq \sum_{1\leq i<j\leq n} p_i p_j \sum_{k=1}^{n-1} \|\Delta x_k\| . \end{aligned} \tag{3.4}$$

Now, if we put

$$I = \sum_{1\leq i<j\leq n} p_i p_j,$$

we observe that

$$1 = \sum_{i,j=1}^{n} p_i p_j = 2\sum_{1\leq i<j\leq n} p_i p_j + \sum_{i=1}^{n} p_i^2 = 2I + \sum_{i=1}^{n} p_i^2$$

from where we deduce

$$I = \frac{1}{2}\sum_{i=1}^{n} p_i (1-p_i) .$$

Now, using (3.3) - (3.4) we deduce the desired inequality (2.2). ∎

Corollary 6. *With the above assumptions for F and x_i $(i = 1, ..., n)$, we have the inequality*

$$\left| F\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \frac{1}{n}\sum_{i=1}^{n} F(x_i)\right| \leq L \cdot \frac{n-1}{n}\sum_{k=1}^{n-1}\|\Delta x_k\| . \tag{3.5}$$

The proof is obvious by the above theorem, choosing $p_i = \frac{1}{n}$ $(i = 1, ..., n)$.

The following corollary provides a counterpart of the generalised triangle inequality.

Corollary 7. *Let $(X, \|\cdot\|)$ be a normed space and $x_i \in X$, $p_i \geq 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\| - \left\|\sum_{i=1}^{n} p_i x_i\right\| \leq \sum_{i=1}^{n} p_i (1 - p_i) \sum_{k=1}^{n-1}\|\Delta x_k\| \tag{3.6}$$

and in particular

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq (n-1)\sum_{k=1}^{n-1}\|\Delta x_k\| . \tag{3.7}$$

The proof is obvious by Theorem 5 on choosing $F : X \to \mathbb{R}$, $F(X) = \|x\|$ which is $L-$Lipschitzian with $L = 1$ as

$$|\|x\| - \|y\|| \leq \|x - y\| \quad \text{for all } x, y \in X.$$

3.2. **Applications for Csiszár $\phi-$Divergence.** The following theorem holds.

Theorem 6. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be $L-$Lipschitzian on $\mathbb{R}_+$. Then for all $p, q \in \mathbb{R}^n_+$, we have the inequality*

$$\left| I_\phi(p, q) - Q_n \phi\left(\frac{P_n}{Q_n}\right)\right| \leq \frac{L}{Q_n}\sum_{i=1}^{n} q_i (Q_n - q_i)\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| . \tag{3.8}$$

Proof. We apply inequality (3.2) for $F = \phi$ and $p_i = \frac{q_i}{Q_n}$, $x_i = \frac{p_i}{q_i}$ $(i = 1, \ldots, n)$ to get

$$\left|\phi\left(\frac{P_n}{Q_n}\right) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right)\right| \leq L\sum_{i=1}^{n}\frac{q_i (Q_n - q_i)}{Q_n^2}\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| .$$

from where we obtain (3.8). ∎

Corollary 8. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be $L-$Lipschitzian and normalised. Then for any $p, q \in \mathbb{R}^n_+$ with $P_n = Q_n$, we have the inequality*

$$0 \leq |I_\phi(p, q)| \leq \frac{L}{Q_n}\sum_{i=1}^{n} q_i (Q_n - q_i)\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| . \tag{3.9}$$

Corollary 9. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex, with a bounded derivative. Then we have the inequality:*

$$0 \leq I_\phi(p, q) - Q_n \phi\left(\frac{P_n}{Q_n}\right) \leq \frac{\|\phi'\|_\infty}{Q_n}\sum_{i=1}^{n} q_i (Q_n - q_i)\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| . \tag{3.10}$$

Moreover, if ϕ is normalised and $P_n = Q_n$, then

$$0 \leq I_\phi(p,q) \leq \frac{\|\phi'\|_\infty}{Q_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} \left| \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right|. \tag{3.11}$$

Now, let us recall the *Kullback-Leibler distance*

$$KL(p,q) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right). \tag{3.12}$$

We observe that, for the convex mapping $\phi(t) := -\log(t)$, $t > 0$,

$$I_\phi(p,q) = \sum_{i=1}^{n} q_i \left[-\log\left(\frac{p_i}{q_i}\right)\right] = \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right) = KL(q,p). \tag{3.13}$$

The following proposition for the Kullback-Leibler distance holds.

Proposition 6. *Let $p, q \in \mathbb{R}^n_+$ satisfy the condition*

$$0 < m \leq r_k := \frac{p_k}{q_k} \quad \text{for all } k = 1, \dots, n. \tag{3.14}$$

Then, we have the inequality

$$\begin{aligned} 0 &\leq KL(q,p) - Q_n \log\left(\frac{Q_n}{P_n}\right) \\ &\leq \frac{1}{mQ_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k|. \end{aligned} \tag{3.15}$$

Proof. As $\phi(t) = -\log(t)$, then $\phi'(t) = -\frac{1}{t}$, $t > 0$. The restriction of ϕ' in the interval $[m, \infty)$ is bounded and $\|\phi'\|_\infty = \sup_{t\in[m,\infty)} \left|\frac{1}{t}\right| = \frac{1}{m} < \infty$. Applying the inequality (3.10), we deduce (3.15). ∎

Remark 4. *If we assume that $P_n = Q_n$, then $m \leq 1$ and (3.15) becomes*

$$0 \leq KL(q,p) \leq \frac{1}{mQ_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k|. \tag{3.16}$$

We also know that for $\phi(t) = t \log t$, $t > 0$, the Csiszár $\phi-$divergence is

$$\phi(p,q) = KL(p,q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

The following proposition also holds.

Proposition 7. *Let $p, q \in \mathbb{R}^n_+$ satisfy the condition*

$$0 < m \leq r_k \leq M < \infty \quad \text{for all } k = 1, \dots, n. \tag{3.17}$$

Then we have the inequality

$$\begin{aligned} 0 &\leq KL(q,p) - P_n \log\left(\frac{P_n}{Q_n}\right) \\ &\leq \frac{\max\{|\log(Ml)|, |\log(ml)|\}}{Q_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k|. \end{aligned} \tag{3.18}$$

Proof. For the mapping $\phi(t) = t\log t$, $t > 0$, we have $\phi'(t) = \log(t) + 1$. On the interval $[m, M]$ we have

$$\log m + 1 \leq \phi'(t) \leq \log M + 1, \quad t \in [m, M]$$

and hence

$$|\phi'(t)| \leq \max\{|\log(Ml)|, |\log(ml)|\}, \quad t \in [m, M].$$

Applying the inequality (3.10), we deduce (3.18). ∎

Remark 5. *If we assume that* $P_n = Q_n$*, then (2.21) becomes*

$$\begin{aligned} (3.19) \quad 0 &\leq KL(q,p) \\ &\leq \frac{\max\{|\log(Ml)|, |\log(ml)|\}}{Q_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k|. \end{aligned}$$

Let $\phi(t) = \left(\sqrt{t} - 1\right)^2$, $t > 0$. Then I_ϕ gives the *Hellinger distance*

$$He(p,q) = \sum_{i=1}^{n} \left(\sqrt{p_i} - \sqrt{q_i}\right)^2,$$

Using Corollary 9, we may state the following proposition.

Proposition 8. *Assume that* $p, q \in \mathbb{R}_+^n$ *satisfy the condition (3.17). Then we have the inequality*

$$\begin{aligned} (3.20) \quad & 0 \leq He(p,q) - \left(\sqrt{P_n} - \sqrt{Q_n}\right)^2 \\ &\leq \max\left\{\frac{|\sqrt{m} - 1|}{\sqrt{m}}, \frac{\left|\sqrt{M} - 1\right|}{\sqrt{M}}\right\} \frac{1}{Q_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k|. \end{aligned}$$

Proof. As $\phi(t) = \left(\sqrt{t} - 1\right)^2$, $t > 0$, then $\phi'(t) = 1 - \frac{1}{\sqrt{t}}$. If we consider the mapping ϕ' restricted to the interval $[m, M] \subset (0, \infty)$, then we observe that

$$\frac{|\sqrt{m} - 1|}{\sqrt{m}} \leq \phi'(t) \leq \frac{\left|\sqrt{M} - 1\right|}{\sqrt{M}}, \quad t \in [m, M]$$

and thus

$$\|\phi'\|_\infty \leq \max\left\{\frac{|\sqrt{m} - 1|}{\sqrt{m}}, \frac{\left|\sqrt{M} - 1\right|}{\sqrt{M}}\right\}.$$

∎

Remark 6. *If we assume that* $P_n = Q_n$*, then* $m \leq 1 \leq M$ *and (2.23) becomes*

$$\begin{aligned} (3.21) \quad & 0 \leq He(p,q) \\ &\leq \max\left\{\frac{1 - \sqrt{m}}{\sqrt{m}}, \frac{\sqrt{M} - 1}{\sqrt{M}}\right\} \frac{1}{Q_n} \sum_{i=1}^{n} q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k|. \end{aligned}$$

Consider now the mapping $\phi(t) = t^\alpha$, $\alpha > 1$, $t > 0$ and the $\alpha-$*order entropy by Rényi* $\mathrm{Re}_\alpha(p,q) := \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}$.

We can state the following proposition.

Proposition 9. *Let $p, q \in \mathbb{R}_+^n$ be such that*

$$0 < r_k \le M < \infty \quad \text{for all } k = 1, \dots, n. \tag{3.22}$$

Then we have the inequality

$$\begin{aligned} 0 &\le \operatorname{Re}_\alpha(p,q) - P_n^\alpha Q_n^{1-\alpha} \\ &\le \alpha M^{\alpha-1} \frac{1}{Q_n} \sum_{i=1}^n q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k| . \end{aligned} \tag{3.23}$$

In particular, if $P_n = Q_n$, then $M \ge 1$ and (3.23) becomes

$$\begin{aligned} 0 &\le \operatorname{Re}_\alpha(p,q) - Q_n \\ &\le \alpha M^{\alpha-1} \frac{1}{Q_n} \sum_{i=1}^n q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k| . \end{aligned} \tag{3.24}$$

The proof is obvious by Corollary 9 applied for $\phi(t) = t^\alpha$, and we omit the details.

Finally, if we consider the χ^2-*distance*

$$D_{\chi^2}(p,q) := \sum_{i=1}^n \frac{(p_i - q_i)^2}{q_i}$$

obtained from the Csiszár $\phi-$divergence for the choice $\phi(t) = (t-1)^2$, then we can state the following proposition too.

Proposition 10. *Let $p, q \in \mathbb{R}_+^n$ fulfill the conditions of (2.20). Then we have the inequality*

$$\begin{aligned} 0 &\le D_{\chi^2}(p,q) - \frac{1}{Q_n}(P_n - Q_n)^2 \\ &\le 2\max\{|m-1|, |M-1|\} \frac{1}{Q_n} \sum_{i=1}^n q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k| . \end{aligned} \tag{3.25}$$

In particular, if $P_n = Q_n$, then (3.25) becomes

$$\begin{aligned} 0 &\le D_{\chi^2}(p,q) \\ &\le 2\max\{1-m, M-1\} \frac{1}{Q_n} \sum_{i=1}^n q_i (Q_n - q_i) \sum_{k=1}^{n-1} |r_{k+1} - r_k| . \end{aligned} \tag{3.26}$$

4. The Case of l_p-Norm

4.1. **Some Inequalities.** We start with the following result.

Theorem 7. *Let $X, Y, F, (p_i)_{i=\overline{1,n}}$ be as in Theorem 5. Then we have the inequality:*

$$\left| F\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i F(x_i) \right| \le L \sum_{i,j=1}^n p_i p_j |j-i|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}}, \tag{4.1}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. As in the proof of Theorem 3, we have

$$\left|F\left(\sum_{i=1}^{n}p_ix_i\right)-\sum_{j=1}^{n}p_jF\left(x_j\right)\right|\leq 2L\sum_{1\leq i<j\leq n}p_ip_j\left\|x_j-x_i\right\|. \tag{4.2}$$

Also,

$$\sum_{1\leq i<j\leq n}p_ip_j\left\|x_j-x_i\right\|\leq\sum_{1\leq i<j\leq n}p_ip_j\sum_{k=i}^{j-1}\left\|\Delta x_k\right\|. \tag{4.3}$$

Using Hölder's discrete inequality, we may write for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}\sum_{k=i}^{j-1}\left\|\Delta x_k\right\| &\leq \left(\sum_{k=i}^{j-1}1\right)^{\frac{1}{q}}\left(\sum_{k=i}^{j-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}=(j-i)^{\frac{1}{q}}\left(\sum_{k=i}^{j-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}\\ &\leq (j-i)^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}\end{aligned}$$

and then, by (4.3), we get

$$\begin{aligned}\sum_{1\leq i<j\leq n}p_ip_j\left\|x_j-x_i\right\| &\leq \sum_{1\leq i<j\leq n}p_ip_j\left(j-i\right)^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}\\ &= \frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\left|j-i\right|^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Using (4.2) we deduce (4.1). ∎

Corollary 10. *Let $(X,\|\cdot\|)$ be a normed space and $x_i\in X$, $p_i\geq 0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n}p_i=1$. Then we have the inequality:*

$$0\leq\sum_{i=1}^{n}p_i\left\|x_i\right\|-\left\|\sum_{i=1}^{n}p_ix_i\right\|\leq\sum_{i,j=1}^{n}p_ip_j\left|j-i\right|^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|^p\right)^{\frac{1}{p}}. \tag{4.4}$$

4.2. **Applications for Csiszár $\phi-$Divergence.** The following result for Csiszár $f-$divergence holds.

Theorem 8. *Let $\phi:\mathbb{R}_+\to\mathbb{R}$ be $L-$Lipschitzian on $\mathbb{R}_+$. Then for all $p,q\in\mathbb{R}_+^n$, we have the inequality:*

$$\left|I_\phi\left(p,q\right)-Q_n\phi\left(\frac{P_n}{Q_n}\right)\right|\leq\frac{L}{Q_n}\sum_{i,j=1}^{n}q_iq_j\left|j-i\right|^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}}-\frac{p_k}{q_k}\right|^p\right)^{\frac{1}{p}}, \tag{4.5}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We apply inequality (4.1) for $F = \phi$, $p_i = \frac{q_i}{Q_n}$, $x_i = \frac{p_i}{q_i}$ to get

$$\begin{aligned}&\left|\phi\left(\frac{P_n}{Q_n}\right) - \frac{1}{Q_n}\sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right)\right| \\ \le\ & L \sum_{i,j=1}^{n} \frac{q_i}{Q_n} \cdot \frac{q_j}{Q_n} \left|j-i\right|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right|^p\right)^{\frac{1}{p}}\end{aligned}$$

from where we obtain (4.5). ∎

Corollary 11. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be $L-$Lipschitzian and normalised. Then for all $p, q \in \mathbb{R}_+^n$ with $P_n = Q_n$, we have*

$$(4.6)\qquad 0 \le \left|I_\phi(p,q)\right| \le \frac{c}{Q_n}\sum_{i,j=1}^{n} q_i q_j \left|j-i\right|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right|^p\right)^{\frac{1}{p}}.$$

Corollary 12. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex with a bounded derivative. Then*

$$(4.7)\qquad \begin{aligned} 0 &\le I_\phi(p,q) - Q_n \phi\left(\frac{P_n}{Q_n}\right) \\ &\le \frac{\left\|\phi'\right\|_\infty}{Q_n}\sum_{i,j=1}^{n} q_i q_j \left|j-i\right|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Moreover, if ϕ is normalised and $P_n = Q_n$, then

$$(4.8)\qquad 0 \le I_\phi(p,q) \le \frac{\left\|\phi'\right\|_\infty}{Q_n}\sum_{i,j=1}^{n} q_i q_j \left|j-i\right|^{\frac{1}{q}} \left(\sum_{k=1}^{n-1}\left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right|^p\right)^{\frac{1}{p}}.$$

Remark 7. *Inequalities for particular divergences as in the previous two sections can be stated, but we omit the details.*

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