Greek means and the arithmetic-geometric mean

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Chapter 1

Introduction

Our goal in writing this book was twofold. First of all, we make a short incursion in the history of mathematics. The subjects in which we are interested, means and double sequences, are related to names like those of Pythagoras, Archimedes, Heron, Lagrange and Gauss, for giving the most famous of them. The second aim was to give the present stage of development of the problems we are dealing with. Of course, we insist on our own results, which are published in Romanian journals with limited distribution and so they are less known.

Here, as in the rest of the book, for referring to a paper or on a book, we indicate, in brackets, the name of the author(s) and the year of publication. If an author is present in the references with more papers published in the same year, we add a small letter after the corresponding year.

Pappus of Alexandria presented in his books, in the fourth century AD, the main mathematical contributions of the ancient Greeks (see [Pappus, 1932]). Among them we can find the means defined by the Pytagorean school: four well known means, the arithmetic mean, defined by

$$\mathcal{A}(a,b)=\frac{a+b}{2},\ a,b>0,$$

the geometric mean, given by

$$\mathcal{G}(a,b)=\sqrt{ab},\ a,b>0,$$

the harmonic mean, with the expression

$$\mathcal{H}(a,b)=\frac{2ab}{a+b},\ a,b>0,$$

the contraharmonic mean, defined by

$$\mathcal{C}(a,b)=\frac{a^2+b^2}{a+b},\ a,b>0,$$

and six unnamed means. These means are the only ten means which can be defined using the method of proportions, which is attributed to Pythagoras of Samos (569-500 BC) (see [C. Gini, 1958; C. Antoine, 1998]), but also to Eudoxus (see [P. Eymard, J.-P. Lafon, 2004]). Having no access to original sources, we must content ourselves to present such controversies, without taking any adherent position.

The method may be described as follows. Consider a set of three numbers with the property that two of their differences are in the same ratio as two of the initial numbers. More exactly, if the numbers are $a,m,b>0$, the first member of the proportion can be one of the ratios

$$\frac{a-m}{a-b},\frac{a-b}{m-b},\ \text{or}\ \frac{a-m}{m-b},$$

while the second member must be

$$\frac{a}{a},\frac{a}{b},\frac{b}{a},\frac{a}{m},\frac{m}{a},\frac{b}{m},\ \text{or}\ \frac{m}{b}.$$

Thus we have twenty one proportions. Such a proportion defines a mean if $a>b>0$ implies $a>m>b$. Namely, the value of m represents the mean of a and b. As it is stated in [C. Gini, 1958], we get only ten means, the Greek means.

We will define these means in the first part of the book. We give some of their properties and relations, as they are presented in [Silvia Toader, G. Toader, 2002]. A special attention is devoted to the determination of all complementaries of a Greek mean with respect to another, which was done in [G. Toader, 2004; Silvia Toader, G. Toader, 2004, 2004a]. The importance of this problem, for the determination of the common limit of some double sequences, will be presented a few lines later. A last subject developed in this part is devoted to the weighted Greek means, which were defined in [G. Toader, 2005]. For doing this, the first member of the proportions was multiplied by the report $\lambda/(\lambda-1)$, where $\lambda\in(0,1)$ is a parameter. The weighted variants of the arithmetic, harmonic and contraharmonic means are those well known for long time. For the geometric mean is obtained a weighted mean which is different from the classical known variant. The other six weighted means are new.

The second part of the book is devoted to double sequences. First of all we present some classical examples. The oldest was given by Archimedes of Syracuse (287-212 BC) in his book Measurement of the Circle. His problem was the evaluation of the number π, defined as the ratio of the perimeter of a circle to its diameter. Consider a circle of radius 1 and denote by p_n and P_n the half of the perimeters of the inscribed and circumscribed regular polygons with n sides, respectively. As

$$p_n < \pi < P_n,\ n \geq 3,$$

to get an estimation with any accuracy, Archimedes passes from a given n to $2n$, proving his famous inequalities

$$3.1408 < 3\frac{10}{71} < p_{96} < \pi < P_{96} < 3\frac{1}{7} < 3.1429\ .$$

The procedure was so defined as a tongs method. But, as was shown in [G. M. Phillips, 1981], it can be presented also as a double sequence. Denoting $P_{2^k n} = a_k$ and $p_{2^k n} = b_k$, he proved that the sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are given, step by step, by the relations

$$a_{k+1} = \mathcal{H}(a_n, b_n), b_{k+1} = \mathcal{G}(a_{n+1}, b_n), k \geq 0,$$

for some initial values a_0 and b_0. Also it is shown that these sequences are monotonously convergent to a common limit, which for $0 < b_0 < a_0$ has the value

$$\frac{a_0 b_0}{\sqrt{a_0^2 - b_0^2}} \arccos \frac{b_0}{a_0}\ .$$

In Archimedes' case, as

$$a_0 = P_3 = 3\sqrt{3} \text{ and } b_0 = p_3 = 3\sqrt{3}/2\ ,$$

the common limit is π.

The second example of double sequences is furnished by Heron's method of extracting square roots. To compute the geometric root of two numbers a and b, Heron used the arithmetic mean and the harmonic mean. Putting $a_0 = a$ and $b_0 = b$, define

$$a_{k+1} = \mathcal{H}(a_n, b_n), b_{k+1} = \mathcal{A}(a_n, b_n), k \geq 0.$$

It is proved that the sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are monotonously convergent to the common limit $\sqrt{ab}$. Of course, the procedure was also used only as a tongs method, the notion of limit being unknown in Heron's time (fl. c. 60, as it is given in [P. S. Bullen, 2003]).

The third example is Lagrange's method of determination of some irrational integrals. In [J. -L. Lagrange, 1784-85], for the evaluation of an integral of the form

$$\int \frac{N(y^2)dy}{\sqrt{(1+p^2y^2)(1+q^2y^2)}} \, ,$$

where N is a rational function, is defined an iterative method which leads to the rationalization of the function. It is based on the double sequence defined by $a_0 = p$, $b_0 = q$ and

$$a_{k+1} = \mathcal{A}(a_n, b_n), b_{k+1} = \mathcal{G}(a_n, b_n), k \geq 0.$$

The same double sequence was defined in [C. F. Gauss, 1800], which was published only many years later. For $a_0 = \sqrt{2}$ and $b_0 = 1$, he remarked that a_4 and b_4 have the same first eleven decimals as those determined in [L. Euler, 1782] for the integral

$$2\int_0^1 \frac{z^2dz}{\sqrt{1-z^4}}.$$

In fact the sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are monotonously convergent to a common limit, which is now known under the name of arithmetic-geometric mean and it is denoted by $\mathcal{A}\otimes\mathcal{G}(a,b)$. Later Gauss was able to represent it using an elliptic integral, by

$$\mathcal{A}\otimes\mathcal{G}(a,b) = \frac{\pi}{2}\cdot\left[\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\right]^{-1}.$$

Of course, the result is used for the numerical evaluation of the elliptic integral.

After presenting some results related to these special double sequences, we pass to the general definition of double sequences, that is the special means $\mathcal{A}$, $\mathcal{G}$, or $\mathcal{H}$, which appear in the definition of these examples are replaced by arbitrary means M and N. Before doing this, we have to remark that the essential difference between the first example and the other examples is the use of a_{k+1} in the definition of b_{k+1}. So we can define double sequences of type

$$a_{k+1} = M(a_n, b_n), b_{k+1} = N(a_{n+1}, b_n), k \geq 0,$$

which are called in [G. M. Phillips, 1981] Archimedean double sequences, or of the type

$$a_{k+1} = M(a_n, b_n), b_{k+1} = N(a_n, b_n), k \geq 0,$$

named in [D. M. E. Foster, G. M. Phillips, 1984] Gaussian double sequences. It is easy to see that an Archimedean double sequence can be written as a Gaussian one, by changing the mean N. So, in what follows, we can content ourself to discuss only this last case.

The first problem which was studied related to a given double sequence is that of the convergence of the sequences $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ to a common limit. In this case it is proved that the common limit defines a mean, denoted by $M \otimes N$, and called the compound mean of M and N. We give in References a lot of papers which deal with this problem. We mention here only a few of them, [D. M. E. Foster, G. M. Phillips, 1985; G. Toader, 1990, 1991; Iulia Costin, G. Toader, 2004, 2004a], in which one looks after minimal conditions on the means M and N that assure that their compound mean exists.

The second main problem was that of the determination of $M \otimes N$. Using the idea by which Gauss was able to give the representation of $\mathcal{A} \otimes \mathcal{G}$, in [J. M. Borwein, P. B. Borwein, 1987] is proved the following Invariance Principle: if $M \otimes N$ exists and is continuous, then it is the unique mean P satisfying

$$P(M(a, b), N(a, b)) = P(a, b),\ \forall a, b > 0.$$

In this case, the mean P is called $(M, N)-$ invariant. As we can see in the case of $\mathcal{A} \otimes \mathcal{G}$, it is not at all easy to determine a mean invariant with respect to two given means. Thus becomes important the change of the point of view, as it is done in [G. Toader, 1991]. Given the means M and P, if the previous relation is satisfied, the mean N is named complementary of M with respect to P. As we saw before, we have determined all the complementaries of a Greek mean with respect to another. On this way, we are able to construct ninety double sequences with known limits. Among them we get only eight cases in which by composing two Greek means we get again a Greek mean. Of course Heron's example is one of them.

Finally, the rate of convergence of the sequences to the common limit is also studied. In our cases it is quadratic and even faster as suggest the numerical examples.

We hope that this book will contribute to the spreading of the ancient Greek mathematical knowledge about the means. We underline also the modern possibilities of development of the subject, first of all related to the work of C. F. Gauss on double sequences.

Chapter 2

Means

This part is devoted to the study of Greek means. We remind first the method of proportions used by the Pytagorean school for the definition of means. We present the ten means constructed on this way, the Greek means. Then we study some of their properties, define more relations that hold among them and determine all the complementaries of a Greek mean with respect to another. Finally we present weighted variants of the Greek means.

2.1 Greek means

As many other important Greek mathematical contributions, the means defined by the Pytagorean school were presented by Pappus of Alexandria in his books (see[Pappus, 1932]). Some indications about them can be found in the books [C. Gini, 1958; J. M. Borwein, P. B. Borwein, 1986; C. Antoine, 1998]. We present here a variant of the original construction of the means, but also their modern transcriptions. We select some properties of these means and some relations among them, as they are given in [Silvia Toader, G. Toader, 2002].

Pythagoras of Samos (569-500 BC) already knew the arithmetic mean $\mathcal{A}$, the geometric mean $\mathcal{G}$, and the harmonic mean $\mathcal{H}$. To construct them he used the method of proportions: he considered a set of three numbers with the property that two of their differences are in the same ratio as two of the initial numbers.

More exactly, let $a > m > b > 0$. Then m represents:

1. the **arithmetic mean** of a and b if

$$\frac{a-m}{m-b}=\frac{a}{a}\ ;$$

2. the **geometric mean** of a and b if

$$\frac{a-m}{m-b}=\frac{a}{m}=\frac{m}{b}\ ;$$

3. the **harmonic mean** of a and b if

$$\frac{a-m}{m-b}=\frac{a}{b}\ ;$$

Following [C. Antoine, 1998], three other means, including the contraharmonic mean $\mathcal{C}$, were defined by Eudoxus, and finally other four means by Temnoides and Euphranor. In [C. Gini,1958] all these seven means are attributed to Nicomah. Only three of these new means have a name:

4. the **contraharmonic mean** of a and b defined by

$$\frac{a-m}{m-b}=\frac{b}{a}\ ;$$

5. the **first contrageometric mean** of a and b defined by

$$\frac{a-m}{m-b}=\frac{b}{m}\ ;$$

6. the **second contrageometric mean** of a and b defined by the proportion

$$\frac{a-m}{m-b}=\frac{m}{a}\ ;$$

The rest of four no-named means are defined by the relations:

7.

$$\frac{a-m}{a-b}=\frac{b}{a}\ ;$$

8.

$$\frac{a-m}{a-b}=\frac{m}{a}\ ;$$

9.

$$\frac{a-b}{m-b}=\frac{a}{b}\ ;$$

10.
$$\frac{a-b}{m-b} = \frac{m}{b} \ .$$

As it is remarked in [C. Gini, 1958] we can consider more such proportions. In fact, the first member can be one of the ratios

$$\frac{a-m}{a-b}, \frac{a-b}{m-b}, \text{ or } \frac{a-m}{m-b},$$

while the second member must be

$$\frac{a}{a}, \frac{a}{b}, \frac{b}{a}, \frac{a}{m}, \frac{m}{a}, \frac{b}{m}, \text{ or } \frac{m}{b}.$$

Thus we have twenty one proportions but we get no other nontrivial mean.

Solving each of the above relations, as an equation with unknown term m , we get the analytic expressions $M(a,b)$ of the means. For the first four means we use the classical notations. For the other six, we accept the neutral symbols proposed in [J. M. Borwein, P. B. Borwein, 1986]. We get so, in order, the following means:

1.
$$\mathcal{A}(a,b) = \frac{a+b}{2} \ ;$$

2.
$$\mathcal{G}(a,b) = \sqrt{ab} \ ;$$

3.
$$\mathcal{H}(a,b) = \frac{2ab}{a+b} \ ;$$

4.
$$\mathcal{C}(a,b) = \frac{a^2+b^2}{a+b} \ ;$$

5.
$$\mathcal{F}_5(a,b) = \frac{a-b+\sqrt{(a-b)^2+4b^2}}{2} \ ;$$

6.
$$\mathcal{F}_6(a,b) = \frac{b-a+\sqrt{(a-b)^2+4a^2}}{2} \ ;$$

7.
$$\mathcal{F}_7(a,b) = \frac{a^2-ab+b^2}{a} \ ;$$

8.
$$\mathcal{F}_8(a,b) = \frac{a^2}{2a-b}\ ;$$

9.
$$\mathcal{F}_9(a,b) = \frac{b(2a-b)}{a}\ ;$$

10.
$$\mathcal{F}_{10}(a,b) = \frac{b+\sqrt{b(4a-3b)}}{2}\ .$$

Sometimes it is convenient to refer to all means by the neutral notation considering that

$$\mathcal{F}_1 = \mathcal{A},\ \mathcal{F}_2 = \mathcal{G},\ \mathcal{F}_3 = \mathcal{H} \text{ and } \mathcal{F}_4 = \mathcal{C}.$$

The first four expressions of the Greek means are symmetric, that is we can use them also to define the corresponding means for $a < b$. For the other six expressions, we have to replace a with b to define the means on $a < b$.

2.2 Definition and properties of means

There are more definitions of means as we can see in the book [C. Gini, 1958]. The most used definition may be found in the book [G. H. Hardy, J. E. Littlewood, G. Pòlya, 1934] but it was suggested even by Cauchy (as it is stated in [C. Gini, 1958]).

Definition 1 *A* ***mean*** *is defined as a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, which has the property*
$$a \wedge b \leq M(a,b) \leq a \vee b,\ \forall a,b > 0 \tag{2.1}$$
where
$$a \wedge b = \min(a,b) \text{ and } a \vee b = \max(a,b).$$

Regarding the properties of means, of course, each mean is **reflexive**, that is
$$M(a,a) = a,\ \forall a > 0,$$
which will be used also as definition of $M(a,a)$ if it is necessary.

A mean can have additional properties.

Definition 2 *The mean M is called:*
a) ***symmetric*** *if*

$$M(a,b) = M(b,a),\ \forall a,b > 0\ ; \tag{2.2}$$

b) ***homogeneous*** *(of degree one) if*

$$M(ta,tb) = t \cdot M(a,b),\ \forall t,a,b > 0\ ; \tag{2.3}$$

c) ***(strictly) isotone*** *if, for $a, b > 0$*

$$M(a,.) \text{ and } M(.,b)$$

are (strictly) increasing;
d) ***strict at the left*** *if*

$$M(a,b) = a\ \Rightarrow a = b\ , \tag{2.4}$$

strict at the right *if*

$$M(a,b) = b\ \Rightarrow a = b\ , \tag{2.5}$$

and ***strict*** *if is strict at the left and strict at the right.*

In what follows, we shall use the following obvious

Lemma 3 *A mean M is isotone if and only if*

$$M(a,b) \leq M(a',b')\ ,\ \text{for all } a \leq a', b \leq b'. \tag{2.6}$$

Example 4 *Of course, $\wedge$ and $\vee$ are also means. We can consider them as trivial Greek means defined by the proportions*

$$\frac{a-b}{a-m} = \frac{a}{a},$$

respectively

$$\frac{a-b}{m-b} = \frac{a}{a}.$$

They are symmetric, homogeneous and isotone, but are not strict neither at the left, nor at the right.

Remark 5 *In [J. M. Borwein, P. B. Borwein, 1986] are used these means for the definition of the Greek means. Namely, a is replaced by $a \vee b$ and b by $a \wedge b$. So we get expressions of the following type*

$$M(a \vee b, a \wedge b).$$

With this construction, any mean becomes symmetric.

Remark 6 *Simple examples of non symmetric means may be given by the projections Π_1 and Π_2 defined respectively by*

$$\Pi_1(a,b) = a,\ \Pi_2(a,b) = b,\ \forall a, b > 0.$$

Of course Π_1 is strict only at the right while Π_2 is strict only at the left. They cannot be defined by the method of proportions.

Remark 7 *In [Silvia Toader, G. Toader, 2002] is proved that all the functions $\mathcal{F}_i, i = 1, 2, ..., 10$ are means. We shall give this proof later, but we refer at them always as means. All the Greek means are homogeneous and strict. The monotony of the above means is also studied in [Silvia Toader, G. Toader, 2002]. We have the following results.*

Theorem 8 *For $a > b > 0$ the Greek means have the following monotonicities: 1) All the means are increasing with respect to a on (b, ∞). 2) The means $\mathcal{A}, \mathcal{G}, \mathcal{H}, \mathcal{F}_6$, $\mathcal{F}_8$, $\mathcal{F}_9$ and $\mathcal{F}_{10}$ are increasing with respect to b on $(0, a)$, thus they are isotone. 3) For each of the means $\mathcal{C}$, $\mathcal{F}_5$ and $\mathcal{F}_7$ there is a number $0 < p < 1$ such that the mean is decreasing with respect to b on the interval $(0, p \cdot a)$ and increasing on $(p \cdot a, a)$. These means have the values $M(a, 0) = M(a, a) = a$ and $M(a, pa) = qa$, respectively. The values of the constants p, q are given in the following table:*

M	$\mathcal{C}$	$\mathcal{F}_5$	$\mathcal{F}_7$
p	$\sqrt{2}-1$	$2/5$	$1/2$
q	$2(\sqrt{2}-1)$	$4/5$	$3/4$

2.3 Comparison of means

Before proving that all of the above functions are means, we give some inequalities among them. We write

$$M \le N$$

to denote

$$M(a,b) < N(a,b),\ \forall a,b > 0.$$

We say that M is **comparable to** N if

$$M \le N \text{ or } N \le M.$$

In [Silvia Toader, G. Toader, 2002] is proved the following

Theorem 9 *Among the Greek means we have only the following inequalities:*

$$\mathcal{H} \le \mathcal{G} \le \mathcal{A} \le \mathcal{F}_6 \le \mathcal{F}_5 \le \mathcal{C}$$

$$\mathcal{H} \le \mathcal{F}_9 \le \mathcal{F}_{10},\ \mathcal{F}_8 \le \mathcal{F}_7 \le \mathcal{F}_5 \le \mathcal{C}$$

$$\mathcal{F}_8 \le \mathcal{A} \le \mathcal{F}_6 \le \mathcal{F}_5 \le \mathcal{C}$$

and

$$\mathcal{G} \le \mathcal{F}_{10}\ .$$

These inequalities are easy to prove. In the next paragraph will be studied other relations among the Greek means. They are more complicated.

Corollary 10 *The functions $\mathcal{F}_k$, $k = 1, 2, ..., 10$ are means.*

It is enough to prove that

$$\wedge \le \mathcal{H}, \wedge \le \mathcal{F}_8, \mathcal{C} \le \vee \text{ and } \mathcal{F}_{10} \le \vee$$

which are simple computations.

Given two means M and N, we can define the means $M \vee N$ and $M \wedge N$ by

$$M \vee N(a,b) = \max\{M(a,b), N(a,b)\},$$

respectively

$$M \wedge N(a,b) = \min\{M(a,b), N(a,b)\}.$$

Of course, if $M < N$ then $M \wedge N = M,\ M \vee N = N$, but generally we get the inequalities

$$M \wedge N \leq M \leq M \vee N$$

and

$$M \wedge N \leq N \leq M \vee N.$$

Remark 11 *As it is well known (see [P. S. Bullen, 2003]), now are known much more means (of course defined by other methods). We mention here two families of means which will be used later: the power means, defined by*

$$P_n(a,b) = \left(\frac{a^n + b^n}{2}\right)^{\frac{1}{n}}, n \neq 0$$

and the generalized contraharmonic means (or Lehmer means), defined by

$$C_n(a,b) = \frac{a^n + b^n}{a^{n-1} + b^{n-1}}.$$

Of course $P_1 = \mathcal{A}, P_{-1} = \mathcal{H},\ C_2 = \mathcal{C}$ *and usually* $P_0 = \mathcal{G}$ *is also taken. It is known that these families of means are increasing with respect to the parameter* n*, that is*

$$P_n \leq P_m \text{ and } C_n \leq C_m \text{ for } n \leq m.$$

2.4 Weak relations and symmetric angular relations

In [I. J. Schoenberg, 1982; D. M. E. Foster, G. M. Phillips, 1985] a comparability of means on a subset was considered.

Definition 12 *The means* M *and* N *are in the relation*

$$M \leq_D N,$$

where $D \subset \mathbb{R}_+^2$ *, if*

$$M(a,b) \leq N(a,b),\ \forall (a,b) \in D.$$

If the last inequality is strict for $a \neq b$, we write

$$M <_D N.$$

If $M <_D N$ and $N \leq_{D'} M$, where $D' = \mathbb{R}_+^2 \setminus D$, we write

$$M \prec_D N.$$

Remark 13 *If the means M and N are symmetric and $M \prec_D N$ then D has a kind of* ***symmetry****, namely:*

$$(a, b) \in D \Rightarrow (b, a) \in D.$$

For non-symmetric means, in [G. Toader, 1987] was given the following

Definition 14 *The means M and N are in the* ***weak relation***

$$M \prec N$$

if $M \prec_D N$ for $D = \left\{(x, y) \in \mathbb{R}_+^2; x < y\right\}$.

The comparison of homogeneous means can be done only on special sets.

Definition 15 *The set $D \subset \mathbb{R}^2$ is called* ***starshaped*** *if*

$$(a, b) \in D, t > 0 \Rightarrow (ta, tb) \in D.$$

It is easy to prove the following property.

Lemma 16 *If the means M and N are homogeneous and $M \prec_D N$ then the set D is starshaped.*

The simplest relation of this kind was given in [Silvia Toader, G. Toader, 2002]. Let $m > 1$.

Definition 17 *The means M and N are in the* ***symmetric angular relation***

$$M \prec_m N$$

if $M \prec_D N$ for $D = \left\{(x, y) \in \mathbb{R}_+^2;\ y/m < x < my\right\}$.

Theorem 18 *Let*

$$s1 = \frac{1+\sqrt{5}}{2},\ s2 = \frac{2+\sqrt{2}}{2},\ s3 = \frac{3+\sqrt{5}}{2}\ ,$$

and $t1, t2, t3, t4$ *respectively* $t5$ *be the greatest roots of the equations*

$$t^3 - 2t^2 + t - 1 = 0,\ t^3 - t^2 - 2t + 1 = 0,\ t^3 - t^2 - t - 1 = 0,$$

$$t^3 - 3t^2 + 2t - 1 = 0,\ t^3 - 5t^2 + 4t - 1 = 0\ .$$

We have the following angular relations between the Greek means:

$$\mathcal{F}_5 \prec_2 \mathcal{F}_{10}\ ,\ \mathcal{F}_8 \prec_2 \mathcal{H}\ ,\ \mathcal{F}_7 \prec_2 \mathcal{A}\ ,\ \mathcal{A} \prec_2 \mathcal{F}_9,\ \mathcal{A} \prec_3 \mathcal{F}_{10}\ ,$$

$$\mathcal{F}_7 \prec_{s1} \mathcal{H}\ ,\ \mathcal{C} \prec_{s1} \mathcal{F}_9\ ,\ \mathcal{F}_5 \prec_{s2} \mathcal{F}_9\ ,$$

$$\mathcal{F}_8 \prec_{s3} \mathcal{G}\ ,\ \mathcal{G} \prec_{s3} \mathcal{F}_9\ ,\ \mathcal{F}_7 \prec_{t1} \mathcal{G}\ ,\ \mathcal{F}_6 \prec_{t2} \mathcal{F}_9\ ,$$

$$\mathcal{C} \prec_{t3} \mathcal{F}_{10}\ ,\ \mathcal{F}_7 \prec_{t4} \mathcal{F}_6\ ,\ \mathcal{F}_6 \prec_{t4} \mathcal{F}_{10}\ \textit{and}\ \mathcal{F}_8 \prec_{t5} \mathcal{F}_{10}\ .$$

Remark 19 *The approximate values of the above numbers are:*

$$s1 = 1.61803...,\ s2 = 1.70710...,\ t1 = 1.75487...,\ t2 = 1.80193...,$$

$$t3 = 1.83928...,\ t4 = 2.32247...,\ s3 = 2.61803...,\ t5 = 4.07959...\ .$$

Corollary 20 *For each fixed* b*, the interval* (b, ∞) *divides into eleven subintervals, such that on each of them the Greek means are completely ordered. Let us present here the table of these intervals and orders.*

$(b, s1 \cdot b)$	$\mathcal{F}_8$	$\mathcal{F}_7$	$\mathcal{H}$	$\mathcal{G}$	$\mathcal{A}$	$\mathcal{F}_6$	$\mathcal{F}_5$	$\mathcal{C}$	$\mathcal{F}_9$	$\mathcal{F}_{10}$
$(s1 \cdot b, s2 \cdot b)$	$\mathcal{F}_8$	$\mathcal{H}$	$\mathcal{F}_7$	$\mathcal{G}$	$\mathcal{A}$	$\mathcal{F}_6$	$\mathcal{F}_5$	$\mathcal{F}_9$	$\mathcal{C}$	$\mathcal{F}_{10}$
$(s2 \cdot b, t1 \cdot b)$	$\mathcal{F}_8$	$\mathcal{H}$	$\mathcal{F}_7$	$\mathcal{G}$	$\mathcal{A}$	$\mathcal{F}_6$	$\mathcal{F}_9$	$\mathcal{F}_5$	$\mathcal{C}$	$\mathcal{F}_{10}$
$(t1 \cdot b, t2 \cdot b)$	$\mathcal{F}_8$	$\mathcal{H}$	$\mathcal{G}$	$\mathcal{F}_7$	$\mathcal{A}$	$\mathcal{F}_6$	$\mathcal{F}_9$	$\mathcal{F}_5$	$\mathcal{C}$	$\mathcal{F}_{10}$
$(t2 \cdot b, t3 \cdot b)$	$\mathcal{F}_8$	$\mathcal{H}$	$\mathcal{G}$	$\mathcal{F}_7$	$\mathcal{A}$	$\mathcal{F}_9$	$\mathcal{F}_6$	$\mathcal{F}_5$	$\mathcal{C}$	$\mathcal{F}_{10}$
$(t3 \cdot b, 2b)$	$\mathcal{F}_8$	$\mathcal{H}$	$\mathcal{G}$	$\mathcal{F}_7$	$\mathcal{A}$	$\mathcal{F}_9$	$\mathcal{F}_6$	$\mathcal{F}_5$	$\mathcal{F}_{10}$	$\mathcal{C}$
$(2b, t4 \cdot b)$	$\mathcal{H}$	$\mathcal{F}_8$	$\mathcal{G}$	$\mathcal{F}_9$	$\mathcal{A}$	$\mathcal{F}_7$	$\mathcal{F}_6$	$\mathcal{F}_{10}$	$\mathcal{F}_5$	$\mathcal{C}$
$(t4 \cdot b, s3 \cdot b)$	$\mathcal{H}$	$\mathcal{F}_8$	$\mathcal{G}$	$\mathcal{F}_9$	$\mathcal{A}$	$\mathcal{F}_{10}$	$\mathcal{F}_6$	$\mathcal{F}_7$	$\mathcal{F}_5$	$\mathcal{C}$
$(s3 \cdot b, 3b)$	$\mathcal{H}$	$\mathcal{F}_9$	$\mathcal{G}$	$\mathcal{F}_8$	$\mathcal{A}$	$\mathcal{F}_{10}$	$\mathcal{F}_6$	$\mathcal{F}_7$	$\mathcal{F}_5$	$\mathcal{C}$
$(3b, t5 \cdot b)$	$\mathcal{H}$	$\mathcal{F}_9$	$\mathcal{G}$	$\mathcal{F}_8$	$\mathcal{F}_{10}$	$\mathcal{A}$	$\mathcal{F}_6$	$\mathcal{F}_7$	$\mathcal{F}_5$	$\mathcal{C}$
$(t5 \cdot b, \infty)$	$\mathcal{H}$	$\mathcal{F}_9$	$\mathcal{G}$	$\mathcal{F}_{10}$	$\mathcal{F}_8$	$\mathcal{A}$	$\mathcal{F}_6$	$\mathcal{F}_7$	$\mathcal{F}_5$	$\mathcal{C}$

Remark 21 *We can give a similar table for each fixed* a*, with* b *running in the interval* $(0, a)$*. We have only to replace each interval* (sb, tb) *with the interval* $(a/t, a/s)$*, by keeping the order of the means. Of course, the order of the intervals must be reversed. Thus the first interval will be* $(0, a/t5)$*.*

2.5 Weighted Greek means

There are also known **weighted** generalizations of some means (see [P. S. Bullen, 2003]). The most important example is that of the weighted power means $\mathcal{P}_{n,\lambda}$ defined by

$$\mathcal{P}_{n;\lambda}(a,b)=\begin{cases}[\lambda\cdot a^n+(1-\lambda)\cdot b^n]^{1/n}, & n\neq 0\\ a^{\lambda}\cdot b^{1-\lambda} & ,n=0\end{cases},$$

with $\lambda\in[0,1]$ fixed. Of course, for $\lambda=0$ or $\lambda=1$, we have

$$\mathcal{P}_{n;0}=\Pi_2 \text{ respectively } P_{n;1}=\Pi_1,\ \forall n\in\mathbb{R}\,.$$

For $n=1,0$ or -1 we have the weighted arithmetic mean $\mathcal{A}_{\lambda}$, the weighted geometric mean G_{λ}, and the weighted harmonic mean $\mathcal{H}_{\lambda}$.

Weighted generalized contraharmonic means are defined by

$$\mathcal{C}_{n;\lambda}(a,b)=\frac{\lambda\cdot a^n+(1-\lambda)\cdot b^n}{\lambda\cdot a^{n-1}+(1-\lambda)\cdot b^{n-1}}.$$

So, for the first four Greek means we have weighted variants. But how to define such variants of the last six Greek means ?

In [G. Toader, 2005] an answer to this question was given. We remark that in the case of the geometric mean, is obtained a weighted variant which is completely different from

$$\mathcal{P}_{0;\lambda}(a,b)=a^{\lambda}\cdot b^{1-\lambda}.$$

Consider a set of three numbers, $a>m>b>0$. Remember that the arithmetic mean is defined by the proportion

$$\frac{a-m}{m-b}=\frac{a}{a}\ .$$

Take $\lambda\in(0,1)$ and multiply the first member of the proportion by $\lambda/(1-\lambda)$. Now m will give the weighted arithmetic mean of a and b. We shall proceed like this in the first six cases, getting successively:

1. the equality

$$\frac{\lambda}{1-\lambda}\cdot\frac{a-m}{m-b}=\frac{a}{a}$$

gives the **weighted arithmetic mean** of a and b defined, as above, by

$$\mathcal{A}_\lambda(a,b) = \lambda \cdot a + (1-\lambda) \cdot b \;;$$

2. the equality
$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{m-b} = \frac{a}{m}$$

gives the **weighted geometric mean** of a and b, defined by

$$\mathcal{G}_\lambda(a,b) = \frac{1}{2\lambda}\left[\sqrt{(1-2\lambda)^2 \cdot a^2 + 4\lambda(1-\lambda)\,ab} - (1-2\lambda)\,a\right];$$

3. the equality
$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{m-b} = \frac{a}{b}$$

gives the **weighted harmonic mean** of a and b, defined by

$$\mathcal{H}_\lambda(a,b) = \frac{ab}{\lambda \cdot b + (1-\lambda) \cdot a}\;;$$

4. the equality
$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{m-b} = \frac{b}{a}$$

gives the **weighted contraharmonic mean** of a and b, defined by

$$\mathcal{C}_\lambda(a,b) = \frac{\lambda \cdot a^2 + (1-\lambda) \cdot b^2}{\lambda \cdot a + (1-\lambda) \cdot b}$$

5. the equality
$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{m-b} = \frac{b}{m}\;;$$

gives the **first weighted contrageometric mean** of a and b defined by

$$\mathcal{F}_{5,\lambda}(a,b) = \frac{1}{2\lambda}\left[\lambda a - (1-\lambda)\,b + \sqrt{\lambda^2 a^2 - 2\lambda(1-\lambda)\,ab + (1-\lambda)(1+3\lambda)\,b^2}\right]\;;$$

6. the equality
$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{m-b} = \frac{m}{a}$$

gives the **second weighted contrageometric mean** of a and b defined by

$$\mathcal{F}_{6,\lambda}(a,b) = \frac{1}{2(1-\lambda)}\left[(1-\lambda)\,b - \lambda a + \sqrt{\lambda(4-3\lambda)\,a^2 - 2\lambda(1-\lambda)\,ab + (1-\lambda)^2\,b^2}\right].$$

Remark 22 *To define the means on $a < b$, we replace a with b and λ with $1-\lambda$. So $\mathcal{A}_\lambda, \mathcal{H}_\lambda$ and $\mathcal{C}_\lambda$ preserve their expressions but the other do not. Of course, for $\lambda = 1/2$ we get the usual Greek means.*

To prove that all of the above functions represent means, some relations among them are useful.

Theorem 23 *For each $\lambda \in (0,1)$ the following inequalities*

$$\mathcal{H}_\lambda \leq \mathcal{G}_\lambda \leq \mathcal{A}_\lambda \leq \mathcal{F}_{5,\lambda} \leq \mathcal{C}_\lambda$$

and

$$\mathcal{A}_\lambda \leq \mathcal{F}_{6,\lambda} \leq \mathcal{C}_\lambda$$

are valid.

Proof. All the relations can be proved by direct calculation. For example, the inequality

$$\mathcal{A}(a,b) \leq \mathcal{F}_{5,\lambda}(a,b), \text{ for } a > b > 0$$

is equivalent with

$$\sqrt{\lambda^2 a^2 - 2\lambda\,(1-\lambda)\,ab + (1-\lambda)\,(1+3\lambda)\,b^2} \geq$$

$$\geq 2\lambda\,[\lambda a + (1-\lambda)b] - \lambda a + (1-\lambda)b,$$

which, after raising to the second power and collecting the like terms, becomes

$$4\lambda^3\,(1-\lambda)\,(a-b)^2 \geq 0\ ,$$

which is certainly true.

Corollary 24 *The functions $\mathcal{A}_\lambda, \mathcal{G}_\lambda, \mathcal{H}_\lambda, \mathcal{C}_\lambda, \mathcal{F}_{5,\lambda}$ and $\mathcal{F}_{6,\lambda}$ define means for each $\lambda \in (0,1)$.*

Proof. We have only to prove that

$$\wedge\ \leq \mathcal{H}_\lambda \ \text{ and } \ \mathcal{C}_\lambda \leq \vee$$

is true. This can be easily verified.

Remark 25 *Passing at limit, for $\lambda \to 0$ we get the value b, while for $\lambda \to 1$ we have the value a. So, we can extend the definition of the above weighted means by considering them equal with Π_2 for $\lambda = 0$ and with Π_1 for $\lambda = 1$.*

Let again the set of three numbers $a > m > b > 0$ and a parameter $\lambda \in (0, 1)$. Continuing as before, we want to find the rest of four weighted Greek means.

7. the relation

$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{a-b} = \frac{b}{a}$$

gives for m the value

$$\mathcal{F}_{7,\lambda}(a,b) = \frac{\lambda \cdot a^2 + b \cdot (b-a) \cdot (1-\lambda)}{\lambda \cdot a}\ ;$$

8. the relation

$$\frac{\lambda}{1-\lambda} \cdot \frac{a-m}{a-b} = \frac{m}{a}$$

gives

$$\mathcal{F}_{8,\lambda}(a,b) = \frac{\lambda \cdot a^2}{a - (1-\lambda) \cdot b}\ ;$$

9. the relation

$$\frac{\lambda}{1-\lambda} \cdot \frac{m-b}{a-b} = \frac{b}{a}$$

gives

$$\mathcal{F}_{9,\lambda}(a,b) = \frac{b \cdot (a + \lambda \cdot b - b)}{\lambda \cdot a}\ ;$$

10. the relation

$$\frac{\lambda}{1-\lambda} \cdot \frac{m-b}{a-b} = \frac{b}{m}$$

gives

$$\mathcal{F}_{10,\lambda}(a,b) = \frac{\lambda \cdot b + \sqrt{\lambda \cdot b \cdot [\lambda \cdot b + 4 \cdot (1-\lambda) \cdot (a-b)]}}{2 \cdot \lambda}\ .$$

Remark 26 *Passing to limit for $\lambda \to 0$ in $\mathcal{F}_{k,\lambda}, k = 7, ..., 10$, we get the values $-\infty, 0, \infty$ respectively ∞ . This shows that they cannot be used to define means for all the values of λ. In fact, each of them has the property $\mathcal{F}_{k,\lambda}(a,b) \geq b$ for $a > b$ if and only if $\lambda \in [1/2, 1)$. To define them on $a < b$, we replace a with b (we cannot replace also λ with $1-\lambda$). We extend the* *definition of the above weighted means for* $\lambda = 1$ *(passing at limit for* λ*):* $\mathcal{F}_{7,\lambda}$ *and* $\mathcal{F}_{8,\lambda}$ *equal with* $\vee$*, while* $\mathcal{F}_{9,\lambda}$ *and* $\mathcal{F}_{10,\lambda}$ *equal with* $\wedge$*. Also, we define the means for* $\lambda \in [0, 1/2)$*, by* $\mathcal{F}_{k,\lambda} = \mathcal{F}_{k,1-\lambda}, k = 7, ..., 10$ *(only to avoid this restriction on* λ*).*

Some properties of the weighted means were also studied in [G. Toader, 2005]. All the weighted Greek means are **homogeneous**. Relative to the **monotony** of the above means, are given the following results.

Theorem 27 *For* $a > b > 0$ *the weighted Greek means have the following monotonicities: 1) All the means are increasing with respect to* a *on* (b, ∞)*. 2) The means* $\mathcal{A}_\lambda, \mathcal{G}_\lambda, \mathcal{H}_\lambda, \mathcal{F}_{6,\lambda}, \mathcal{F}_{8,\lambda}, \mathcal{F}_{9,\lambda}$ *and* $\mathcal{F}_{10,\lambda}$ *are increasing with respect to* b *on* $(0, a)$*. 3) The means* $\mathcal{C}_\lambda$*,* $\mathcal{F}_{5,\lambda}$ *and* $\mathcal{F}_{7,\lambda}$ *are decreasing with respect to* b *on the interval* $(0, p_\lambda \cdot a)$ *and increasing on* $(p_\lambda \cdot a, a)$*. These means have the values* $M(a, 0) = M(a, a) = a$ *respectively* $M(a, p_\lambda a) = q_\lambda \cdot a$*. The values of the constants* p_λ, q_λ *are given in the following table:*

M	$\mathcal{C}_\lambda$	$\mathcal{F}_{5,\lambda}$	$\mathcal{F}_{7,\lambda}$
p_λ	$\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}}$	$\frac{2\lambda}{1+3\lambda}$	$\frac{1}{2}$
q_λ	$2 \cdot p_\lambda$	$2 \cdot p_\lambda$	$\frac{4-5\lambda}{4(1-\lambda)}, \lambda \leq \frac{1}{2}$ $\frac{5\lambda-1}{4\lambda}, \lambda \geq \frac{1}{2}$

.

Proof. All the results can be verified by the study of the sign of the partial derivatives of the corresponding means. For example, in the case of the mean $\mathcal{C}_\lambda$ we have

$$\frac{\partial \mathcal{C}_\lambda}{\partial b} = \frac{(1-\lambda)\left[-\lambda \cdot a^2 + 2\lambda \cdot ab + (1-\lambda) \cdot b^2\right]}{\left[\lambda \cdot a + (1-\lambda) \cdot b\right]^2}$$

and it is positive if and only if

$$\left[\sqrt{\lambda} \cdot a - \left(\sqrt{\lambda} - 1\right) \cdot b\right] \cdot \left[\sqrt{\lambda} \cdot a - \left(\sqrt{\lambda} + 1\right) \cdot b\right] < 0.$$

As the first factor is positive, we get the equivalent condition

$$b > \frac{\sqrt{\lambda}}{1 + \sqrt{\lambda}} \cdot a \, .$$

Similarly we have

Theorem 28 *For $b > a > 0$ the weighted Greek means have the following monotonicities: 1) All the means are increasing with respect to b on (a, ∞). 2) The means $\mathcal{A}_\lambda, \mathcal{G}_\lambda, \mathcal{H}_\lambda, \mathcal{F}_{6,\lambda}, \mathcal{F}_{8,\lambda}, \mathcal{F}_{9,\lambda}$ and $\mathcal{F}_{10,\lambda}$ are increasing with respect to a on $(0, b)$. 3) The means $\mathcal{C}_\lambda$, $\mathcal{F}_{5,\lambda}$ and $\mathcal{F}_{7,\lambda}$ are decreasing with respect to a on the interval $(0, p'_\lambda \cdot b)$ and increasing on $(p'_\lambda \cdot b, b)$. These means have the values $M(0,b) = M(b,b) = b$ respectively $M(p'_\lambda \cdot b, b) = q'_\lambda \cdot b$. The values of the constants p'_λ, q'_λ are given in the following table:*

M	$\mathcal{C}_\lambda$	$\mathcal{F}_{5,\lambda}$	$\mathcal{F}_{7,\lambda}$
p'_λ	$\frac{\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}$	$\frac{2(1-\lambda)}{4-3\lambda}$	$\frac{1}{2}$
q'_λ	$2 \cdot p'_\lambda$	$2 \cdot p'_\lambda$	$\frac{4-5\lambda}{4(1-\lambda)}, \lambda \leq \frac{1}{2}$ $\frac{5\lambda-1}{4\lambda}, \lambda \geq \frac{1}{2}$

.

As concerns the asymptotic behavior of the given means, is proved the following

Theorem 29 *For a fixed value of b, the weighted Greek means have the following asymptotes:*

$i)$ the mean $\mathcal{H}_\lambda$ has a horizontal asymptote of equation

$$y = \frac{b}{1-\lambda}$$

$ii)$ the mean $\mathcal{F}_{9,\lambda}$ has a horizontal asymptote of equation:

$$y = \begin{cases} \frac{b}{1-\lambda} & \text{if } \lambda \leq \frac{1}{2} \\ \frac{b}{\lambda} & \text{if } \lambda \geq \frac{1}{2} \end{cases};$$

$iii)$ the means $\mathcal{G}_\lambda$ and $\mathcal{F}_{10,\lambda}$ have asymptotic directions 0;
$iv)$ the mean $\mathcal{F}_{8,\lambda}$ has the inclined asymptote with equation:

$$y = \begin{cases} (1-\lambda) \cdot (a + \lambda \cdot b) & \text{if } \lambda \leq \frac{1}{2} \\ \lambda \cdot [a + (1-\lambda) \cdot b] & \text{if } \lambda \geq \frac{1}{2} \end{cases};$$

$v)$ the mean $\mathcal{F}_{6,\lambda}$ has the inclined asymptote with equation

$$y = \frac{\sqrt{4\lambda - 3\lambda^2} - \lambda}{2} \cdot \left(\frac{a}{1-\lambda} + \frac{b}{\sqrt{4\lambda - 3\lambda^2}} \right);$$

vi) the means $\mathcal{C}_\lambda$ and $\mathcal{F}_{5,\lambda}$ have the inclined asymptote with equation

$$y = a - \frac{1-\lambda}{\lambda} \cdot b \;;$$

vi) the mean $\mathcal{F}_{7,\lambda}$ has the inclined asymptote with equation

$$y = \begin{cases} a - \frac{\lambda}{1-\lambda} \cdot b \text{ if } \lambda \leq \frac{1}{2} \\ a - \frac{1-\lambda}{\lambda} \cdot b \text{ if } \lambda \geq \frac{1}{2} \end{cases} .$$

Proof. Let us prove this statement for the mean $\mathcal{F}_{5,\lambda}$. We have

$$\lim_{a\to\infty} \frac{\mathcal{F}_{5,\lambda}(a,b)}{a} = 1$$

and then

$$\lim_{a\to\infty} \left[\mathcal{F}_{5,\lambda}(a,b) - a\right] = \frac{\lambda - 1}{\lambda} \cdot b \;.$$

Looking after new relations among the weighted Greek means, we get only two global relations.

Theorem 30 *For every* $\lambda \in (0,1)$ *the inequalities*

$$\mathcal{F}_{9,\lambda} \leq \mathcal{F}_{10,\lambda}$$

and

$$\mathcal{F}_{8,\lambda} \leq \mathcal{F}_{7,\lambda}$$

hold.

Proof. For $\lambda \geq 1/2$ and $a > b$, each of the above inequalities is equivalent with the relation $\lambda a \geq (1-\lambda) b$.

Corollary 31 *The functions* $\mathcal{F}_{k,\lambda}$ *are means for* $k = 7, 8, 9$ *and* 10.

2.6 Angular relations

To give the relation between $\mathcal{F}_{5,\lambda}$ and $\mathcal{F}_{6,\lambda}$ in [G. Toader, 2005] was defined a new notion, which is more general than the symmetric angular relation.

Definition 32 *For $m, n \geq 0$, consider the set*

$$D_{m,n} = \{(a,b) \in \mathbb{R}_+^2 \ : \ \min(m,n) \cdot a \leq b \leq \max(m,n) \cdot a \ \}.$$

We say that the means M and N are in the ***angular relation***

$$M \prec_{m,n} N$$

if

$$M \prec_{D_{m,n}} N.$$

Theorem 33 *For every $\lambda \in (0,1)$ holds the angular inequality*

$$\mathcal{F}_{5,\lambda} \prec_{1,m} \mathcal{F}_{6,\lambda} \ , \ \textit{where } m = \frac{\lambda}{1-\lambda}.$$

Proof. The inequality $\mathcal{F}_{6,\lambda}(a,b) > \mathcal{F}_{5,\lambda}(a,b)$ is equivalent with the relation $(a-b) \cdot (m \cdot a - b) < 0$. For $\lambda > 1/2$, as $m > 1$, we have no solution with $a > b$, thus

$$\mathcal{F}_{5,\lambda}(a,b) \leq \mathcal{F}_{6,\lambda}(a,b) \text{ if and only if } a \leq b \leq m \cdot a \ .$$

For $\lambda = 1/2$ we have $m = 1$, thus

$$\mathcal{F}_{6,\lambda} \leq \mathcal{F}_{5,\lambda} \ ,$$

which is equivalent with

$$\mathcal{F}_{5,\lambda} \prec_{1,1} \mathcal{F}_{6,\lambda}.$$

Finally, for $\lambda < 1/2$, as $m < 1$, we have

$$\mathcal{F}_{5,\lambda}(a,b) \leq \mathcal{F}_{6,\lambda}(a,b) \text{ if and only if } m \cdot a \leq b \leq a \ .$$

To present the next results, we allow angular relations for infinite value of the parameters m or n. Also we denote by

$$\beta = \frac{\sqrt{5}-1}{2},$$

the inverse of the **golden section**.

Theorem 34 *Among the weighted Greek means we have the angular inequalities $M \prec_{m,n} N$ with*

$$m=\begin{cases} m_1, \lambda\in\left(0,\frac{1}{2}\right] \\ m_2, \lambda\in\left(\frac{1}{2},\beta\right) \\ 1, \lambda\in[\beta,1) \end{cases},\ n=\begin{cases} 1, \lambda\in(0,1-\beta] \\ n_2, \lambda\in\left(1-\beta,\frac{1}{2}\right) \\ n_3, \lambda\in\left[\frac{1}{2},1\right) \end{cases},$$

where the value of m_1 and m_2 are given in the following table:

M	N	m_1	m_2
$\mathcal{H}_\lambda$	$\mathcal{F}_{9,\lambda}$	0	$\frac{2\lambda-1}{\lambda(1-\lambda)}$
$\mathcal{A}_\lambda$	$\mathcal{F}_{8,\lambda}$	0	$\frac{2\lambda-1}{\lambda(1-\lambda)}$
$\mathcal{A}_\lambda$	$\mathcal{F}_{9,\lambda}$	$1-\lambda$	$\frac{\lambda^2}{1-\lambda}$
$\mathcal{A}_\lambda$	$\mathcal{F}_{10,\lambda}$	$\frac{1-\lambda}{2-\lambda}$	$\frac{\lambda^3}{(1-\lambda)^2(1+\lambda)}$
$\mathcal{C}_\lambda$	$\mathcal{F}_{9,\lambda}$	$\frac{\sqrt{5\lambda^2-8\lambda+4}-\lambda}{2(1-\lambda)}$	$\frac{\lambda\left(\sqrt{5}-1\right)}{2(1-\lambda)}$
$\mathcal{G}_\lambda$	$\mathcal{F}_{9,\lambda}$	$\frac{2-\lambda-\sqrt{5\lambda^2-8\lambda+4}}{2\lambda^2}$	$\frac{1+\lambda-(1-\lambda)\sqrt{5}}{2(1-\lambda)}$
$\mathcal{F}_{5,\lambda}$	$\mathcal{F}_{9,\lambda}$	$\frac{1-\sqrt{4\lambda^3-4\lambda^2+1}}{2\lambda^2}$	$\frac{1-\sqrt{1-\lambda}}{1-\lambda}$

while n_1 and n_2 are:

M	N	n_2	n_3
$\mathcal{H}_\lambda$	$\mathcal{F}_{9,\lambda}$	$\frac{\lambda(1-\lambda)}{2\lambda-1}$	∞
$\mathcal{A}_\lambda$	$\mathcal{F}_{8,\lambda}$	$\frac{\lambda(1-\lambda)}{2\lambda-1}$	∞
$\mathcal{A}_\lambda$	$\mathcal{F}_{9,\lambda}$	$\frac{\lambda}{(1-\lambda)^2}$	$\frac{1}{\lambda}$
$\mathcal{A}_\lambda$	$\mathcal{F}_{10,\lambda}$	$\frac{\lambda^2(2-\lambda)}{(1-\lambda)^3}$	$\frac{1+\lambda}{\lambda}$
$\mathcal{C}_\lambda$	$\mathcal{F}_{9,\lambda}$	$\frac{2\lambda}{(1-\lambda)\left(\sqrt{5}-1\right)}$	$\frac{2\lambda}{\sqrt{5\lambda^2-2\lambda+1}-1+\lambda}$
$\mathcal{G}_\lambda$	$\mathcal{F}_{9,\lambda}$	$\frac{2\lambda}{2-\lambda-\lambda\sqrt{5}}$	$\frac{2(1-\lambda)}{1+\lambda-\sqrt{1-2\lambda+5\lambda^2}}$
$\mathcal{F}_{5,\lambda}$	$\mathcal{F}_{9,\lambda}$	$\frac{\lambda}{1-\sqrt{\lambda}}$	$\frac{2(1-\lambda)^2}{1-\sqrt{1-4\lambda+8\lambda^2-4\lambda^3}}$

.

Proof. Consider the inequality

$$\mathcal{F}_{9,\lambda}(a,b)\geq\mathcal{H}_\lambda(a,b).$$

It is easy to verify it for $a\geq b$ and $\lambda\in(0,1/2]$, as well as for $a<b$ and $\lambda\in[1/2,1)$. For $a\geq b$ and $\lambda\in(1/2,1)$ the inequality is equivalent with the relation $(1-2\lambda)a+\lambda(1-\lambda)b\geq 0$. This cannot hold if $\lambda\leq 1-\beta$.

In the case of $a < b$ and $\lambda \in (1/2, 1)$, the inequality is equivalent with $(1-2\lambda)b - \lambda(1-\lambda)a \leq 0$. This cannot hold if $\lambda \geq \beta$. The proof of the other cases is of the same type.

Theorem 35 *Among the weighted Greek means we have the following angular inequalities $M \prec_{m,n} N$ with*

$$m = \begin{cases} 1, \lambda \in (0, 1-\beta] \\ m_2, \lambda \in \left(1-\beta, \frac{1}{2}\right) \\ m_3, \lambda \in \left[\frac{1}{2}, 1\right) \end{cases}, \quad n = \begin{cases} n_1, \lambda \in \left(0, \frac{1}{2}\right] \\ n_2, \lambda \in \left(\frac{1}{2}, \beta\right) \\ 1, \lambda \in [\beta, 1) \end{cases},$$

where the value of m_1 and m_2 are:

M	N	m_2	m_3
$\mathcal{F}_{7,\lambda}$	$\mathcal{A}_\lambda$	$\frac{(1-\lambda)^2}{\lambda}$	λ
$\mathcal{F}_{8,\lambda}$	$\mathcal{H}_\lambda$	$\frac{(1-\lambda)^2}{\lambda}$	λ
$\mathcal{F}_{8,\lambda}$	$\mathcal{G}_\lambda$	$\frac{2-\lambda-\lambda\sqrt{5}}{2\lambda}$	$\frac{1+\lambda-\sqrt{5\lambda^2-2\lambda+1}}{2(1-\lambda)}$
$\mathcal{F}_{7,\lambda}$	$\mathcal{H}_\lambda$	$\frac{(1-\lambda)\left(\sqrt{5}-1\right)}{2\lambda}$	$\frac{\lambda-1+\sqrt{5\lambda^2-2\lambda+1}}{2\lambda}$
$\mathcal{F}_{7,\lambda}$	$\mathcal{F}_{5,\lambda}$	$\frac{\sqrt{1-2\lambda}\left(\sqrt{4\lambda^3-4\lambda^2+1}-\sqrt{1-2\lambda}\right)}{2\lambda^2}$	0

while n_1 and n_2 are:

M	N	n_1	n_2
$\mathcal{F}_{7,\lambda}$	$\mathcal{A}_\lambda$	$\frac{1}{1-\lambda}$	$\frac{1-\lambda}{\lambda^2}$
$\mathcal{F}_{8,\lambda}$	$\mathcal{H}_\lambda$	$\frac{1}{1-\lambda}$	$\frac{1-\lambda}{\lambda^2}$
$\mathcal{F}_{8,\lambda}$	$\mathcal{G}_\lambda$	$\frac{2\lambda}{2-\lambda-\sqrt{5\lambda^2-8\lambda+4}}$	$\frac{2(1-\lambda)}{1+\lambda-(1-\lambda)\sqrt{5}}$
$\mathcal{F}_{7,\lambda}$	$\mathcal{H}_\lambda$	$\frac{2(1-\lambda)}{\sqrt{5\lambda^2-8\lambda+4}-\lambda}$	$\frac{2(1-\lambda)}{\lambda\left(\sqrt{5}-1\right)}$
$\mathcal{F}_{7,\lambda}$	$\mathcal{F}_{5,\lambda}$	∞	$\frac{2(1-\lambda)^2}{\sqrt{2\lambda-1}\left(\sqrt{6\lambda-4\lambda^2-1}-\sqrt{2\lambda-1}\right)}$

.

Proof. The inequality

$$\mathcal{G}_\lambda(a,b) \geq \mathcal{F}_{8,\lambda}(a,b)$$

is also studied in four circumstances. i) For $a \geq b$ and $\lambda \in (0, 1/2]$, the inequality is equivalent with the relation

$$\left[2\lambda b - \left(2-\lambda+\lambda\sqrt{5}\right)\cdot a\right]\cdot\left[2\lambda b - \left(2-\lambda-\lambda\sqrt{5}\right)\cdot a\right] \leq 0.$$

As the first factor is negative, the second must be positive. This cannot hold if $\lambda \leq 1-\beta$. ii) For $a \geq b$ and $\lambda \in (1/2,1)$ the inequality is equivalent with the relation.

$$\left[2\left(\lambda-1\right)b+\left(1+\lambda+\sqrt{5\lambda^2-2\lambda+1}\right)\cdot a\right]\cdot$$

$$\left[2\left(\lambda-1\right)b+\left(1+\lambda-\sqrt{5\lambda^2-2\lambda+1}\right)\cdot a\right]\geq 0.$$

The first factor is positive, thus the second must be also positive. iii) In the case $a<b$ and $\lambda \in (0,1/2]$, the inequality is equivalent with

$$\left[2\lambda a-\left(2-\lambda+\sqrt{5\lambda^2-8\lambda+4}\right)\cdot b\right]\cdot$$

$$\left[2\lambda a-\left(2-\lambda-\sqrt{5\lambda^2-8\lambda+4}\right)\cdot b\right]\leq 0.$$

The first factor is negative, thus the second must be positive. iv) In the case of $a<b$ and $\lambda \in (1/2,1)$, the inequality is equivalent with

$$\left[2(1-\lambda)a-\left(1+\lambda+(1-\lambda)\sqrt{5}\right)\cdot b\right]\cdot$$

$$\left[2(1-\lambda)a-\left(1+\lambda-(1-\lambda)\sqrt{5}\right)\cdot b\right]\leq 0.$$

The first factor is negative, thus the second must be positive. It cannot be so if $\lambda \geq \beta$.The proof of the other results is of the same type.

The other relations among the weighted Greek means are more complicated. As was shown before, if $M \prec_D N$, then D is starshaped for homogeneous means M and N. This can be described also by the function f defined by

$$f(x)=M(1,x)-N(1,x)\ .$$

If it is positive on $[m,n]$, then $M(a,b)\geq N(a,b)$ for every (a,b) in the angle $D_{m,n}$. So, if f has at most two changes of its sign, between M and N there is an angular inequality. If f has more changes of the sign, the relation between M and N becomes more complicated but it can be described in the same manner. This is the situation with the other relations among the weighted Greek means.

Remark 36 *In [G. Toader, 1989] are defined other generalizations of* $\mathcal{G}$ *and* $\mathcal{H}$*:*

- the generalized weighted geometric mean

$$\mathcal{G}_{\lambda,\mu}(a,b) = \sqrt{\lambda a^2 + (1-\lambda-\mu)ab + \mu b^2} \text{ for } 0 \le \lambda, \mu \le 1;$$

- the generalized weighted harmonic mean

$$\mathcal{H}_{\lambda,\mu,\sigma,\tau}(a,b) = \frac{\lambda a^2 + (\sigma+\tau-\lambda-\mu)ab + \mu b^2}{\sigma a + \tau b} \text{ for } 0 \le \lambda \le \sigma, 0 \le \mu \le \tau.$$

It is easy to verify that if $p < q$ we have

$$\mathcal{A}_q \prec \mathcal{A}_p,\ \mathcal{G}_q \prec \mathcal{G}_p,\ \mathcal{H}_q \prec \mathcal{H}_p\ .$$

The following characterizations are also proved:

$$\mathcal{G}_{\lambda,\mu} < \mathcal{G}_{\lambda',\mu'} \ \Leftrightarrow\ \lambda' - \lambda = \mu' - \mu > 0;$$

$$\mathcal{H}_{\lambda,\mu,\upsilon,\sigma} < \mathcal{H}_{\lambda',\mu',\upsilon',\sigma'} \ \Leftrightarrow\ \frac{\lambda}{\nu} < \frac{\lambda'}{\nu'}, \frac{\mu}{\sigma} < \frac{\mu'}{\sigma'}, \frac{\mu+\nu-\lambda}{\nu+\sigma} = \frac{\mu'+\nu'-\lambda'}{\nu'+\sigma'};$$

$$\mathcal{G}_{\lambda,\mu} < \mathcal{A}_\nu \ \Leftrightarrow\ \sqrt{\lambda} + \sqrt{\mu} < 1, \nu = \frac{1+\lambda-\mu}{2};$$

$$\mathcal{H}_{\lambda,\mu,\upsilon,\sigma} < \mathcal{A}_\tau \ \Leftrightarrow\ \frac{\lambda}{\nu} + \frac{\mu}{\sigma} < 1, \tau = \frac{\lambda-\mu+\sigma}{\nu+\sigma}.$$

$$\mathcal{G}_{\lambda,\mu} \prec \mathcal{G}_{\lambda',\mu'} \Leftrightarrow \lambda \ge \lambda', \mu \le \mu',$$

$$\mathcal{G}_{\lambda,\mu} \prec \mathcal{A}_\nu \Leftrightarrow \nu \le \min\left\{\sqrt{\lambda}, 1-\sqrt{\mu}\right\},$$

$$\mathcal{H}_{\lambda,\mu,\nu,\sigma} \prec \mathcal{A}_\tau \Leftrightarrow \tau \le \min\left\{\frac{\lambda}{\nu}, \frac{\lambda-\mu+\sigma}{\nu+\sigma}, 1-\frac{\mu}{\sigma}\right\}.$$

2.7 Operations with means

We use ordinary notations for operations with functions. For example $M \cdot N$ is defined by

$$(M \cdot N)(a,b) = M(a,b) \cdot N(a,b),\ \forall a,b > 0.$$

Of course, if M and N are means, the result of the operation with functions is not a mean. We have to combine more operations with functions to get a (partial) operation with means.

One of the first definition of a partial operation with means was given in [F. G. Tricomi, 1970] where are characterized the linear combinations

$$r\mathcal{A} + s\mathcal{G} + (1 - r - s)\mathcal{H}$$

which are means. The special case $s = 1 - r$ is studied in detail in [W. Janous, 2001] under the name of generalized Heronian means.

Remark 37 *Using operations with means, we can characterize the relation $M \prec_m N$ by*

$$(N - M) \cdot (\vee - m\wedge) \geq 0.$$

Remark 38 *For the means $\mathcal{G}_{\lambda,\mu}$ and $\mathcal{H}_{\lambda,\mu,\sigma,\tau}$ defined before, we have*

$$\mathcal{H}_{\lambda,\mu,\sigma,\tau} = \frac{\mathcal{G}^2_{\frac{\lambda}{\sigma+\tau},\frac{\mu}{\sigma+\tau}}}{\mathcal{A}_{\frac{\sigma}{\sigma+\tau}}}.$$

This is a mean only for $0 \leq \lambda \leq \sigma, 0 \leq \mu \leq \tau$, though $\mathcal{G}_{\lambda/(\sigma+\tau),\mu/(\sigma+\tau)}$ is a mean for $0 \leq \lambda, \mu \leq \sigma + \tau$.

Remark 39 *Using operations with $\wedge$ and $\vee$, we can give the Greek means as follows:*

$$\mathcal{A} = \frac{\vee + \wedge}{2},\ \mathcal{G} = \sqrt{\vee\wedge},\ \mathcal{H} = \frac{2 \vee \wedge}{\vee + \wedge},\ \mathcal{C} = \frac{\vee^2 + \wedge^2}{\vee + \wedge},$$

$$\mathcal{F}_5 = \frac{1}{2}\left[\vee - \wedge + \sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right],\ \mathcal{F}_6 = \frac{1}{2}\left[\wedge - \vee + \sqrt{(\vee - \wedge)^2 + 4\vee^2}\right],$$

$$\mathcal{F}_7 = \frac{\vee^2 - \vee \wedge + \wedge^2}{\vee},\ \mathcal{F}_8 = \frac{\vee^2}{2 \vee - \wedge},$$

$$\mathcal{F}_9 = \frac{\wedge (2 \vee - \wedge)}{\vee} \text{ and } \mathcal{F}_{10} = \frac{1}{2}\left[\wedge + \sqrt{\wedge (4 \vee - 3\wedge)}\right].$$

In the next paragraph, we shall use the following method of **composition** of means. Given three means M, N and P, the expression

$$M(N, P)(a, b) = M(N(a, b), P(a, b)),\ \forall a, b > 0$$

defines always a mean $M(N, P)$.

2.8 Invariant and complementary means

In [C. Gini, 1958] two means M and N are called **complementary** (with respect to $\mathcal{A}$) if $M + N = 2 \cdot \mathcal{A}$. We remark that for every mean M, the function $2 \cdot \mathcal{A} - M$ is again a mean. Thus the complementary of every mean M exists and it is denoted by ${}^{c}M$. The most interesting example of mean defined on this way is the contraharmonic mean given by $\mathcal{C} = {}^{c}\mathcal{H}$.

More fruitful seems to be another notion considered also in [C. Gini, 1958]. Two means M and N are called **inverses** (with respect to $\mathcal{G}$) if $M \cdot N = \mathcal{G}^2$. Again, for every (nonvanishing) mean M, the expression $\mathcal{G}^2/M$ gives a mean, the inverse of M, which we denote by ${}^{i}M$. If the mean M is homogenous, we have

$$ {}^{i}M(a,b) = \frac{1}{M\left(\frac{1}{b}, \frac{1}{a}\right)} $$

and this is used in [J. M. Borwein, P. B. Borwein, 1986] as definition of the inverse. For example we have

$$ {}^{i}\mathcal{A} = \mathcal{H} \; . $$

In [G. Toader, 1991] was proposed a generalization of complementariness and of inversion.

Definition 40 *A mean N is called **complementary to** M **with respect to** P (or $P-$**complementary** to M) if it verifies*

$$ P(M,N) = P. $$

Remark 41 *The definition was given again in [J. Matkowski, 1999]. Previously in [J. M. Borwein, P. B. Borwein, 1987], in the same case, the mean P is called $(M,N)-$**invariant**.*

Remark 42 *Of course, $\mathcal{A}-$complementary means complementary and $\mathcal{G}-$complemen- tary means inverse.*

Remark 43 *The $P-$complementary of a given mean does not necessarily exist nor is unique. For example the Π_1-symmetric of Π_1 is any mean M , but no mean $M \neq \Pi_1$ has a Π_1-symmetric. If a given mean has a unique $P-$complementary mean N, denote $N = M^{(P)}$.*

The following existence theorem is proved in [J. Matkowski, 1999].

Theorem 44 *Let P be a fixed symmetric mean which is continuous and strictly isotone. Then every mean M has a unique $P-$complementary mean N.*

Remark 45 *For every mean M we have*

$$M^{(M)} = M, \Pi_1^{(M)} = \Pi_2, M^{(\Pi_2)} = \Pi_2, M^{(\vee)} = \vee^{(M)} = \vee, M^{(\wedge)} = \wedge^{(M)} = \wedge$$

and if P is a symmetric mean then

$$\wedge^{(P)} = \vee \ , \ \left(M^{(P)}\right)^{(P)} = M \ .$$

In [G. Toader, 2004] was studied the complementariness with respect to the Greek means. We denote the complementary of M with respect to $\mathcal{F}_k$ by $M^{(\mathcal{F}k)}, k = 5, ..., 10$.

Theorem 46 *We have successively*

$$M^{(\mathcal{A})} = 2\mathcal{A} - M \,;$$

$$M^{(\mathcal{G})} = \frac{\mathcal{G}^2}{M} \,;$$

$$M^{(\mathcal{H})} = \frac{M \cdot \mathcal{H}}{2M - \mathcal{H}} \,;$$

$$M^{(\mathcal{C})} = \frac{1}{2} \cdot \left(\mathcal{C} + \sqrt{\mathcal{C}^2 + 4M\mathcal{C} - 4M^2}\right) \,;$$

$$M^{(\mathcal{F}5)} = \begin{cases} \frac{1}{2}\left[\mathcal{F}_5 + \sqrt{\mathcal{F}_5 \cdot (5\mathcal{F}_5 - 4M)}\right] \ , \ \textit{if } \mathcal{F}_5 \leq M \\ \\ \mathcal{F}_5 + M - \frac{M^2}{\mathcal{F}_5} \qquad , \ \textit{if } \ \mathcal{F}_5 \geq M \end{cases} ;$$

$$M^{(\mathcal{F}6)} = \begin{cases} \mathcal{F}_6 + M - \frac{M^2}{\mathcal{F}_6} \qquad , \ \textit{if } \ \mathcal{F}_6 \leq M \\ \\ \frac{1}{2}\left[\mathcal{F}_6 + \sqrt{\mathcal{F}_6(5\mathcal{F}_6 - 4M)}\right] \ , \ \textit{if } \mathcal{F}_6 \geq M \end{cases} ;$$

$$M^{(\mathcal{F}7)} = \begin{cases} \frac{1}{2}\left[M + \sqrt{M(4\mathcal{F}_7 - 3M)}\right] \qquad , \ \textit{if } \ \mathcal{F}_7 \leq M \\ \\ \frac{1}{2}\left[M + \mathcal{F}_7 + \sqrt{\mathcal{F}_7^2 + 2M\mathcal{F}_7 - 3M^2}\right] \ , \ \textit{if } \mathcal{F}_7 \geq M \end{cases} ;$$

$$M^{(\mathcal{F}8)} = \begin{cases} 2M - \frac{M^2}{\mathcal{F}_8} & , \ \text{if } \ \mathcal{F}_8 \leq M \\ \\ \mathcal{F}_8 + \sqrt{\mathcal{F}_8(\mathcal{F}_8 - M)}\, , & \text{if } \mathcal{F}_8 \geq M \end{cases} ;$$

$$M^{(\mathcal{F}9)} = \begin{cases} M - \sqrt{M(M - \mathcal{F}_9)}\, , & \text{if } \mathcal{F}_9 \leq M \\ \\ \frac{M^2}{2M - \mathcal{F}_9} & , \ \text{if } \ \mathcal{F}_9 \geq M \end{cases} ;$$

$$M^{(\mathcal{F}10)} = \begin{cases} \frac{1}{2}\left[M + \mathcal{F}_{10} - \sqrt{M^2 + 2M\mathcal{F}_{10} - 3\mathcal{F}_{10}^2}\right] \, , & \text{if } \mathcal{F}_{10} \leq M \\ \\ M - \mathcal{F}_{10} + \frac{\mathcal{F}_{10}^2}{M} & , \ \text{if } \ \mathcal{F}_{10} \geq M \end{cases} .$$

Proof. Let us find, for instance, the complementary of M with respect to $\mathcal{F}_5$. If we denote it by N, it must verify the relation $\mathcal{F}_5(M, N) = \mathcal{F}_5$. i) Assuming that $N \leq M$, we get the condition

$$M - N + \sqrt{(M - N)^2 + 4N^2} = 2\mathcal{F}_5 \ ,$$

or

$$N^2 - N\mathcal{F}_5 + M\mathcal{F}_5 - \mathcal{F}_5^2 = 0.$$

The discriminant of this equation, $\Delta = \mathcal{F}_5(5\mathcal{F}_5 - 4M)$, is always positive because $5\mathcal{F}_5 > 4\vee$. If we choose $N = \frac{1}{2}\left[\mathcal{F}_5 + \sqrt{\mathcal{F}_5(5\mathcal{F}_5 - 4M)}\right]$, we get a mean if $\mathcal{F}_5 \leq M$. Indeed, in this case we have $\wedge \leq N \leq M$. The first relation is equivalent with

$$M \leq \mathcal{F}_5 + \wedge - \frac{\wedge^2}{\mathcal{F}_5} = \vee \ .$$

ii) In the case $N \geq M$, we have the condition

$$N - M + \sqrt{(M - N)^2 + 4M^2} = 2\mathcal{F}_5 \ ,$$

that is

$$N = \mathcal{F}_5 + M - \frac{M^2}{\mathcal{F}_5}.$$

This is a mean if $\mathcal{F}_5 \geq M$, as $M \leq N \leq \vee$. The last relation is equivalent with

$$M \geq \frac{1}{2}\left[\mathcal{F}_5 + \sqrt{\mathcal{F}_5(5\mathcal{F}_5 - 4\vee)}\right] = \wedge \ .$$

The other cases can be proved similarly. ■

Remark 47 *If a mean M is not comparable with $\mathcal{F}_i$ (for some $i = 5, \ldots, 10$), then $M^{\mathcal{F}i}$ has two expressions, depending on the relation between M and $\mathcal{F}_i$ in the given point. For example, we have:*

$$M^{(\mathcal{F}9)}(a,b) = \begin{cases} M(a,b) - \sqrt{M(a,b)\left[M(a,b) - \mathcal{F}_9(a,b)\right]} & \text{if } \mathcal{F}_9(a,b) \le M(a,b) \\ \frac{M^2(a,b)}{2M(a,b) - \mathcal{F}_9(a,b)} & \text{if } \mathcal{F}_9(a,b) \ge M(a,b) \end{cases} .$$

In [Silvia Toader, G. Toader, 2004, 2004a] is given the complete list of complementary means of a Greek mean with respect to another. Most of them are expressed using operations with the special means $\vee$ and $\wedge$.

Corollary 48 *The complementaries of the Greek means are:*

$$\mathcal{G}^{(\mathcal{A})} = \mathcal{C}_{3/2},\ \mathcal{H}^{(\mathcal{A})} = \mathcal{C},\ \mathcal{C}^{(\mathcal{A})} = \mathcal{H}\ ,$$

$$\mathcal{F}_5^{(\mathcal{A})} = \frac{1}{2}\left[\vee + 3\wedge - \sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right] ,$$

$$\mathcal{F}_6^{(\mathcal{A})} = \frac{1}{2}\left[3\vee + \wedge - \sqrt{(\vee - \wedge)^2 + 4\vee^2}\right] ,$$

$$\mathcal{F}_7^{(\mathcal{A})} = \mathcal{F}_9\ ,\ \mathcal{F}_8^{(\mathcal{A})} = \frac{\vee^2 + \vee\wedge - \wedge^2}{2\vee - \wedge}\ ,\ \mathcal{F}_9^{(\mathcal{A})} = \mathcal{F}_7\ ,$$

$$\mathcal{F}_{10}^{(\mathcal{A})} = \frac{1}{2}\left[2\vee + \wedge - \sqrt{\wedge(4\vee - 3\wedge)}\right] .$$

Corollary 49 *The inverses of the Greek means are:*

$$\mathcal{A}^{(\mathcal{G})} = \mathcal{H}\ , \mathcal{H}^{(\mathcal{G})} = \mathcal{A}\ ,\ \mathcal{C}^{(\mathcal{G})} = \frac{\vee\wedge(\vee + \wedge)}{\vee^2 + \wedge^2}\ ,$$

$$\mathcal{F}_5^{(\mathcal{G})} = \frac{\vee}{2\wedge}\left[\sqrt{(\vee - \wedge)^2 + 4\wedge^2} - \vee + \wedge\right] ,$$

$$\mathcal{F}_6^{(\mathcal{G})} = \frac{\wedge}{2\vee}\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} + \vee - \wedge\right] ,$$

$$\mathcal{F}_7^{(\mathcal{G})} = \frac{\vee^2\wedge}{\vee^2 - \vee\wedge + \wedge^2}\ ,\ \mathcal{F}_8^{(\mathcal{G})} = \mathcal{F}_9\ ,\ \mathcal{F}_9^{(\mathcal{G})} = \mathcal{F}_8\ ,$$

$$\mathcal{F}_{10}^{(\mathcal{G})} = \frac{\vee}{2(\vee - \wedge)}\left[\sqrt{\wedge(4\vee - 3\wedge)} - \wedge\right] .$$

Corollary 50 *The complementary of the Greek means with respect to $\mathcal{H}$ are:*

$$\mathcal{A}^{(\mathcal{H})} = \mathcal{C}^{(\mathcal{G})} , \; \mathcal{G}^{(\mathcal{H})} = \mathcal{C}_{3/2}^{(\mathcal{G})} , \; \mathcal{C}^{(\mathcal{H})} = \mathcal{C}_{3}^{(\mathcal{G})} ,$$

$$\mathcal{F}_5^{(\mathcal{H})} = \frac{\vee \wedge \left[\vee^2 + \vee \wedge + 2 \wedge^2 + \vee \sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right]}{2\left(\vee^3 + \vee \wedge^2 + \wedge^3\right)} ,$$

$$\mathcal{F}_6^{(\mathcal{H})} = \frac{\vee \wedge \left[2 \vee^2 + \vee \wedge + \wedge^2 + \wedge\sqrt{(\vee - \wedge)^2 + 4\vee^2}\right]}{2\left(\vee^3 + \vee^2 \wedge + \wedge^3\right)} ,$$

$$\mathcal{F}_7^{(\mathcal{H})} = \frac{\vee \wedge \left(\vee^2 - \vee \wedge + \wedge^2\right)}{\vee^3 - \vee^2 \wedge + \wedge^3} ,$$

$$\mathcal{F}_8^{(\mathcal{H})} = \mathcal{F}_7^{(\mathcal{G})} , \; \mathcal{F}_9^{(\mathcal{H})} = \frac{\vee \wedge \left(2 \vee - \wedge\right)}{\vee^2 + \vee \wedge - \wedge^2} ,$$

$$\mathcal{F}_{10}^{(\mathcal{H})} = \frac{\vee \wedge \left[2 \vee^2 + \vee \wedge - 2 \wedge^2 + \vee \sqrt{\wedge \left(4 \vee - 3\wedge\right)}\right]}{2\left(\vee^3 + \vee^2 \wedge - \wedge^3\right)} .$$

Corollary 51 *The complementary of the Greek means with respect to $\mathcal{C}$ are:*

$$\mathcal{A}^{(\mathcal{C})} = \frac{\mathcal{P}_2^2 + \mathcal{P}_4^2}{2\mathcal{A}} , \; \mathcal{G}^{(\mathcal{C})} = \frac{1}{2}\left(\mathcal{C} + \sqrt{\mathcal{C}^2 + 4\mathcal{C}\mathcal{G} - 4\mathcal{G}^2}\right) ,$$

$$\mathcal{H}^{(\mathcal{C})} = \frac{1}{\mathcal{A}}\left(\mathcal{P}_2^2 + \sqrt{\mathcal{P}_2^4 + 4\mathcal{G}^2\mathcal{P}_2^2 - 4\mathcal{G}^4}\right) ,$$

$$\mathcal{F}_5^{(\mathcal{C})} = \frac{1}{2\left(\vee + \wedge\right)} \left[\vee^2 + \wedge^2 + \right.$$

$$\left. \sqrt{\vee^4 + 2 \vee^2 \wedge^2 - 8 \vee \wedge^3 - 7 \wedge^4 + 4 \wedge^2 \sqrt{(\vee - \wedge)^2 + 4\wedge^2}}\right] ,$$

$$\mathcal{F}_6^{(\mathcal{C})} = \frac{1}{2\left(\vee + \wedge\right)} \left[\vee^2 + \wedge^2 + \right.$$

$$\left. \sqrt{\wedge^4 + 2 \vee^2 \wedge^2 - 8 \vee^3 \wedge - 7 \vee^4 + 4 \vee^2 \sqrt{(\vee - \wedge)^2 + 4\vee^2}}\right] ,$$

$$\mathcal{F}_7^{(\mathcal{C})} = \frac{1}{2 \vee \left(\vee + \wedge\right)} \left[\vee^3 + \vee \wedge^2 + \sqrt{\vee^6 + 6 \vee^4 \wedge^2 - 4 \vee^3 \wedge^3 + \vee^2 \wedge^4 + 4 \vee \wedge^5 - 4\wedge^6}\right] ,$$

$$\mathcal{F}_8^{(\mathcal{C})} = \frac{1}{2\,(\vee + \wedge)\,(2\vee - \wedge)} \left[\left(\vee^2 + \wedge^2\right)(2\vee - \wedge)\right.$$

$$\left. + \sqrt{8\vee^6 - 8\vee^5\wedge + 9\vee^4\wedge^2 - 4\vee^3\wedge^3 + 2\vee^2\wedge^4 - 4\vee\wedge^5 + \wedge^6}\right] ,$$

$$\mathcal{F}_9^{(\mathcal{C})} = \frac{1}{2\vee(\vee + \wedge)} \left[\vee^3 + \vee\wedge^2\right.$$

$$\left. + \sqrt{\vee^6 + 8\vee^5\wedge - 10\vee^4\wedge^2 - 12\vee^3\wedge^3 + 17\vee^2\wedge^4 + 4\vee\wedge^5 - 4\wedge^6}\right] ,$$

$$\mathcal{F}_{10}^{(\mathcal{C})} = \frac{1}{2\,(\vee + \wedge)} \left[\vee^2 + \wedge^2\right.$$

$$\left. + \sqrt{\vee^4 - 2\vee^3\wedge - 2\vee^2\wedge^2 + 2\vee\wedge^3 + 5\vee^4 + 2\vee(\vee - \wedge)\sqrt{\wedge(4\vee - 3\wedge)}}\right] .$$

Corollary 52 *Let* $s2 = 1 + \sqrt{2}/2$. *The complementary of the Greek means with respect to* $\mathcal{F}_5$ *are:*

$$\mathcal{A}^{(\mathcal{F}\,5)} = \frac{1}{8\wedge^2}\left[\vee^3 + \vee^2\wedge + 7\vee\wedge^2 - \wedge^3 - \left(\vee^2 + 2\vee\wedge - 3\wedge^2\right)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right] ;$$

$$\mathcal{G}^{(\mathcal{F}\,5)} = \frac{\vee - \wedge}{2\wedge}\left[\vee + \wedge - \sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right] + \sqrt{\vee\wedge} ,$$

$$\mathcal{H}^{(\mathcal{F}\,5)} = \frac{1}{2\,(\vee + \wedge)^2}\left[5\vee^3 + \vee^2\wedge + 3\vee\wedge^2 - \wedge^3\right.$$

$$\left. + \left(\wedge^2 + 2\vee\wedge - 3\vee^2\right)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right] ;$$

$$\mathcal{C}^{\,(\mathcal{F}\,5)} = \frac{1}{4}\left[\vee - \wedge + \sqrt{(\vee - \wedge)^2 + 4\wedge^2} + \sqrt{\frac{2}{\vee + \wedge}}\right.$$

$$\left. \cdot\sqrt{\vee^3 - \vee^2\wedge + \vee\wedge^2 + 19\wedge^3 + (\vee^2 - 9\wedge^2)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}}\right] ;$$

$$\mathcal{F}_6^{(\mathcal{F}\,5)} = 2\vee + \frac{1}{4\wedge^2}\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} - 3\vee - \wedge\right]$$

$$\cdot\left[\wedge^2 + 2\vee\wedge - \vee^2 + (\vee - \wedge)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right] ;$$

$$\mathcal{F}_7^{(\mathcal{F}\,5)} = \frac{1}{2\vee^2\wedge^2}\left[\vee^5 - 3\vee^4\wedge + 5\vee^3\wedge^2 - 5\vee^2\wedge^3 + 5\vee\wedge^4 - \wedge^5\right.$$

$$\left. - \left(\vee^4 - 2\vee^3\wedge + 2\vee^2\wedge^2 - 2\vee\wedge^3 + \wedge^4\right)\sqrt{(\vee-\wedge)^2 + 4\wedge^2}\right],$$

$$\mathcal{F}_8^{(\mathcal{F}\,5)} = \frac{1}{2\wedge^2(2\vee-\wedge)^2}\left[\vee^5 - \vee^4\wedge + 8\vee^3\wedge^2 - 10\vee^2\wedge^3 + 5\vee\wedge^4 - \wedge^5\right.$$

$$\left. - \left(\vee^4 - 4\vee^2\wedge^2 + 4\vee\wedge^3 - \wedge^4\right)\sqrt{(\vee-\wedge)^2 + 4\wedge^2}\right];$$

$$\mathcal{F}_9^{(\mathcal{F}\,5)} = \frac{1}{4}\left[\vee - \wedge + \sqrt{(\vee-\wedge)^2 + 4\wedge^2} + \sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{5\vee^3 - 18\vee^2\wedge + 27\vee\wedge^2 - 4\wedge^3 + (5\vee^2 - 13\vee\wedge + 4\wedge^2)\sqrt{(\vee-\wedge)^2 + 4\wedge^2}}\right],$$

if $\vee \leq s2 \cdot \wedge$ *while*

$$\mathcal{F}_9^{(\mathcal{F}\,5)} = \frac{1}{2\vee^2}\left[5\vee^3 - 5\vee^2\wedge + 3\vee\wedge^2 - \wedge^3 - \left(3\vee^2 - 4\vee\wedge + \wedge^2\right)\sqrt{(\vee-\wedge)^2 + 4\wedge^2}\right],$$

if $\vee \geq s2 \cdot \wedge$

$$\mathcal{F}_{10}^{(\mathcal{F}\,5)} = \frac{1}{4}\left[\vee - \wedge + \sqrt{(\vee-\wedge)^2 + 4\wedge^2} + \sqrt{2}\right.$$

$$\left.\cdot\sqrt{10\wedge^2 + \left(5\vee - 7\wedge - 2\sqrt{\wedge(4\vee - 3\wedge)}\right)\left(\vee - \wedge + \sqrt{(\vee-\wedge)^2 + 4\wedge^2}\right)}\right],$$

if $\vee \leq 2\wedge$ *while*

$$\mathcal{F}_{10}^{(\mathcal{F}\,5)} = \wedge + \frac{1}{4\wedge}\left[\vee + \wedge - \sqrt{(\vee-\wedge)^2 + 4\wedge^2}\right]\cdot\left[2\vee - 3\wedge + \sqrt{\wedge(4\vee - 3\wedge)}\right],$$

if $\vee \geq 2\wedge$.

Corollary 53 *Let* $t2 > 1$ *respectively* $t4 > 1$ *the roots of the equations*

$$t^3 - t^2 - 2t + 1 = 0,\ t^3 - 3t^2 + 2t - 1 = 0\,.$$

The complementary of the Greek means with respect to $\mathcal{F}_6$ are:

$$\mathcal{A}^{(\mathcal{F}6)} = \frac{1}{4}\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} - \vee + \wedge + \sqrt{2}\right.$$

$$\left.\cdot\sqrt{17\vee^2 - 10\vee\wedge + 3\wedge^2 - (7\vee - 3\wedge)\sqrt{(\vee-\wedge)^2 + 4\vee^2}}\right] ;$$

$$\mathcal{G}^{(\mathcal{F}6)} = \frac{1}{4}\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} - \vee + \wedge + \sqrt{2}\right.$$

$$\left.\cdot\sqrt{15\vee^2 - 10\vee\wedge + 5\wedge^2 + 4\sqrt{\vee\wedge}(\vee-\wedge) - \left(5\vee - 5\wedge + 4\sqrt{\vee\wedge}\right)\sqrt{(\vee-\wedge)^2 + 4\vee^2}}\right] ;$$

$$\mathcal{H}^{(\mathcal{F}6)} = \frac{1}{4}\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} - \vee + \wedge + \sqrt{\frac{2}{\vee + \wedge}}\right.$$

$$\left.\cdot\sqrt{15\vee^3 + 13\vee^2\wedge - 13\vee\wedge^2 + 5\wedge^3 - (5\vee^2 + 8\vee\wedge - 5\wedge^2)\sqrt{(\vee-\wedge)^2 + 4\vee^2}}\right] ;$$

$$\mathcal{C}^{(\mathcal{F}6)} = \frac{\wedge}{2\vee^2(\vee + \wedge)^2}\left[2\vee^4 + \vee^3\wedge + 5\vee^2\wedge^2 - \vee\wedge^3 + \wedge^4\right.$$

$$\left. + \left(2\vee^3 - \vee^2\wedge - \wedge^3\right)\sqrt{(\vee-\wedge)^2 + 4\vee^2}\right] ;$$

$$\mathcal{F}_5^{(\mathcal{F}6)} = 2\wedge + \frac{1}{4\vee^2}\left[\sqrt{(\vee - \wedge)^2 + 4\wedge^2} - \vee - 3\wedge\right]$$

$$\cdot\left[\vee^2 + 2\vee\wedge - \wedge^2 + (\wedge - \vee)\sqrt{(\vee-\wedge)^2 + 4\vee^2}\right] ,$$

$$\mathcal{F}_7^{(\mathcal{F}6)} = \frac{1}{2}\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} - \vee + \wedge + \frac{1}{\sqrt{\vee}}\right.$$

$$\left.\cdot\sqrt{38\vee^3 - 36\vee^2\wedge + 26\vee\wedge^2 - 3\wedge^3 - (18\vee^2 - 18\vee\wedge + 8\wedge^2)\sqrt{(\vee-\wedge)^2 + 4\vee^2}}\right] ,$$

if $\vee \leq t4 \cdot \wedge$ *while*

$$\mathcal{F}_7^{(\mathcal{F}6)} = \frac{1}{4\vee^4}\left[3\vee^5 - 5\vee^4\wedge + 9\vee^3\wedge^2 - 5\vee^2\wedge^3 + 3\vee\wedge^4 - \wedge^5\right.$$

$$+\left(\vee^4+2\vee^3\wedge-3\vee^2\wedge^2+2\vee\wedge^3-\wedge^4\right)\sqrt{(\vee-\wedge)^2+4\vee^2}\Big]\ ;$$

if $\vee \geq t4\cdot\wedge$

$$\mathcal{F}_8^{(\mathcal{F}6)}=\frac{1}{2}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{\frac{2}{2\vee-\wedge}}\right.$$

$$\left.\cdot\sqrt{34\vee^3-39\vee^2\wedge+20\vee\wedge^2-5\wedge^3-(14\vee^2-15\vee\wedge+5\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right]\ ;$$

$$\mathcal{F}_9^{(\mathcal{F}6)}=\frac{1}{2\vee^4}\Big[\left(\vee^4-4\vee^2\wedge^2+4\vee\wedge^3-\wedge^4\right)\sqrt{(\vee-\wedge)^2+4\vee^2}$$
$$-\vee^5+5\vee^4\wedge-6\vee^3\wedge^2+8\vee^2\wedge^3-5\vee\wedge^4+\wedge^5\Big]\ ,$$

if $\vee \leq t2\cdot\wedge$ *while*

$$\mathcal{F}_9^{(\mathcal{F}6)}=\frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{15\vee^3-2\vee^2\wedge-7\vee\wedge^2+4\wedge^3-(5\vee^2+3\vee\wedge-4\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}}\right]\ ,$$

if $\vee \geq t2\cdot\wedge$

$$\mathcal{F}_{10}^{(\mathcal{F}6)}=\frac{1}{4\vee^2}\Big[\left(2\vee^2-2\vee\wedge+\wedge^2\right)\sqrt{(\vee-\wedge)^2+4\vee^2}+\left(2\vee^2-\vee\wedge+\wedge^2\right)$$

$$\cdot\sqrt{\wedge(4\vee-3\wedge)}-\wedge\sqrt{\wedge(4\vee-3\wedge)}\sqrt{(\vee-\wedge)^2+4\vee^2}-2\vee^3+2\vee^2\wedge+3\vee\wedge^2-\wedge^3\Big]\ ,$$

if $\vee \leq t4\cdot\wedge$ *while*

$$\mathcal{F}_{10}^{(\mathcal{F}6)}=\frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{2}\right.$$

$$\left.\cdot\sqrt{10\vee^2-\left(\sqrt{(\vee-\wedge)^2+4\vee^2}+\wedge-\vee\right)\cdot\left(2\sqrt{\wedge(4\vee-3\wedge)}+5\vee-3\wedge\right)}\right]\ ,$$

if $\vee \geq t4\cdot\wedge$.

Corollary 54 *Let* $s1 = \left(1+\sqrt{5}\right)/2$ *and* $t1 > 1$ *respectively* $t4 > 1$ *the roots of the equations*

$$t^3 - 2t^2 + t - 1 = 0,\ t^3 - 3t^2 + 2t - 1 = 0.$$

The complementary of the Greek means with respect to $\mathcal{F}_7$ *are:*

$$\mathcal{A}^{(\mathcal{F}7)} = \frac{1}{4}\left[\vee + \wedge + \frac{1}{\sqrt{\vee}}\sqrt{(\vee+\wedge)\left(5\vee^2 - 11\vee\wedge + 8\wedge^2\right)}\right],$$

if $\vee \le 2\wedge$ *while*

$$\mathcal{A}^{(\mathcal{F}7)} = \frac{1}{4\vee}\left[3\vee^2 - \vee\wedge + 2\wedge^2 + \sqrt{5\vee^4 - 14\vee^3\wedge + 9\vee^2\wedge^2 - 4\vee\wedge^3 + 4\wedge^4}\right],$$

if $\vee \ge 2\wedge$;

$$\mathcal{G}^{(\mathcal{F}7)} = \frac{1}{2\vee}\left[\vee^2 - \vee\wedge + \wedge^2 + \vee\sqrt{\vee\wedge}\right.$$

$$\left. + \sqrt{\vee^4 - 5\vee^3\wedge + 3\vee^2\wedge^2 - 2\vee\wedge^3 + \wedge^4 + 2\vee\sqrt{\vee\wedge}\left(\vee^2 - \vee\wedge + \wedge^2\right)}\right.$$

if $\vee \le t1 \cdot \wedge$ *while*

$$\mathcal{G}^{(\mathcal{F}7)} = \frac{1}{2}\left[\sqrt{\vee\wedge} + \sqrt{4\sqrt{\frac{\wedge}{\vee}}\left(\vee^2 - \vee\wedge + \wedge^2\right) - 3\vee\wedge}\right],$$

if $\vee \ge t1 \cdot \wedge$

$$\mathcal{H}^{(\mathcal{F}7)} = \frac{1}{\vee+\wedge}\left[\vee\wedge + \sqrt{\wedge\left(2\vee^3 - 3\vee^2\wedge + 2\wedge^3\right)}\right],$$

if $\vee \le s1 \cdot \wedge$ *while*

$$\mathcal{H}^{(\mathcal{F}7)} = \frac{1}{2\vee(\vee+\wedge)}\left[\vee^3 + 2\vee^2\wedge + \wedge^3\right.$$

$$\left. + \sqrt{\vee^6 + 4\vee^5\wedge - 12\vee^4\wedge^2 + 2\vee^3\wedge^3 + 4\vee^2\wedge^4 + \wedge^6}\right],$$

if $\vee \ge s1 \cdot \wedge$;

$$\mathcal{C}^{(\mathcal{F}7)} = \frac{1}{2\vee(\vee+\wedge)}\left[\vee\left(\vee^2 + \wedge^2\right) + \sqrt{\vee\left(\vee^2+\wedge^2\right)\left(\vee^3 - 3\vee\wedge^2 + 4\wedge^3\right)}\right];$$

$$\mathcal{F}_5^{(\mathcal{F}7)} = \frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\wedge^2}+\vee-\wedge+\sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{(\vee^2-\vee\wedge+4\wedge^2)\sqrt{(\vee-\wedge)^2+4\wedge^2}+\vee(\vee^2-2\vee\wedge-\wedge^2)}\right]\ ;$$

$$\mathcal{F}_6^{(\mathcal{F}7)} = \frac{1}{4}\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee+\wedge+\sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{(7\vee^2-7\vee\wedge+4\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}-13\vee^3+14\vee^2\wedge-11\vee\wedge^2+4\wedge^3}\right]\ ,$$

if $\vee \leq t4\cdot\wedge$, *while*

$$\mathcal{F}_6^{(\mathcal{F}7)} = \frac{1}{4\vee}\left[\vee\sqrt{(\vee-\wedge)^2+4\vee^2}+\vee^2-\vee\wedge+2\wedge^2+\sqrt{2}\right.$$

$$\left.\cdot\sqrt{\vee(5\vee^2-5\vee\wedge+2\wedge^2)\sqrt{(\vee-\wedge)^2+4\vee^2}-9\vee^4+6\vee^3\wedge-\vee^2\wedge^2-2\vee\wedge^3+2\wedge^4}\right]\ ;$$

if $\vee \geq t4\cdot\wedge$;

$$\mathcal{F}_8^{(\mathcal{F}7)} = \frac{1}{2\vee(2\vee-\wedge)}\left[3\vee^3-3\vee^2\wedge+3\vee\wedge^2-\wedge^3\right.$$

$$\left.+\sqrt{5\vee^6-18\vee^5\wedge+27\vee^4\wedge^2-24\vee^3\wedge^3+15\vee^2\wedge^4-6\vee\wedge^5+\wedge^6}\right]\ ;$$

$$\mathcal{F}_9^{(\mathcal{F}7)} = \frac{1}{2\vee}\left[\wedge(2\vee-\wedge)+\sqrt{\wedge(2\vee-\wedge)(4\vee^2-10\vee\wedge+7\wedge^2)}\right]\ ,$$

if $\vee \leq 2\wedge$ *while*

$$\mathcal{F}_9^{(\mathcal{F}7)} = \frac{\vee+\wedge}{2}+\frac{1}{2\vee}\sqrt{\vee^4+2\vee^3\wedge-15\vee^2\wedge^2+16\vee\wedge^3-4\wedge^4}\ ,$$

if $\vee \geq 2\wedge$

$$\mathcal{F}_{10}^{(\mathcal{F}7)} = \frac{1}{4}\left[\sqrt{\wedge(4\vee-3\wedge)}+\wedge+\sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{(4\vee^2-7\vee\wedge+4\wedge^2)\sqrt{\wedge(4\vee-3\wedge)}-\wedge(2\vee^2+\vee\wedge-4\wedge^2)}\right]\ ,$$

if $\vee \leq t4 \cdot \wedge$, *while*

$$\mathcal{F}_{10}^{(\mathcal{F}7)} = \frac{1}{4\vee}\left[2\vee^2 - \vee\wedge + 2\wedge^2 + \vee\sqrt{\wedge(4\vee - 3\wedge)} + \sqrt{2}\right.$$

$$\left.\cdot\sqrt{2\vee^4 - 8\vee^3\wedge + 7\vee^2\wedge^2 - 2\vee\wedge^3 + 2\wedge^4 + \vee(2\vee^2 - 5\vee\wedge + 2\wedge^2)\sqrt{\wedge(4\vee - 3\wedge)}}\right],$$

if $\vee \geq t4 \cdot \wedge$.

Corollary 55 *Let* $s3 = \left(3 + \sqrt{5}\right)/2$ *and* $t5 > 1$ *the root of the equation*

$$t^3 - 5t^2 + 4t - 1 = 0 .$$

The complementary of the Greek means with respect to $\mathcal{F}_8$ *are:*

$$\mathcal{A}^{(\mathcal{F}8)} = \frac{1}{4\vee^2}\left(2\vee^3 + \vee^2\wedge + \wedge^3\right);$$

$$\mathcal{G}^{(\mathcal{F}8)} = 2\sqrt{\vee\wedge} - \frac{\wedge(2\vee - \wedge)}{\vee},$$

if $\vee \leq s3 \cdot \wedge$ *while*

$$\mathcal{G}^{(\mathcal{F}8)} = \frac{\vee}{2\vee - \wedge}\left[\vee + \sqrt{\vee^2 - \sqrt{\vee\wedge}(2\vee - \wedge)}\right],$$

if $\vee \geq s3 \cdot \wedge$

$$\mathcal{H}^{(\mathcal{F}8)} = \frac{4\wedge}{(\vee + \wedge)^2} \cdot \left(\vee^2 - \vee\wedge + \wedge^2\right),$$

if $\vee \leq 2\wedge$ *while*

$$\mathcal{H}^{(\mathcal{F}8)} = \frac{\vee}{2\vee - \wedge}\left[\vee + \sqrt{\frac{\vee}{\vee + \wedge}} \cdot \sqrt{\vee^2 - 3\vee\wedge + 2\wedge^2}\right],$$

if $\vee \geq 2\wedge$;

$$\mathcal{C}^{(\mathcal{F}8)} = \frac{\wedge(\vee^2 + \wedge^2)}{\vee^2(\vee + \wedge)^2}\left(3\vee^2 - 2\vee\wedge + \wedge^2\right);$$

$$\mathcal{F}_5^{(\mathcal{F}8)} = \frac{\wedge}{2\vee^2}\left[3\vee^2 - 8\vee\wedge + 3\wedge^2 + (3\vee - \wedge)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right];$$

$$\mathcal{F}_6^{(\mathcal{F}8)} = \frac{1}{2\vee^2}\left[\left(4\vee^2 - 3\vee\wedge + \wedge^2\right)\sqrt{(\vee - \wedge)^2 + 4\vee^2} - 8\vee^3 + 9\vee^2\wedge - 4\vee\wedge^2 + \wedge^3\right];$$

$$\mathcal{F}_7^{(\mathcal{F}8)} = \frac{\wedge}{\vee^4}\left(\vee^2 - \vee\wedge + \wedge^2\right)\left(3\vee^2 - 3\vee\wedge + \wedge^2\right) \ ;$$

$$\mathcal{F}_9^{(\mathcal{F}8)} = \frac{\wedge\left(2\vee - \wedge\right)}{\vee^4}\left(2\vee^3 - 4\vee^2\wedge + 4\vee\wedge^2 - \wedge^3\right) \ ,$$

if $\vee \leq s3 \cdot \wedge$ *while*

$$\mathcal{F}_9^{(\mathcal{F}8)} = \frac{\wedge}{\left(2\vee - \wedge\right)\sqrt{\vee}}\left[\vee\sqrt{\vee} + \sqrt{\left(\vee - \wedge\right)\left(\vee^2 - 3\vee\wedge + \wedge^2\right)}\right] \ ,$$

if $\vee \geq s3 \cdot \wedge$

$$\mathcal{F}_{10}^{(\mathcal{F}8)} = \frac{\sqrt{\wedge}}{2\vee^2}\left[\left(2\vee^2 - 2\vee\wedge + \wedge^2\right)\sqrt{4\vee - 3\wedge} - \left(2\vee^2 - 4\vee\wedge + \wedge^2\right)\sqrt{\wedge}\right] \ ,$$

if $\vee \leq t5 \cdot \wedge$, *while*

$$\mathcal{F}_{10}^{(\mathcal{F}8)} = \frac{\vee}{2\vee - \wedge}\left[\vee + \frac{1}{\sqrt{2}}\sqrt{2\vee^2 - 2\vee\wedge + \wedge^2 - \left(2\vee - \wedge\right)\sqrt{\wedge\left(4\vee - 3\wedge\right)}}\right] \ ,$$

if $\vee \geq t5 \cdot \wedge$.

Corollary 56 *Let* $s1 = \left(1 + \sqrt{5}\right)/2 \quad s2 = 1 + \sqrt{2}/2$ *and* $s3 = \left(3 + \sqrt{5}\right)/2$ *while* $t2 > 1$ *be the root of the equation*

$$t^3 - t^2 - 2t + 1 = 0 \ .$$

The complementary of the Greek means with respect to $\mathcal{F}_9$ *are:*

$$\mathcal{A}^{(\mathcal{F}9)} = \frac{\vee\left(\vee + \wedge\right)^2}{4\left(\vee^2 - \vee\wedge + \wedge^2\right)} \ ,$$

if $\vee \leq 2\wedge$, *while*

$$\mathcal{A}^{(\mathcal{F}9)} = \frac{1}{2}\left[\vee + \wedge - \sqrt{\frac{\vee + \wedge}{\vee}\left(\vee^2 - 3\vee\wedge + 2\wedge^2\right)}\right] \ ,$$

if $\vee \geq 2\wedge$;

$$\mathcal{G}^{(\mathcal{F}9)} = \frac{\vee^2\sqrt{\wedge}}{2\vee\sqrt{\vee} - \left(2\vee - \wedge\right)\sqrt{\wedge}} \ ,$$

if $\vee \leq s3 \cdot \wedge$, *while*

$$\mathcal{G}^{(\mathcal{F}9)} = \sqrt{\wedge}\left[\sqrt{\vee} - \sqrt{\vee - \sqrt{\frac{\wedge}{\vee}}\,(2\vee - \wedge)}\right],$$

if $\vee \geq s3 \cdot \wedge$;

$$\mathcal{H}^{(\mathcal{F}9)} = \frac{4\vee^3\wedge}{(\vee + \wedge)\,(2\vee^2 - \vee\wedge + \wedge^2)},$$

$$\mathcal{C}^{(\mathcal{F}9)} = \frac{\vee\,(\vee^2 + \wedge^2)^2}{(\vee + \wedge)\,(2\vee^3 - 2\vee^2\wedge + \vee\wedge^2 + \wedge^3)},$$

if $\vee \leq s1 \cdot \wedge$, *while*

$$\mathcal{C}^{(\mathcal{F}9)} = \frac{\sqrt{\vee^2 + \wedge^2}}{\vee + \wedge}\left[\sqrt{\vee^2 + \wedge^2} - \sqrt{\frac{\vee^3 - 2\vee^2\wedge + \wedge^3}{\vee}}\right],$$

if $\vee \geq s1 \cdot \wedge$;

$$\mathcal{F}_5^{(\mathcal{F}9)} = \frac{\vee\left[2\vee^3 - 3\vee^2\wedge + 6\vee\wedge^2 - 3\wedge^3 + (2\vee^2 - \vee\wedge + \wedge^2)\sqrt{(\vee - \wedge)^2 + 4\wedge^2}\right]}{2\,(4\vee^3 - 6\vee^2\wedge + 6\vee\wedge^2 - \wedge^3)},$$

if $\vee \leq s2 \cdot \wedge$, *while*

$$\mathcal{F}_5^{(\mathcal{F}9)} = \frac{1}{2}\left[\vee - \wedge + \sqrt{(\vee - \wedge)^2 + 4\wedge^2} - \sqrt{\frac{2}{\vee}}\cdot\right.$$

$$\left.\sqrt{(\vee^2 - 3\vee\wedge + \wedge^2)\sqrt{(\vee - \wedge)^2 + 4\wedge^2} + \vee^3 - 4\vee^2\wedge + 6\vee\wedge^2 - \wedge^3}\right],$$

if $\vee \geq s2 \cdot \wedge$;

$$\mathcal{F}_6^{(\mathcal{F}9)} = \frac{\vee}{2\,(4\vee^4 - 4\vee^3\wedge + 2\vee^2\wedge^2 + 2\vee\wedge^3 - \wedge^4)}\left[-2\vee^4 + 8\vee^3\wedge\right.$$

$$\left.-7\vee^2\wedge^2 + 4\vee\wedge^3 - \wedge^4 + \left(2\vee^3 - 2\vee^2\wedge + 3\vee\wedge^2 - \wedge^3\right)\sqrt{(\vee - \wedge)^2 + 4\vee^2}\right],$$

if $\vee \leq t2 \cdot \wedge$, *while*

$$\mathcal{F}_6^{(\mathcal{F}\,9)} = \frac{1}{2}\left[\sqrt{\left(\vee - \wedge\right)^2 + 4\vee^2} - \vee + \wedge - \sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{3\vee^3 - 2\vee\wedge^2 + \wedge^3 - \left(\vee^2 + \vee\wedge - \wedge^2\right)\sqrt{\left(\vee - \wedge\right)^2 + 4\vee^2}}\right] ,$$

if $\vee \geq t2 \cdot \wedge$;

$$\mathcal{F}_7^{(\mathcal{F}\,9)} = \frac{\left(\vee^2 - \vee\wedge + \wedge^2\right)^2}{\vee\left(2\vee^2 - 4\vee\wedge + 3\wedge^2\right)} ,$$

if $\vee \leq 2\wedge$, *while*

$$\mathcal{F}_7^{(\mathcal{F}\,9)} = \frac{\sqrt{\vee^2 - \vee\wedge + \wedge^2}}{\vee}\left[\sqrt{\vee^2 - \vee\wedge + \wedge^2} - \sqrt{\vee^2 - 3\vee\wedge + 2\wedge^2}\right] ,$$

if $\vee \geq 2\wedge$;

$$\mathcal{F}_8^{(\mathcal{F}\,9)} = \frac{\vee^5}{\left(2\vee - \wedge\right)\left(2\vee^3 - 4\vee^2\wedge + 4\vee\wedge^2 - \wedge^3\right)} ,$$

if $\vee \leq s3 \cdot \wedge$, *while*

$$\mathcal{F}_8^{(\mathcal{F}\,9)} = \frac{1}{2\vee - \wedge}\left[\vee^2 - \sqrt{\vee\left(\vee - \wedge\right)\left(\vee^2 - 3\vee\wedge + \wedge^2\right)}\right] ,$$

if $\vee \geq s3 \cdot \wedge$;

$$\mathcal{F}_{10}^{(\mathcal{F}\,9)} = \frac{\sqrt{\wedge}}{2}\left[\sqrt{\wedge} + \sqrt{4\vee - 3\wedge} - \sqrt{\frac{2}{\vee}}\right.$$

$$\left.\cdot\sqrt{2\vee^2 - 3\vee\wedge + \wedge^2 - \left(\vee - \wedge\right)\sqrt{\wedge\left(4\vee - 3\wedge\right)}}\right] .$$

Corollary 57 *Let* $t3 > 1, t4 > 1$ *respectively* $t5 > 1$ *the roots of the equations*

$$t^3 - t^2 - t - 1 = 0,\ t^3 - 3t^2 + 2t - 1 = 0,\ t^3 - 5t^2 + 4t - 1 = 0 .$$

The complementary of the Greek means with respect to $\mathcal{F}_{10}$ *are:*

$$\mathcal{A}^{(\mathcal{F}\,10)} = \frac{1}{2\left(\vee + \wedge\right)}\left[\vee^2 + 5\vee\wedge - 2\wedge^2 - \left(\vee - \wedge\right)\sqrt{\wedge\left(4\vee - 3\wedge\right)}\right] ,$$

if $\vee \leq 3\wedge$, *while*

$$\mathcal{A}^{(\mathcal{F}\,10)} = \frac{1}{4}\left[\vee + 2\wedge + \sqrt{\wedge\,(4\vee - 3\vee)}\right.$$

$$\left. - \sqrt{\vee^2 - 8\vee\wedge + 9\wedge^2 + 2\,(\vee - 2\wedge)\sqrt{\wedge\,(4\vee - 3\wedge)}}\right] ,$$

if $\vee \geq 3\wedge$;

$$\mathcal{G}^{(\mathcal{F}\,10)} = \frac{1}{2}\sqrt{\frac{\wedge}{\vee}}\left[4\vee - \wedge - \sqrt{\vee\wedge} - \left(\sqrt{\vee} - \sqrt{\wedge}\right)\sqrt{4\vee - 3\wedge}\right] ;$$

$$\mathcal{H}^{(\mathcal{F}\,10)} = \frac{1}{4\vee(\vee + \wedge)}\left[2\vee^3 + 9\vee^2\wedge - 2\vee\wedge^2 - \wedge^3 - \left(\vee^2 - \wedge^2\right)\sqrt{\wedge\,(4\vee - 3\wedge)}\right] ;$$

$$\mathcal{C}^{(\mathcal{F}\,10)} = \frac{\vee}{2\,(\vee + \wedge)\,(\vee^2 + \wedge^2)}\left[2\vee^3 + \vee^2\wedge + 6\vee\wedge^2 - \wedge^3 - \left(\vee^2 - \wedge^2\right)\sqrt{\wedge\,(4\vee - 3\wedge)}\right] ,$$

if $\vee \leq t3 \cdot \wedge$, *while*

$$\mathcal{C}^{(\mathcal{F}\,10)} = \frac{1}{4\,(\vee + \wedge)}\left[2\vee^2 + \vee\wedge + 3\wedge^2 + (\vee + \wedge)\sqrt{\wedge\,(4\vee - 3\wedge)} - \sqrt{2}\cdot\right.$$

$$\left.\sqrt{4\vee^4 - 4\vee^3\wedge - 3\vee^2\wedge^2 + 2\vee\wedge^3 + 5\wedge^4 + (2\vee^3 - \vee^2\wedge - 4\vee\wedge^2 - \wedge^3)\sqrt{4\wedge\vee - 3\wedge^2}}\right]$$

if $\vee \geq t3 \cdot \wedge$;

$$\mathcal{F}_5^{(\mathcal{F}\,10)} = 2\vee - \wedge + \frac{1}{4\wedge}\left[2\vee + \wedge + \sqrt{\wedge\,(4\vee - 3\wedge)}\right]\cdot\left[\sqrt{(\vee - \wedge)^2 + 4\wedge^2} - \vee - \wedge\right] ;$$

if $\vee \leq 2\wedge$, *while*

$$\mathcal{F}_5^{(\mathcal{F}\,10)} = \frac{1}{4}\left[\vee + \sqrt{\wedge\,(4\vee - 3\wedge)} + \sqrt{(\vee - \wedge)^2 + 4\wedge^2} - \sqrt{2}\right.$$

$$\left.\cdot\sqrt{\left[\vee + \sqrt{\wedge\,(4\vee - 3\wedge)}\right]\cdot\left[\sqrt{(\vee - \wedge)^2 + 4\wedge^2} + \vee - 4\wedge\right] - \wedge\,(3\vee - 5\wedge)}\right] ,$$

if $\vee \geq 2\wedge$;

$$\mathcal{F}_6^{(\mathcal{F}\,10)} = \frac{1}{4\vee^2}\left[\wedge\sqrt{\wedge\,(4\vee - 3\wedge)} + \vee^2 + 2\vee\wedge\right]\cdot\left[\sqrt{(\vee - \wedge)^2 + 4\vee^2} + \vee - \wedge\right]$$

$$-\frac{1}{2}\sqrt{\wedge\,(4\vee-3\wedge)}-\frac{3\vee}{4}+\frac{\wedge^3}{4\vee^2}\ ,$$

if $\vee \leq t4\cdot\wedge$, *while*

$$\mathcal{F}_6^{(\mathcal{F}\,10)}=\frac{1}{4}\Bigg[2\wedge-\vee+\sqrt{\wedge\,(4\vee-3\wedge)}+\sqrt{(\vee-\wedge)^2+4\vee^2}-\sqrt{2}$$

$$\cdot\sqrt{\left[2\wedge-\vee+\sqrt{\wedge\,(4\vee-3\wedge)}\right]\cdot\left[\sqrt{(\vee-\wedge)^2+4\vee^2}-\vee-2\wedge\right]+2\vee^2-9\vee\wedge+9\wedge^2}\ ,$$

if $\vee \geq t4\cdot\wedge$;

$$\mathcal{F}_7^{(\mathcal{F}\,10)}=\frac{1}{2\vee(\vee^2-\vee\wedge+\wedge^2)}\big[2\vee^4-3\vee^3\wedge+6\vee^2\wedge^2-5\vee\wedge^3$$

$$+2\wedge^4-\vee(\vee-\wedge)^2\cdot\sqrt{\wedge\,(4\vee-3\wedge)}\Big]\ ,$$

if $\vee \leq t4\cdot\wedge$, *while*

$$\mathcal{F}_7^{(\mathcal{F}\,10)}=\frac{1}{4\vee}\Big[2\vee^2-\vee\wedge+2\wedge^2+\vee\sqrt{\wedge\,(4\vee-3\wedge)}-\sqrt{2}$$

$$\cdot\sqrt{2\vee^4-2\vee^3\wedge+\vee^2\wedge^2-2\vee\wedge^3+2\wedge^4+\vee(2\vee^2-5\vee\wedge+2\wedge^2)\sqrt{\wedge\,(4\vee-3\wedge)}}\Bigg],$$

if $\vee \geq t4\cdot\wedge$;

$$\mathcal{F}_8^{(\mathcal{F}\,10)}=\frac{1}{2\vee^2(2\vee-\wedge)}\big[2\vee^4+6\vee^3\wedge-11\vee^2\wedge^2+6\vee\wedge^3$$

$$-\wedge^4-\left(2\vee^3-5\vee^2\wedge+4\vee\wedge^2-\wedge^3\right)\sqrt{\wedge\,(4\vee-3\wedge)}\Big]\ ,$$

if $\vee \leq t5\cdot\wedge$, *while*

$$\mathcal{F}_8^{(\mathcal{F}\,10)}=\frac{1}{4(2\vee-\wedge)}\Big\{2\vee^2+2\vee\wedge-\wedge^2+(2\vee-\wedge)\sqrt{\wedge\,(4\vee-3\wedge)}$$

$$-\sqrt{2}\cdot\big[2\vee^4-20\vee^3\wedge+34\vee^2\wedge^2-18\vee\wedge^3+3\wedge^4$$

$$+\left(4\vee^3-14\vee^2\wedge+12\vee\wedge^2-3\wedge^3\right)\sqrt{\wedge\,(4\vee-3\wedge)}\Big]^{1/2}\Bigg\},$$

if $\vee \geq t5\cdot\wedge$;

$$\mathcal{F}_9^{(\mathcal{F}\,10)}=\frac{1}{2\vee(2\vee-\wedge)}\left[2\vee^3+5\vee^2\wedge-7\vee\wedge^2+2\wedge^3-\vee(\vee-\wedge)\sqrt{\wedge\,(4\vee-3\wedge)}\right]\ .$$

Theorem 58 *Among the Greek means we have only the following relations:*

$$\mathcal{H}^{(\mathcal{A})} = \mathcal{C}\ ,\ \mathcal{C}^{(\mathcal{A})} = \mathcal{H}\ ,\ \mathcal{F}_7^{(\mathcal{A})} = \mathcal{F}_9\ ,\ \mathcal{F}_9^{(\mathcal{A})} = \mathcal{F}_7\ ,$$

$$\mathcal{A}^{(\mathcal{G})} = \mathcal{H}\ ,\ \mathcal{H}^{(\mathcal{G})} = \mathcal{A}\ ,\ \mathcal{F}_8^{(\mathcal{G})} = \mathcal{F}_9\ \text{ and } \mathcal{F}_9^{(\mathcal{G})} = \mathcal{F}_8\ .$$

2.9 Partial derivatives of means

Regarding the first order partial derivatives of means, in [Silvia Toader, 2002] was proved the following results.

Theorem 59 *If M is a differentiable mean then*

$$M_a(c,c) + M_b(c,c) = 1. \tag{2.7}$$

Proof. Indeed, Taylor's formula of degree one for M gives

$$M(a+t,b+t) = M(a,b) + t[M_a(a,b) + M_b(a,b)] + O(t^2),$$

for t in a neighborhood of zero. Taking $a = b = c$ we get

$$c + t = c + t[M_a(c,c) + M_b(c,c)] + O(t^2),$$

thus (2.7). ■

Theorem 60 *If M is a differentiable mean then*

$$M_a(c,c) \geq 0. \tag{2.8}$$

Proof. For $t > 0$ we have

$$M(c,c) = c \leq M(c+t,c) \leq c+t$$

and so

$$M_a(c,c) = \lim_{t\to 0, t>0} \frac{M(c+t,c) - M(c,c)}{t} \geq 0.$$

■

Remark 61 *Similarly we prove that*

$$M_b(c,c) \geq 0.$$

Using (2.7) we deduce that

$$0 \leq M_a(c,c) \leq 1.$$

This property doesn't hold in an arbitrary point.

Remark 62 *If M is symmetric we know the value of the first order partial derivatives for $a = b$. It was proved in [D. M. E. Foster, G. M. Phillips, 1984].*

Theorem 63 *If M is a symmetric differentiable mean then*

$$M_a(c,c) = M_b(c,c) = 1/2. \tag{2.9}$$

Proof. If M is symmetric we have

$$M_a(c,c) = \lim_{t\to 0} \frac{M(c+t,c) - M(c,c)}{t} = \lim_{t\to 0} \frac{M(c,c+t) - M(c,c)}{t} = M_b(c,c),$$

thus (2.7) gives (2.9). ■

Example 64 *This property is valid for the Greek means*

$$\mathcal{A}, \mathcal{G}, \mathcal{H}, \mathcal{C}, \mathcal{F}_5 \text{ and } \mathcal{F}_6,$$

but the other are not differentiable.

In [Silvia Toader, 2002] are given also some results on second order partial derivatives of generalized means.

Theorem 65 *If M is a twice differentiable mean then*

$$M_{aa}(c,c) + 2M_{ab}(c,c) + M_{bb}(c,c) = 0. \tag{2.10}$$

Proof. We can use the same idea as in the previous proofs. Indeed, Taylor's formula of degree two for M gives

$$M(a+t,b+t) = M(a,b) + t[M_a(a,b) + M_b(a,b)]$$
$$+t^2[M_{aa}(a,b) + 2M_{ab}(a,b) + M_{bb}(a,b)]/2 + O(t^3),$$

for t in a neighborhood of zero. Taking $a = b = c$ and using the formula (2.7) we get (2.10). ■

Corollary 66 *If M is a symmetric mean then*

$$M_{ab}(c,c) = -M_{aa}(c,c). \tag{2.11}$$

Proof. As above $M_{aa}(c,c) = M_{bb}(c,c)$ and so (2.10) gives (2.11). ■

In [Silvia Toader, 2002] was shown that most of the "usual" symmetric means have also the property

$$M_{aa}(c,c) = \frac{\alpha}{c},\ \alpha \in \mathbb{R}\ . \tag{2.12}$$

For the first six Greek means, which are differentiable, we obtain the following values

M	$\mathcal{A}$	$\mathcal{G}$	$\mathcal{H}$	$\mathcal{C}$	$\mathcal{F}_5$	$\mathcal{F}_6$
α	0	-$\frac{1}{4}$	-$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

.

Chapter 3

Double sequences

We begin this part by presenting some classical examples of double sequences. They are related to the method of Archimedes for evaluation of π, Heron's method of extracting square roots, Lagrange's procedure of determination of the integral of some irrational functions and Gauss' approximation of some elliptic integral. For the definition of these double sequences are used the arithmetic mean, the geometric mean and the harmonic mean.

Then we define general double sequences using two arbitrary means. We study conditions on these means that assure the convergence of the sequences to a common limit and present methods for the determination of this limit. The rate of the convergence is finally studied.

3.1 Measurement of the circle

The undisputed leader of Greek scientists was Archimedes of Syracuse (287-212 BC). He is well known for many discoveries and inventions in physics and engineering, but also for his contributions to mathematics. One of them is the evaluation of π, developed especially in his book *Measurement of the Circle* . We use for its presentation the paper [G. M. Phillips, 1981] and the book [G. M. Phillips, 2000].

As it is known, π was defined as the ratio of the perimeter of a given circle to its diameter. Consider a circle of radius 1. Let p_n and P_n denote, respectively, half of the perimeters of the inscribed and circumscribed regular polygons with n sides. As Archimedes remarked, for every $n \geq 3$,

$$p_n < \pi < P_n \, .$$

To get an estimation with any accuracy, Archimedes passes from a given n to $2n$. By simple geometrical considerations, he obtained the relations

$$p_{2n}^2 = \frac{2np_n^2}{n + \sqrt{n^2 - p_n^2}}$$

and

$$P_{2n} = \frac{2nP_n}{n + \sqrt{n^2 + P_n^2}} \ .$$

Beginning with inscribed and circumscribed regular hexagons, with $p_6 = 3$ and $P_6 = 2\sqrt{3}$, then using four times the above formulas, he proved his famous inequalities

$$3.1408 < 3\frac{10}{71} < p_{96} < \pi < P_{96} < 3\frac{1}{7} < 3.1429 \ .$$

In [G. M. Phillips, 1981, 2000], Archimedes' method is developed otherwise. As

$$p_n = n \cdot \sin\frac{\pi}{n} \text{ and } P_n = n \cdot \tan\frac{\pi}{n} \ ,$$

it follows

$$p_n + P_n = n \cdot \sin\frac{\pi}{n} \cdot \left(1 + \frac{1}{\cos\frac{\pi}{n}}\right) = 2n \cdot \tan\frac{\pi}{n} \cdot \cos^2\frac{\pi}{2n} \ .$$

Amplifying the second member with p_n, implies that

$$p_n + P_n = 2 \cdot p_n \cdot P_n \cdot \frac{\cos^2\frac{\pi}{2n}}{2n \cdot \sin\frac{\pi}{2n} \cdot \cos\frac{\pi}{2n}} = \frac{2 \cdot p_n \cdot P_n}{P_{2n}} \ ,$$

thus

$$P_{2n} = \mathcal{H}(P_n, p_n) \ .$$

Other simple calculations yield

$$p_n \cdot P_{2n} = 2n^2 \cdot \sin\frac{\pi}{n} \cdot \tan\frac{\pi}{2n} = 4n^2 \cdot \sin^2\frac{\pi}{2n} = p_{2n}^2 \ ,$$

or

$$p_{2n} = \mathcal{G}(P_{2n}, p_n) \ .$$

Renounce at the geometrical origins of the terms P_n and p_n. Replace the sequences $(P_{2^k n})_{k\geq 0}$ and $(p_{2^k n})_{k\geq 0}$ by two positive sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, defined by

$$a_{n+1} = \mathcal{H}(a_n, b_n) \text{ and } b_{n+1} = \mathcal{G}(a_{n+1}, b_n) \ , \ n \geq 0 \ , \tag{3.1}$$

for some initial values a_0 and b_0 arbitrarily chosen. The main property of these sequences is given in the following

Theorem 67 *The sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, defined by (3.1) are monotonously convergent to a common limit $\mathcal{H} \boxtimes \mathcal{G}(a_0, b_0)$.*

Proof. As

$$\wedge \leq \mathcal{H} \leq \mathcal{G} \leq \vee, \tag{3.2}$$

by induction it follows that

$$0 < a_0 < a_1 < \cdots < a_n < b_n < \cdots < b_1 < b_0 \ .$$

The sequence $(a_n)_{n\geq 0}$ is thus monotonic increasing and bounded above by b_0. So, it has a limit, say α. Similarly, the sequence $(b_n)_{n\geq 0}$ is monotonic decreasing and bounded below by a_0. It has so the limit β. Passing at limit in the relation $b_{n+1} = \mathcal{G}(a_{n+1}, b_n)$, we get $\beta = \mathcal{G}(\alpha, \beta)$, thus $\alpha = \beta$. Similar results, with the a's and b's interchanged, can be obtained in the case $0 < b_0 < a_0$. ■

In the next two theorems, it is given the value of the common limit of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. It was determined in [G. M. Phillips, 1981].

Theorem 68 *If $0 < b_0 < a_0$, the common limit of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ is*

$$\mathcal{H} \boxtimes \mathcal{G}(a_0, b_0) = \frac{a_0 b_0}{\sqrt{a_0^2 - b_0^2}} \arccos \frac{b_0}{a_0} \ .$$

Proof. As in Archimedes' case, put

$$a_0 = \lambda \cdot \tan\theta \text{ and } \ b_0 = \lambda \cdot \sin\theta \ ,$$

where $\lambda > 0$ and $0 < \theta < \frac{\pi}{2}$. So

$$\cos\theta = \frac{b_0}{a_0} \text{ and } \ \sin\theta = \frac{b_0}{\lambda} \ ,$$

thus

$$\theta = \arccos \frac{b_0}{a_0} \text{ and } \lambda = \frac{a_0 b_0}{\sqrt{a_0^2 - b_0^2}} \,.$$

It is easy to see that

$$a_1 = 2\lambda \cdot \tan \frac{\theta}{2} \text{ and } \; b_1 = 2\lambda \cdot \sin \frac{\theta}{2} \,,$$

and generally, by an induction argument

$$a_n = 2^n \lambda \cdot \tan \frac{\theta}{2^n} \text{ and } \; b_n = 2^n \lambda \cdot \sin \frac{\theta}{2^n} \,.$$

Of course

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda \cdot \theta \,,$$

which gives the desired result. ■

Remark 69 *Regarding the notation $\mathcal{H} \boxtimes \mathcal{G}$ we will see later the general definition.*

Corollary 70 *In Archimedes' case, as*

$$a_0 = P_3 = 3\sqrt{3} \textit{ and } b_0 = p_3 = 3\sqrt{3}/2 \,,$$

the common limit is π.

Remark 71 *To illustrate the resulting approximation process of π, we use the following table:*

n	a_n	b_n
0	*5.1961...*	*2.5980...*
1	*3.4641...*	*3.0000...*
2	*3.2153...*	*3.1058...*
3	*3.1596...*	*3.1326...*
4	*3.1460...*	*3.1393...*
5	*3.1427...*	*3.1410...*
6	*3.1418...*	*3.1414...*

Theorem 72 *If $0 < a_0 < b_0$, the common limit of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ is*

$$\mathcal{H} \boxtimes \mathcal{G}(a_0, b_0) = \frac{a_0 b_0}{\sqrt{b_0^2 - a_0^2}} \cosh^{-1} \left(\frac{b_0}{a_0} \right) \,.$$

Proof. In this case, we can put

$$a_0 = \lambda \cdot \tanh\theta \text{ and } \; b_0 = \lambda \cdot \sinh\theta \; .$$

So

$$\cosh\theta = \frac{b_0}{a_0} \text{ and } \; \sinh\theta = \frac{b_0}{\lambda}$$

which gives

$$\theta = \cosh^{-1}\left(\frac{b_0}{a_0}\right)$$

and from the basic relation $\cosh^2\theta - \sinh^2\theta = 1$,

$$\lambda = \frac{a_0 b_0}{\sqrt{b_0^2 - a_0^2}} \; .$$

We have

$$a_1 = 2\lambda \cdot \tanh\frac{\theta}{2} \text{ and } \; b_1 = 2\lambda \cdot \sinh\frac{\theta}{2} \; ,$$

and generally, by an induction argument

$$a_n = 2^n\lambda \cdot \tanh\frac{\theta}{2^n} \text{ and } \; b_n = 2^n\lambda \cdot \sinh\frac{\theta}{2^n} \; .$$

Again

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda \cdot \theta \; ,$$

which gives the desired result. ■

In [G. M. Phillips, 1981] was proved the following result.

Theorem 73 *The rate of convergence of the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ can be evaluated by the relation*

$$\lim_{n\to\infty} \frac{\mathcal{H} \boxtimes \mathcal{G}(a_0,b_0) - a_{n+1}}{\mathcal{H} \boxtimes \mathcal{G}(a_0,b_0) - a_n} = \lim_{n\to\infty} \frac{\mathcal{H} \boxtimes \mathcal{G}(a_0,b_0) - b_{n+1}}{\mathcal{H} \boxtimes \mathcal{G}(a_0,b_0) - b_n} = \frac{1}{4} \; , \; \forall a_0, b_0 \; .$$

Proof. Assume $0 < b_0 < a_0$ and consider $b_n = 2^n\lambda \cdot \sin\frac{\theta}{2^n}$, as it was given above. Using the MacLaurin's formula for the sinus function, we have

$$b_n = 2^n\lambda \cdot \left(\frac{\theta}{2^n} - \frac{\theta^3}{6\cdot 2^{3n}} + \frac{\theta^4}{24\cdot 2^{4n}} \cdot \sin\frac{\theta\cdot t_n}{2^n}\right) \; , \; t_n \in (0,1) \; ,$$

or

$$\lambda\theta - b_n = \lambda \cdot \left(\frac{\theta^3}{6 \cdot 2^{2n}} - \frac{\theta^4}{24 \cdot 2^{3n}} \cdot \sin \frac{\theta \cdot t_n}{2^n} \right) .$$

So

$$\frac{\lambda\theta - b_{n+1}}{\lambda\theta - b_n} = \frac{\frac{1}{2^2} - \frac{\theta}{4 \cdot 2^{n+3}} \cdot \sin \frac{\theta \cdot t_{n+1}}{2^{n+1}}}{1 - \frac{\theta}{4 \cdot 2^n} \cdot \sin \frac{\theta \cdot t_n}{2^n}} ,$$

which gives the desired result in this case. The other three cases can be treated similarly. ∎

The result can be also given as follows.

Theorem 74 *The error of the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ tend to zero asymptotically like $1/4^n$.*

3.2 Heron's method of extracting square roots

As it is shown in [P. S. Bullen, 2003] (following [T. Heath, 1921; Z. Chajoth, 1932; A. Pasche, 1946, 1948]), Heron used the iteration of the arithmetic and harmonic means of two numbers to compute their geometric mean.

To find the square root of a positive number x, we will choose two numbers a, b with $0 < a < b$ and $ab = x$. Putting $a_o = a,\ b_o = b$ we define

$$a_{n+1} = \mathcal{H}(a_n, b_n),\ b_{n+1} = \mathcal{A}(a_n, b_n),\ n \geq 0$$

and get the following result.

Theorem 75 *The sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are convergent to the common limit*

$$\mathcal{H} \otimes \mathcal{A}(a, b) = \mathcal{G}(a, b) = \sqrt{x} .$$

Proof. We know that

$$\wedge \leq \mathcal{H} \leq \mathcal{A} \leq \vee. \tag{3.3}$$

It follows that

$$a_n < a_{n+1} < b_{n+1} < b_n\ ,\ n \geq 0.$$

Also, it is easy to see that

$$b_{n+1} - a_{n+1} = \frac{(b_n - a_n)^2}{2(b_n + a_n)} < \frac{b_n - a_n}{2},$$

thus

$$b_n - a_n < \frac{b - a}{2^n}, \ n > 0.$$

Taking into account the obvious relation

$$a_n b_n = a_{n-1} b_{n-1} = ... = a_o b_o = ab = x,$$

we deduce that the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are convergent and

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \sqrt{x} = \mathcal{G}(a, b).$$

■

Remark 76 *Some comments on this iteration are given also in [J. J. Mathieu, 1879; T. Nowicki, 1998]. Regarding the notation $\mathcal{H} \otimes \mathcal{A}$ we will see later the general definitions. We can illustrate Heron's approximation process by computing $\sqrt{2}$. Starting with the values $a = 1, b = 2$ we get the following table:*

n	a_n	b_n
0	*1.00000...*	*2.00000...*
1	*1.33333...*	*1.50000...*
2	*1.41176...*	*1.41666...*
3	*1.41420...*	*1.41421...*

Heron's method has been extended to roots of higher order in [A. N. Nikolaev, 1925; H. Ory, 1938; C. Georgakis, 2002]. Also an iterative method for approximating higher order roots by using square roots has been given in [D. Vythoulkas, 1949].

3.3 Lagrange and the definition of the $\mathcal{AGM}$

A similar algorithm was developed in [J. L. Lagrange, 1784-1785] for the resolution of another problem. We can use [D. A. Cox, 1984] for the presentation of the above mentioned paper.

Lagrange intended to determine integrals of the form

$$\int \frac{N(y^2)dy}{\sqrt{(1+p^2y^2)(1+q^2y^2)}} , \tag{3.4}$$

where N is a rational function and $p > q > 0$. Using the substitutions

$$p' = \mathcal{A}\,(p,q)\,,\ q' = \mathcal{G}\,(p,q)\,.$$

and

$$y' = \frac{\sqrt{2}}{p+q}\sqrt{pqy^2 - 1 + \sqrt{(1+p^2y^2)(1+q^2y^2)}}\ ,$$

he was showing that

$$\frac{dy}{\sqrt{(1+p^2y^2)(1+q^2y^2)}} = \frac{dy'}{\sqrt{(1+p'^2y'^2)(1+q'^2y'^2)}} \tag{3.5}$$

and

$$y = y'\sqrt{\frac{1+p'^2y'^2}{1+q'^2y'^2}}\ , \tag{3.6}$$

thus

$$\int \frac{N(y^2)dy}{\sqrt{(1+p^2y^2)(1+q^2y^2)}} = \int N\left(\frac{y'^2(1+p'^2y'^2)}{1+q'^2y'^2}\right)\frac{dy'}{\sqrt{(1+p'^2y'^2)(1+q'^2y'^2)}} =$$

$$\int \frac{N'(y'^2)dy'}{\sqrt{(1+p'^2y'^2)(1+q'^2y'^2)}}$$

where N' is again a rational function.

The approximation method defined by Lagrange is based on the following double sequence. Starting with the terms

$$a_0 = p,\ b_0 = q,$$

we define

$$a_{n+1} = \mathcal{A}(a_n, b_n),\ b_{n+1} = \mathcal{G}(a_n, b_n),\ n \geq 0\ . \tag{3.7}$$

Let us make also the following notations:

$$N_0 = N,\ y_0 = y,$$

$$y_{n+1} = \frac{\sqrt{2}}{a_n + b_n}\sqrt{a_n b_n y_n^2 - 1 + \sqrt{(1 + a_n^2 y_n^2)(1 + b_n^2 y_n^2)}}$$

and

$$N_{n+1}(y_{n+1}^2) = N_n\left(\frac{y_{n+1}^2(1 + a_{n+1}^2 y_{n+1}^2)}{1 + b_{n+1}^2 y_{n+1}^2}\right) .$$

Taking into account the above formulas, the integral

$$\int \frac{N_0(y_0^2)dy_0}{\sqrt{(1 + a_0^2 y_0^2)(1 + b_0^2 y_0^2)}} ,$$

becomes, step by step,

$$\int \frac{N_n(y_n^2)dy_n}{\sqrt{(1 + a_n^2 y_n^2)(1 + b_n^2 y_n^2)}} , \ n = 1, 2, ... \tag{3.8}$$

On the other hand, using the relation between the means $\mathcal{A}$ and $\mathcal{G}$ we can prove the following results.

Theorem 77 *For every initial values* a_0, b_0 , *the sequences* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *defined by (3.7) have the following properties:*

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \geq b_{n+1} \geq b_n \geq \cdots \geq b_2 \geq b_1 ; \tag{3.9}$$

$$0 \leq a_n - b_n \leq \frac{|a_0 - b_0|}{2^n} \tag{3.10}$$

Proof. We know that

$$\wedge \leq \mathcal{G} \leq \mathcal{A} \leq \vee. \tag{3.11}$$

For $n > 0$, we have

$$a_n \geq \mathcal{A}(a_n, b_n) = a_{n+1} \geq b_{n+1} = \mathcal{G}(a_n, b_n) \geq b_n ,$$

which gives (3.9). Concerning the second relation, we have for the beginning

$$0 \leq a_1 - b_1 = \frac{a_0 + b_0}{2} - \sqrt{a_0 b_0} \leq \frac{a_0 + b_0}{2} - \min(a_0, b_0) = \frac{|a_0 - b_0|}{2} .$$

Then, from $b_{n+1} \geq b_n$ we obtain

$$a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = \frac{a_n - b_n}{2} ,$$

which by induction gives (3.10). ■

Remark 78 *We can add to the relations (3.9) the inequalities*

$$\min(a_0, b_0) \leq \mathcal{G}(a_0, b_0) = b_1 \leq a_1 = \mathcal{A}(a_0, b_0) \leq \max(a_0, b_0) \ . \tag{3.12}$$

Corollary 79 *The sequences* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *defined by (3.7) are convergent to a common limit* $l = \mathcal{M}(a_0, b_0)$.

Proof. From (3.9) follows that the sequence $(a_n)_{n\geq 1}$ is decreasing and bounded below, while $(b_n)_{n\geq 1}$ is increasing and bounded above. Thus both are convergent. From (3.10) it follows that the limits are equal. ■

Remark 80 *Generally the convergence is much faster than it is suggested by (3.10). To illustrate this, let us remember the evaluation of* $\mathcal{M}(\sqrt{2}, 1)$ *given in [C. F. Gauss, 1800]. Using the following table:*

n	a_n	b_n
0	1.41421356237309504880 2	1.00000000000000000000 0
1	1.20710678118654752440 1	1.18920711500272106671 7
2	1.19815694809463429555 9	1.19812352149312012260 7
3	1.19814023479387720908 3	1.19814023467730720579 8
4	1.19814023473559220744 1	1.19814023473559220743 9

Gauss found

$$\mathcal{M}(\sqrt{2}, 1) = 1.1981402347355922074... \tag{3.13}$$

Thus he obtained 19 accurate places in four iterations.

Remark 81 *In [G. Almkvist, B. Berndt, 1988] it is given another quantitative measure of the rapidity of convergence of the sequences* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$. *Define*

$$c_n = \sqrt{a_n^2 - b_n^2} \ , \ n \geq 0$$

and observe that

$$c_{n+1} = \frac{a_n - b_n}{2} = \frac{c_n^2}{4 \cdot a_{n+1}} < \frac{c_n^2}{4 \cdot M(a, b)} \ .$$

Thus $(c_n)_{n\geq 0}$ *tends to* 0 *quadratically. Remember that more generally, the convergence of the sequence* $(\alpha_n)_{n\geq 0}$ *to* L *is of the* m*th order if there exist the constants* $C > 0$ *and* $m \geq 1$ *such that*

$$|\alpha_{n+1} - L| \leq C \cdot |\alpha_n - L|^m \ , \ n \geq 0 \ .$$

Remark 82 *From (3.9) and (3.12) we get*

$$\min(a_0, b_0) \leq \mathcal{M}(a_0, b_0) \leq \max(a_0, b_0) .$$

This implies that $\mathcal{M}$ defines a mean, which is called the ***arithmetic-geometric mean*** *(or $\mathcal{AGM}$) and is denoted also by $\mathcal{A} \otimes \mathcal{G}$.*

Corollary 83 *If $N' = 1$, the sequence of integrals (3.8) tends to the easy computable integral*

$$\int \frac{dy}{1 + l^2 y^2} .$$

Proof. Indeed, we have $N_n = 1$, for all $n \geq 0$ and $a_n,\ b_n \to l$ for $n \to \infty$. ■

3.4 Elliptic integrals

Return to the double sequence $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$, defined by Lagrange for the initial values

$$a_0 = a,\ b_0 = b$$

by the recurrences

$$a_{n+1} = \mathcal{A}(a_n, b_n),\ b_{n+1} = \mathcal{G}(a_n, b_n),\ n \geq 0 .$$

As we have seen, they have the common limit $\mathcal{M}(a, b)$, which represents the $\mathcal{AGM}$ of a and b.

From the definition of $\mathcal{M}(a, b)$ we see that it has two obvious properties

$$\mathcal{M}(a, b) = \mathcal{M}(a_1, b_1) = \mathcal{M}(a_2, b_2) = ...$$

and

$$\mathcal{M}(\lambda a, \lambda b) = \lambda \mathcal{M}(a, b) .$$

But the determination of $\mathcal{M}(a, b)$ is not at all a simple exercise. We have seen that Gauss calculated $\mathcal{M}(\sqrt{2}, 1)$ with high accuracy. In fact he was able to prove much more.

Theorem 84 *If $a \geq b > 0$, then*

$$\mathcal{M}(a, b) = \frac{\pi}{2} \cdot \left[\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \right]^{-1} . \tag{3.14}$$

Proof. Denote

$$I(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \ . \tag{3.15}$$

The key step is to show that

$$I(a_1, b_1) = I(a, b) \ . \tag{3.16}$$

To prove this, Gauss introduced a new variable θ' such that

$$\sin\theta = \frac{2a\sin\theta'}{a + b + (a-b)\sin^2\theta'} \tag{3.17}$$

and remarked that

$$(a^2\cos^2\theta + b^2\sin^2\theta)^{-1/2} d\theta = (a_1^2\cos^2\theta' + b_1^2\sin^2\theta')^{-1/2} d\theta' \ . \tag{3.18}$$

But the details are rather complicated, even as they were given in [C. C. J. Jacobi, 1881; P. Eymard, J.-P. Lafon, 2004]. Much simpler is the proof of D. J. Newman given in [T. H. Ganelius, W. K. Hayman, D. J. Newman, 1982] and then in [I. J. Schoenberg, 1982]. Changing the variable in (3.15) by the substitution

$$x = b \cdot \tan\theta \ ,$$

we have

$$dx = b \cdot \frac{d\theta}{\cos^2\theta} \ ,$$

or

$$\frac{d\theta}{\cos\theta} = \frac{dx}{\sqrt{b^2 + x^2}} \ .$$

So we have

$$I(a,b) = \int_0^{\pi/2} \frac{1}{\sqrt{a^2 + b^2\tan^2\theta}} \cdot \frac{d\theta}{\cos\theta} = \int_0^{\infty} \frac{1}{\sqrt{a^2 + x^2}} \cdot \frac{dx}{\sqrt{b^2 + x^2}} \ .$$

Denoting

$$J(a,b) = \int_0^{\infty} \frac{dx}{\sqrt{(a^2 + x^2)(b^2 + x^2)}} \ , \tag{3.19}$$

we get the equality

$$I(a,b) = J(a,b) \ . \tag{3.20}$$

So for obtaining (3.16) we have to prove

$$J(a_1, b_1) = J(a, b) \; . \tag{3.21}$$

We have

$$J(a_1, b_1) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a_1^2 + t^2)(b_1^2 + t^2)}}$$

$$= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2 + 2ab + b^2 + 4t^2)(ab + t^2)}} \; .$$

Changing the variable by the substitution

$$t = \frac{x^2 - ab}{2x} \; ,$$

we have

$$dt = \frac{1}{2} \left(1 + \frac{ab}{x^2} \right) dx \; ,$$

thus

$$J(a_1, b_1) = \int_0^{\infty} \frac{(x^2 + ab)\, dx}{\sqrt{(a^2x^2 + b^2x^2 + a^2b^2 + x^4)(a^2b^2 + 2abx^2 + x^4)}} = J(a, b) \; .$$

Iterating (3.16) gives us

$$I(a, b) = I(a_1, b_1) = I(a_2, b_2) = \cdots$$

so that

$$I(a, b) = \lim_{n\to\infty} I(a_n, b_n) = I(l, l) = \frac{\pi}{2 \cdot l} \; ,$$

where

$$l = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \mathcal{M}(a, b) \; .$$

Thus

$$\mathcal{M}(a, b) = \frac{\pi}{2 \cdot I(a, b)} \; ,$$

which is (3.14). ■

Remark 85 *In a similar manner, iterating (3.21), or using (3.20), we get the second representation*

$$\mathcal{M}(a, b) = \frac{\pi}{2 \cdot J(a, b)} \; . \tag{3.22}$$

Remark 86 *As it is shown in [D. A. Cox, 1984], setting*

$$py = \tan\theta \ ,$$

one obtains

$$\frac{dy}{\sqrt{(1+p^2y^2)(1+q^2y^2)}} = \frac{d\theta}{\sqrt{p^2\cos^2\theta + q^2\sin^2\theta}} \ ,$$

so that the relation (3.5) gives (3.18). Thus Lagrange not only could have defined the $\mathcal{AGM}$, he could have also proved the above theorem effortlessly. D. A. Cox has the opinion that "unfortunately, none of this happened; Lagrange never realized the power of what he had discovered".

3.5 Gaussian double sequences

In the previous paragraphs of this chapter where defined more double sequences. Starting from two positive numbers a and b, which we denoted also by $a_o = a$ and $b_o = b$, we have defined a double sequence $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, by

$$a_{n+1} = \mathcal{H}(a_n, b_n) \text{ and } b_{n+1} = \mathcal{A}(a_n, b_n) \ , \ n \geq 0 \ , \tag{3.23}$$

and another by

$$a_{n+1} = \mathcal{A}(a_n, b_n), \ b_{n+1} = \mathcal{G}(a_n, b_n), \ n \geq 0. \tag{3.24}$$

As we saw, the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are monotonously convergent to a common limit which we denoted by $\mathcal{H} \otimes \mathcal{A}(a,b)$ in the case of the relations (3.23) and by $\mathcal{M}(a,b) = \mathcal{A} \otimes \mathcal{G}(a,b)$ in that of the relations (3.24).

In what follows the means $\mathcal{A}, \mathcal{G}$ and $\mathcal{H}$ will be replaced by arbitrary means M and N. We look for minimal conditions on these means to assure for the resulting double sequences similar properties with those of the original double sequences.

Consider two means M and N and two initial values $a, b > 0$.

Definition 87 *The pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by*

$$a_{n+1} = M(a_n, b_n) \text{ and } b_{n+1} = N(a_n, b_n) \ , \ n \geq 0 \ , \tag{3.25}$$

*where $a_0 = a, b_0 = b$, is called a **Gaussian double sequence**.*

Without auxiliary conditions on the means M and N the sequences can be divergent.

Example 88 *Take $M = \Pi_2$ and $N = \Pi_1$. It follows that $a_{n+1} = b_n$ and $b_{n+1} = a_n$, $n \geq 0$, thus the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are divergent unless $a = b$.*

However, in [I. Costin, G. Toader, 2004] is proved the following

Lemma 89 *Given a Gaussian double sequence (3.25), if we denote*

$$\underline{a_n} = a_n \wedge b_n \text{ and } \overline{b_n} = a_n \vee b_n \ , \ n \geq 0,$$

then the sequences $\left(\underline{a_n}\right)_{n\geq 0}$ and $\left(\overline{b_n}\right)_{n\geq 0}$ are monotonic convergent.

Proof. For each $n \geq 0$, we have

$$\underline{a_n} \leq a_{n+1} = M(a_n, b_n) \leq \overline{b_n}$$

and

$$\underline{a_n} \leq b_{n+1} = N(a_n, b_n) \leq \overline{b_n},$$

thus

$$\underline{a_0} \leq \underline{a_n} \leq \underline{a_{n+1}} \leq \overline{b_{n+1}} \leq \overline{b_n} \leq \overline{b_0}$$

and the conclusion follows. ■

Remark 90 *If the sequences $\left(\underline{a_n}\right)_{n\geq 0}$ and $\left(\overline{b_n}\right)_{n\geq 0}$ are convergent to a common limit, then the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are also convergent to the same limit which lies between $a \wedge b$ and $a \vee b$.*

Definition 91 *The mean M is **compoundable in the sense of Gauss** (or **G-compoundable**) with the mean N if the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by (3.25) are convergent to a common limit $M \otimes N(a, b)$ for each $a, b > 0$. If M is G-compoundable with N and also N is G-compoundable with M we say that M and N are **G-compoundable.***

Following the previous remark, the function $M \otimes N$ defines a mean which is called **Gaussian compound mean** (or **G-compound mean**) and $\otimes$ is called **Gaussian product.**

The study of G-compoundability has a reach history. It begun with some results for homogeneous means in [G. Andreoli, 1957; P. J. Myrberg, 1958, 1958a; G. Allasia, 1969-70; F. G. Tricomi, 1975]. Then it was continued with the case of comparable, strict and continues means, in works like [I. J. Schoenberg, 1982; D. M. E. Foster, G. M. Phillips, 1985; J. Wimp, 1985; J. M. Borwein, P. B. Borwein, 1987; G. Toader, 1987]. The results and the proofs are very similar with those of the original algorithm of Gauss. With a more sophisticated method the result was proven for non-symmetric means in [D. M. E. Foster, G. M. Phillips, 1986].

Theorem 92 *If the means M and N are continuous and strict at the left then M and N are G-compoundable.*

Remark 93 *We have also a variant for means which are strict at the right. As shows the example of the means Π_1 and Π_2 which are not G-compoundable (in any order), the result is not valid if we assume one mean to be strict at the left and the other strict at the right. But, as was proved in [G. Toader, 1990, 1991], we can G-compose a strict mean with any mean. The proof is very similar with that of [D. M. E. Foster, G. M. Phillips, 1986].*

Theorem 94 *If one of the means M and N is continuous and strict then M and N are G-compoundable.*

Proof. From (3.25) follows that the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ lie in the closed interval determined by a and b. By the Bolzano-Weierstrass theorem, there are the subsequences $(a_{n_k})_{k\geq 0}$, $(b_{n_k})_{k\geq 0}$ and the points $\alpha, \beta, \alpha', \beta'$ such that

$$\lim_{k\to\infty} a_{n_k} = \alpha,\ \lim_{k\to\infty} b_{n_k} = \beta,\ \lim_{k\to\infty} a_{n_{k+1}} = \alpha',\ \lim_{k\to\infty} b_{n_{k+1}} = \beta'.$$

We can prove that $\alpha = \beta$. Indeed, suppose $\alpha < \beta$. From (3.25) follows that

$$\alpha \leq \alpha' \leq \beta,\ \alpha \leq \beta' \leq \beta.$$

We show that

$$\alpha' = \alpha \text{ or } \alpha' = \beta. \tag{3.26}$$

If $\alpha < \alpha' \leq \beta' \leq \beta$, we choose $0 < r < (\alpha' - \alpha)/2$ and $K = K_r$ such that

$$\left|a_{n_{k+1}} - \alpha'\right| < r,\ \left|b_{n_{k+1}} - \alpha'\right| < r,\ \forall k \geq K.$$

So

$$a_{n_{k+1}} > \alpha' - r > (\alpha' + \alpha)/2$$

and

$$b_{n_{k+1}} > \beta' - r > (\alpha' + \alpha)/2.$$

It follows that

$$a_{n_k} > (\alpha' + \alpha)/2 > \alpha, \ \forall k \geq K,$$

which is inconsistent with the hypothesis that $\lim_{k\to\infty} a_{n_k} = \alpha$. If $\alpha \leq \beta' \leq \alpha' < \beta$ we obtain a similar contradiction by choosing $0 < r < (\beta - \alpha')/2$. Analogously we can prove that

$$\beta' = \alpha \text{ or } \beta' = \beta \tag{3.27}$$

holds. Now, if M is continuous and strict, using (3.26) we get

$$\alpha = M(\alpha, \beta) \text{ or } \beta = M(\alpha, \beta),$$

thus $\alpha = \beta$. If N is continuous and strict, we use (3.27) to arrive at the same conclusion. The hypothesis $\alpha > \beta$ gives analogously $\alpha = \beta$. Hence

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} b_{n_k} = \alpha,$$

which leads to

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha.$$

■

Remark 95 *Using this result, one can G-compose a "good" mean (that is a continuous and strict mean) even with a "bad" one. But products like* $\Pi_1 \otimes \Pi_2$, $\Pi_2 \otimes \Pi_1$, $\vee \otimes \wedge$ *or* $\wedge \otimes \vee$ *does not exist.*

In [J. M. Borwein, P. B. Borwein, 1987] it was proven the following invariance principle, a generalization of the method which was used by Gauss in the case of classical $\mathcal{AGM}$ for proving its integral representation. It will be also used in the next paragraph for the determination of some G-compound means.

Theorem 96 *(**Invariance Principle**) Suppose that $M \otimes N$ exists and is continuous. Then $M \otimes N$ is the unique mean P satisfying*

$$P(M(a,b), N(a,b)) = P(a,b) \tag{3.28}$$

for all $a, b > 0$.

Proof. Iteration of (3.28) shows that

$$P(a,b) = P(a_n, b_n) = \lim_{n\to\infty} P(a_n, b_n).$$

Thus

$$P(a,b) = P(M \otimes N(a,b), M \otimes N(a,b)) = M \otimes N(a,b)$$

since $P(c,c) = c$. ■

Remark 97 *The relation (3.28) is usually called* ***Gauss' functional equation.***

3.6 Determination of G-compound means

As we saw, the arithmetic-geometric G-compound mean can be represented by

$$\mathcal{A} \otimes \mathcal{G}(a,b) = \frac{\pi}{2} \cdot \left[\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} \right]^{-1}.$$

The proof is based on the corresponding Gauss' functional equation (3.16).

The invariance principle gives the following

Corollary 98 *If M is G-compoundable with N, then*

$$M \otimes N = P$$

if and only if P is $(M,N)-$invariant.

The assumption on G-compoundability is essential.

Example 99 *We have*

$$\Pi_1^{(\vee)} = \vee$$

but $\Pi_1 \otimes \vee$ does not exist.

Corollary 100 *If M is continuous and strict then*

$$M \otimes N = P$$

if and only if

$$N = M^{(P)}.$$

Using this result and the remark 45, we deduce that for every continuous strict mean M we have

$$M \otimes M = M, \Pi_1 \otimes M = \Pi_1, M \otimes \Pi_2 = \Pi_2,$$

$$M \otimes \vee = \vee \otimes M = \vee \text{ and } M \otimes \wedge = \wedge \otimes M = \wedge$$

Also we have ninety interesting G-compound means related to Greek means. To give them, remember the notations

$$\mathcal{A} = \mathcal{F}_1,\ \mathcal{G} = \mathcal{F}_2,\ \mathcal{H} = \mathcal{F}_3 \text{ and } \mathcal{C} = \mathcal{F}_4\ .$$

Corollary 101 *For each $i, j = 1, 2, ..., 10$, with $i \neq j$, we have*

$$\mathcal{F}_i \otimes \mathcal{F}_i^{(\mathcal{F}\, j)} = \mathcal{F}_j\ .$$

We get so ninety double sequences with known limit.

Of course

$$N \otimes M(a, b) = M \otimes N(b, a),$$

so that, if M and N are symmetric

$$N \otimes M = M \otimes N.$$

But, for example

$$\Pi_1 \otimes \mathcal{G} \neq \mathcal{G} \otimes \Pi_1.$$

Indeed, as we saw $\Pi_1 \otimes \mathcal{G} = \Pi_1$ but $\mathcal{G} \otimes \Pi_1 = \mathcal{G}_{2/3}$ as $\mathcal{G}^{\mathcal{P}(0,2/3)} = \Pi_1$.

In [G. Toader, 1987] we proved that for every $\lambda, \mu \in (0, 1)$ we have

$$\mathcal{A}_\lambda \otimes \mathcal{A}_\mu = \mathcal{A}_{\frac{\mu}{1-\lambda+\mu}}\ .$$

Also, in [G. Toader, 1991] we proved that for every $\lambda \in (0, 1)$ we have

$$\mathcal{A}_\lambda \otimes \mathcal{H}_{1-\lambda} = \mathcal{G}.$$

3.7 Rate of convergence of G-compound means

Assume that the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by (3.25) have the common limit α. We examine the **rate of convergence** of the sequences to their limit. We consider the **errors** of the sequences

$$\delta_n = a_n - \alpha \ , \ \varepsilon_n = b_n - \alpha \ , \ n \geq 0.$$

To study them, we use the Taylor formulas for M and N in the point (α, α). Supposing that the means M and N have continuous partial derivatives up to the second order, then we obtain the equality

$$\delta_{n+1} = M_a(\alpha, \alpha)\delta_n + [1 - M_a(\alpha, \alpha)]\, \varepsilon_n + O(\delta_n^2 + \varepsilon_n^2) \ . \tag{3.29}$$

and the similar relation

$$\varepsilon_{n+1} = N_a(\alpha, \alpha)\delta_n + [1 - N_a(\alpha, \alpha)]\, \varepsilon_n + O(\delta_n^2 + \varepsilon_n^2) \ . \tag{3.30}$$

Using them, in [D. M. E. Foster, G. M. Phillips, 1986] is proved the following result.

Theorem 102 *If there is no integer $k \geq 0$ for which $a_k = b_k$ and if*

$$0 < M_a(\alpha, \alpha), N_a(\alpha, \alpha) < 1 \text{ and } M_a(\alpha, \alpha) \neq N_a(\alpha, \alpha)$$

then, as $n \to \infty$,

$$\delta_{n+1} = [M_a(\alpha, \alpha) - N_a(\alpha, \alpha)]\delta_n + O(\delta_n^2)$$

and

$$\varepsilon_{n+1} = [M_a(\alpha, \alpha) - N_a(\alpha, \alpha)]\varepsilon_n + O(\varepsilon_n^2).$$

Proof. From (3.29) and (3.30) we have

$$\delta_n - \delta_{n+1} = [1 - M_a(\alpha, \alpha)](\delta_n - \varepsilon_n) + O(\delta_n^2 + \varepsilon_n^2)$$

and

$$\varepsilon_n - \varepsilon_{n+1} = -N_a(\alpha, \alpha)(\delta_n - \varepsilon_n) + O(\delta_n^2 + \varepsilon_n^2).$$

Hence

$$\frac{\varepsilon_n - \varepsilon_{n+1}}{\delta_n - \delta_{n+1}} = -\frac{N_a(\alpha,\alpha)}{1 - M_a(\alpha,\alpha)} + O(|\delta_n| + |\varepsilon_n|).$$

Thus we have

$$\varepsilon_n - \varepsilon_{n+1} = -\frac{N_a(\alpha,\alpha)}{1 - M_a(\alpha,\alpha)}(\delta_n - \delta_{n+1}) + O(|\delta_n| + |\varepsilon_n|)(\delta_n - \delta_{n+1})$$

and adding this relation from n to $n+p-1$, the authors prove that

$$\varepsilon_n - \varepsilon_{n+p} = -\frac{N_a(\alpha,\alpha)}{1 - M_a(\alpha,\alpha)}(\delta_n - \delta_{n+p}) + O(|\delta_n| + |\varepsilon_n|)(\delta_n - \delta_{n+p}).$$

Letting $p \to \infty$ we deduce that

$$\varepsilon_n = -\frac{N_a(\alpha,\alpha)}{1 - M_a(\alpha,\alpha)} \cdot \delta_n + O(|\delta_n| + |\varepsilon_n|) \cdot \delta_n..$$

Substituting it in (3.29) and (3.30) we get the desired results. ■

Supposing that the means M and N have continuous partial derivatives up to the third order, then we have the relation

$$\delta_{n+1} = M_a(\alpha,\alpha)\delta_n + [1 - M_a(\alpha,\alpha)]\,\varepsilon_n \tag{3.31}$$

$$+\frac{1}{2}\left[M_{aa}(\alpha,\alpha)\delta_n - M_{bb}(\alpha,\alpha)\varepsilon_n\right](\delta_n - \varepsilon_n) + O(|\delta_n|^3 + |\varepsilon_n|^3)\ ,$$

and the similar relation

$$\varepsilon_{n+1} = N_a(\alpha,\alpha)\delta_n + [1 - N_a(\alpha,\alpha)]\,\varepsilon_n$$

$$+\frac{1}{2}\left[N_{aa}(\alpha,\alpha)\delta_n - N_{bb}(\alpha,\alpha)\varepsilon_n\right](\delta_n - \varepsilon_n) + O(|\delta_n|^3 + |\varepsilon_n|^3).$$

Using them in [D. M. E. Foster, G. M. Phillips, 1986] was further proven that:

Theorem 103 *If there is no integer $k \geq 0$ for which $a_k = b_k$ and if*

$$0 < M_a(\alpha,\alpha), N_a(\alpha,\alpha) < 1 \text{ and } M_a(\alpha,\alpha) = N_a(\alpha,\alpha)$$

then, as $n \to \infty$,

$$2[1 - M_a(\alpha,\alpha)]\delta_{n+1} = \{[1 - M_a(\alpha,\alpha)][M_{aa}(\alpha,\alpha) - N_{aa}(\alpha,\alpha)]$$

$$+M_a(\alpha,\alpha)[M_{bb}(\alpha,\alpha)-N_{bb}(\alpha,\alpha)]\}\delta_n^2+O\left(|\delta_n|^3\right)$$

and

$$2M_a(\alpha,\alpha)\varepsilon_{n+1}=-\{[1-M_a(\alpha,\alpha)][M_{aa}(\alpha,\alpha)-N_{aa}(\alpha,\alpha)]$$
$$+M_a(\alpha,\alpha)[M_{bb}(\alpha,\alpha)-N_{bb}(\alpha,\alpha)]\}\varepsilon_n^2+O\left(|\varepsilon_n|^3\right)$$

Proof. Indeed, from the previous results we have

$$2(\delta_{n+1}-\varepsilon_{n+1})=\left[\left(M_{aa}(\alpha,\alpha)-N_{aa}(\alpha,\alpha)\right)\delta_n-\left(M_{bb}(\alpha,\alpha)-N_{bb}(\alpha,\alpha)\right)\varepsilon_n\right](\delta_n-\varepsilon_n)$$
$$+O(|\delta_n|^3+|\varepsilon_n|^3).$$

Using the fact that

$$M_a(\alpha,\alpha)\delta_n+\left[1-M_a(\alpha,\alpha)\right]\varepsilon_n=O(\delta_n^2+\varepsilon_n^2)$$

and

$$M_a(\alpha,\alpha)\delta_{n+1}+\left[1-M_a(\alpha,\alpha)\right]\varepsilon_{n+1}=O(\delta_{n+1}^2+\varepsilon_{n+1}^2)=O(\delta_n^4+\varepsilon_n^4),$$

we get the desired results. ■

Corollary 104 *If M and N are symmetric means then*

$$\delta_{n+1}=\left[M_{aa}(\alpha,\alpha)-N_{aa}(\alpha,\alpha)\right]\cdot\delta_n^2+O\left(|\delta_n|^3\right)$$

and

$$\varepsilon_{n+1}=-\left[M_{aa}(\alpha,\alpha)-N_{aa}(\alpha,\alpha)\right]\cdot\varepsilon_n^2+O\left(|\varepsilon_n|^3\right).$$

Proof. In this case

$$M_a(\alpha,\alpha)=N_a(\alpha,\alpha)=\frac{1}{2}$$

and

$$M_{aa}(\alpha,\alpha)=M_{bb}(\alpha,\alpha),\ N_{aa}(\alpha,\alpha)=N_{bb}(\alpha,\alpha).$$

■

Remark 105 *As we saw, most of the "usual" symmetric means have the property (2.12). For such means we have the following*

Corollary 106 *If M and N are symmetric means such that*

$$M_{aa}(\alpha,\alpha)=\frac{c}{\alpha},\ N_{aa}(\alpha,\alpha)=\frac{d}{\alpha}\ ,\ c,d\in\mathbb{R}\ .$$

then

$$\delta_{n+1}=\frac{c-d}{\alpha}\cdot\delta_n^2+O\left(|\delta_n|^3\right)$$

and

$$\varepsilon_{n+1}=-\frac{c-d}{\alpha}\cdot\varepsilon_n^2+O\left(|\varepsilon_n|^3\right).$$

Remark 107 *In the special case of the $\mathcal{AGM}$ we have*

$$\delta_{n+1}=\frac{1}{4\alpha}\cdot\delta_n^2+O\left(|\delta_n|^3\right)\ ,\ \varepsilon_{n+1}=-\frac{1}{4\alpha}\cdot\varepsilon_n^2+O\left(|\varepsilon_n|^3\right).$$

3.8 Archimedean double sequences

As we saw, Archimedes' polygonal method of evaluation of π, was interpreted in [G. M. Phillips, 1981, 2000], as a double sequence

$$a_{n+1}=\mathcal{H}(a_n,b_n)\text{ and }b_{n+1}=\mathcal{G}(a_{n+1},b_n)\ ,\ n\geq 0\ . \tag{3.32}$$

More generally, let us consider two means M and N and two initial values $a,b>0$.

Definition 108 *The pair of sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by*

$$a_{n+1}=M(a_n,b_n)\text{ and }b_{n+1}=N(a_{n+1},b_n)\ ,\ n\geq 0\ , \tag{3.33}$$

*where $a_0=a,b_0=b$, is called an **Archimedean double sequence**.*

In [I. Costin, G. Toader, 2002] is given the following

Lemma 109 *For every means M and N and every two initial values $a,b>0$, the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ defined by (3.33) converge monotonically.*

Proof. If $a \le b$ we can show by induction that

$$a_n \le a_{n+1} \le b_{n+1} \le b_n \ , \ n = 0, 1, ... \tag{3.34}$$

Indeed, assume that $a_n \le b_n$ (which holds for $n = 0$). From (3.33) and the definition of means we have

$$a_n \le a_{n+1} = M(a_n, b_n) \le b_n \ .$$

Then, by the same reason, we have

$$a_{n+1} \le b_{n+1} = M(a_{n+1}, b_n) \le b_n$$

and (3.34) was proven. So $(a_n)_{n\ge 0}$ is increasing and bounded above by $b = b_0$, thus it has a limit $\alpha(a, b) \le b$. Similarly, $(b_n)_{n\ge 0}$ is monotonic decreasing, bounded below by $a = a_0$ and has a limit $\beta(a, b) \ge a$. The case $a > b$ is similar and this completes the proof. ■

Remark 110 *The trivial example:*

$$a_{n+1} = \Pi_1(a_n, b_n) = a_n, \ b_{n+1} = \Pi_2(a_{n+1}, b_n) = b_n \ , \ n \ge 0 \ ,$$

shows that, without some auxiliary assumptions on the means M and N, the sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ can have different limits.

Definition 111 *The mean M is **compoundable in the sense of Archimedes** (or **A-compoundable**) with the mean N if the sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ defined by (3.33) are convergent to a common limit $M \boxtimes N(a, b)$ for each $a, b > 0$.*

Remark 112 *From the proof of the previous Lemma we deduce that*

$$a \wedge b \le M \boxtimes N(a, b) \le a \vee b \ , \ \forall a, b > 0 \ ,$$

that is, if M is A-compoundable with N, then $M \boxtimes N$ is a mean.

Definition 113 *The mean $M \boxtimes N$ is called **Archimedean compound mean** (or **A-compound mean)** of M and N and $\boxtimes$ is called the **Archimedean product.***

A rather general result was proved in [D. M. E. Foster, G. M. Phillips, 1984].

Theorem 114 *If the means M and N are continuous, symmetrical, and strict, then M is A-compoundable with N.*

These hypotheses were later weakened in [I. Costin, G. Toader, 2004a] where was proved that it is enough that only one of the two means have some properties like those from the theorem of Foster and Phillips.

Theorem 115 *If the mean M is continuous and strict at the left, or N is continuous and strict at the right, then M is A-compoundable with N.*

Proof. Assume that M is continuous and strict at the left. From the previous Lemma we deduce that for every $a, b \in J$, the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, defined by (3.33), have the limits α respectively β. From the first relation of (3.33) and the continuity of M we deduce that $\alpha = M(\alpha, \beta)$. As M is strict at the left it follows that $\alpha = \beta$ for every $a, b \in J$. The case N continuous and strict at the right is similar and so the proof is complete. ■

Remark 116 *The mean Π_1 is continuous and strict at the right, but it is not strict at the left. So, it is a "good" mean for the A-compounding at the right, but it is a "bad" mean for the A-compounding at the left. For a similar reason, Π_2 is good for the left A-compounding, but it is bad for the right A-compounding. For example, Π_2 is A-compoundable with Π_1 and*

$$\Pi_2 \boxtimes \Pi_1 = \Pi_2 \ ,$$

but, as we saw, $\Pi_1$is not A-compoundable with Π_2.

Remark 117 *As*

$$M \boxtimes N = M \otimes N(M, \Pi_2)$$

we can apply the results proved for Gaussian products also for the Archimedean products, making the above substitution.

Chapter 4

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