A note on certain Euler–Mascheroni type sequences
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Abstract. Expressions of type $x_n = \sum_{k=1}^{n} \frac{1}{a_k} - \log a_n$ ($a_n > 0$) will be called of Euler–Mascheroni type, as for $a_k \equiv k$ we obtain a sequence of approximations of the Euler–Mascheroni constant $\gamma$. The aim of this note is to solve two open problems posed by K. Kashihara [1] related to the convergence or divergence of $(x_n)$ when $a_n = p_n$ (nth prime), and $a_n = S(n)$ (Smarandache function). An analogous result on the Smarandache ceil function is pointed out, too.

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1 Introduction

Let $(a_n)$ be a sequence of strictly positive real numbers, and construct the new sequence $(x_n)$ defined by

$$x_n = \sum_{k=1}^{n} \frac{1}{a_k} - \log a_n \quad (n = 1, 2, \ldots)$$

(1)

For $a_k = k$ ($k = 1, 2, \ldots$) one obtains $x_n = \sum_{k=1}^{n} \frac{1}{k} - \log n$, which gives the well-known Euler sequence (or Euler–Mascheroni sequence), having as limit the Euler–Mascheroni constant $\gamma$ (see [3]).

In his book K. Kashihara [1] (see p. 42) posed the problems of convergence or divergence of sequence $(x_n)$ given by (1) for the particular cases $a_k = p_k$, the $k$th prime; as well as $a_k = S(k)$, the Smarandache function value. We will prove the following:

Theorem 1. The sequence $(x_n^1)$ given by

$$(x_n^1) = \sum_{k=1}^{n} \frac{1}{p_k} - \log p_n$$

(2)

is divergent, being unbounded from below. The sequence $(x_n^2)$ given by

$$(x_n^2) = \sum_{k=1}^{n} \frac{1}{S(k)} - \log S(n)$$

(3)
is divergent, being unbounded from above.

2 Proof of the Theorem

An old result of P. Chebyshev (see e.g. [2]) states that

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O(1),
\]

where \( p \) denote primes. This means that \( \left( \sum_{p \leq p_n} \frac{1}{p} - \log \log p_n \right) \) is a convergent sequence. Remarking that

\[
x_n^1 = \left( \sum_{p \leq p_n} \frac{1}{p} - \log \log p_n \right) + \log \log p_n - \log p_n,
\]

and by \( \log \log p_n - \log p_n = \log \left( \log p_n \right) \), since \( \log p_n \to 0 \) as \( n \to \infty \) we get that \( x_n^1 \to -\infty \) as \( n \to \infty \).

This proves the first part of the theorem.

For the second part, put \( n = m! \). Then, since \( S(n) = \min\{k \geq 1 : n|k!\} \), we have \( S(n) = m \), and

\[
x_n^2 = \left( 1 + \frac{1}{2} + \cdots + \frac{1}{m} - \log m \right) + \sum_{k < n, k \neq \ell, \ell < n} \frac{1}{S(k)},
\]

because for \( k = l! \), \( l < m \) one has \( S(k) = \ell \). Now, the last sum is greater than \( \sum_{p < n} \frac{1}{p} \), as for primes \( k = p < n \) one has \( S(k) = S(p) = p \), and \( p \neq \ell! \). It is well known that \( \sum_{p=1}^{\infty} \frac{1}{p} = +\infty \), so as \( m \to \infty \), clearly \( (x_n^2) \) becomes unbounded from above, since the term \( 1 + \frac{1}{2} + \cdots + \frac{1}{m} - \log m \) is bounded.

Remarks

1) For many improvements of (4) see our monograph [2].

2) For generalized Euler-Mascheroni constants, see our paper [3].

3) The above proof shows that \( S(n) \) may be replaced by any function having the properties \( S(k!) = k \) and \( S(p) = p \) (\( p \) prime).

4) Let \( S_2(n) = \min\{m \geq 1 : n|m^2\} \) be the "Smarandache ceil function" of order 2. By defining

\[
x_n^3 = \sum_{k=1}^{n} \frac{1}{S_2(k)} - \log S_2(n)
\]

we can prove similarly that \( (x_n^3) \) is an unbounded (from above) sequence. Even, a more precise
result holds true. Indeed, recently Wang Xiaoying [4] proved that

$$\sum_{n \leq x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \log^2 x + A_1 \log x + A_2 + \mathcal{O}(x^{-\frac{1}{4} + \varepsilon}). \quad (6)$$

Since \( \sqrt{n} \leq S_2(n) \leq n \), we have \( \log S_2(n) = \mathcal{O}(\log n) = \mathcal{O}(\log^2 n) \), so by (6) it follows that

$$\frac{x^3}{\log^2 n} \sim \frac{3}{2\pi^2} \text{ as } n \to \infty. \quad (7)$$

References


