A note on $f$-minimum functions

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Abstract. For a given arithmetical function $f : \mathbb{N} \to \mathbb{N}$, let $F : \mathbb{N} \to \mathbb{N}$ be defined by $F(n) = \min\{m \geq 1 : n|f(m)\}$, if this exists. Such functions, introduced in [4], will be called as the $f$-minimum functions. If $f$ satisfies the property $a \leq b \Rightarrow f(a)|f(b)$, we shall prove that $F(ab) = \max\{F(a), F(b)\}$ for $(a, b) = 1$. For a more restrictive class of functions, we will determine $F(n)$ where $n$ is an even perfect number. These results are generalizations of theorems from [10], [1], [3], [6].

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1 Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers, and $f : \mathbb{N} \to \mathbb{N}$ a given arithmetical function, such that for each $n \in \mathbb{N}$ there exists
at least an \( m \in \mathbb{N} \) such that \( n|f(m) \). In 1999 and 2000 [4], [5], as a common generalization of many arithmetical functions, we have defined the application \( F : \mathbb{N} \to \mathbb{N} \) given by

\[
F(n) = \min\{m \geq 1 : n|f(m)\},
\]

called as the "\( f \)-minimum function". Particularly, for \( f(m) = m! \) one obtains the Smarandache function (see [10], [1])

\[
S(n) = \min\{m \geq 1 : n|m!\}
\]  

Moree and Roskam [2], and independently the author [4], [5], have considered the Euler minimum function

\[
E(n) = \min\{m \geq 1 : n|\varphi(n)\}
\]

where \( \varphi \) is Euler’s totient. Many other particular cases of (1), as well as, their "dual" or analogues functions have been studied in the literature; for a survey of concepts and results, see [9].

In 1980 Smarandache discovered the following basic property of \( S(n) \) given by (2):

\[
S(ab) = \max\{S(a), S(b)\} \text{ for } (a, b) = 1.
\]  

Our aim in what follows is to extend property (4) to a general class of \( f \)-minimum functions. Further, for a subclass we will be able to determine \( F(n) \) for even perfect numbers \( n \).
2 Main results

Theorem 1. Suppose that $F$ of (1) is well defined. Then for distinct primes $p_i$, and arbitrary $\alpha_i \geq 1 \ (i = 1, r)$ one has

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \geq \max\{F(p_i^{\alpha_i}) : i = 1, r\}. \quad (5)$$

The second result offers a reverse inequality:

Theorem 2. With the notations of Theorem 1 suppose that $f$ satisfies the following divisibility condition:

$$a | b \Rightarrow f(a) | f(b) \quad (a, b \geq 1) \quad (\ast)$$

Then one has

$$F\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq \text{l.c.m.}\{F(p_i^{\alpha_i}) : i = 1, r\}, \quad (6)$$

where l.c.m. denotes the least common multiple.

By replacing (\ast) with another condition, a more precise result is obtainable:

Theorem 3. Suppose that $f$ satisfies the condition:

$$a \leq b \Rightarrow f(a) | f(b) \quad (a, b \geq 1) \quad (\ast\ast)$$

Then

$$F(mn) = \max\{F(m), F(n)\} \text{ for } (m, n) = 1. \quad (7)$$

Finally, we shall prove the following:
Theorem 4. Suppose that $f$ satisfies (**) and the following two assumptions:

(i) $n|f(n)$; (ii) For each prime $p$ and $m < p$ we have $p \nmid f(n)$.  (8)

Let $k$ be an even perfect number. Then

\[ F(k) = k/2^s, \text{ where } 2^s\parallel k. \]  (9)

Remarks. 1) The function $\varphi$ satisfies property (*). Then relation (6) gives a result for the Euler minimum function $E(n)$ (see [7], [8]).

2) Let $f(m) = m!$. Then clearly (**) holds true. Thus (7) extends relation (4). For another example, let $f(m) = l.c.m.\{1, 2, \ldots, m\}$. Then the function $F$ given by (1) satisfies again (7), proved e.g. in [1].

3) If $f(n) = n!$, then both (i) and (ii) of (8) are satisfied. This relation (9) for $F \equiv S$ follows. This was first proved in [3] (see also [6]).

3 Proofs of theorems

Theorem 1. There is no loss of generality to prove (5) for $r = 2$. Let $p^\alpha, q^\beta$ be two distinct prime powers. Then

\[ F(p^\alpha q^\beta) = \min\{n \geq 1 : p^\alpha q^\beta|f(m)\} = m_0, \]

so $p^\alpha q^\beta|f(m_0)$. This is equivalent to $p^\alpha |f(m_0), q^\beta|f(m_0)$. By definition (1) we get $m_0 \geq F(p^\alpha)$ and $m_0 \geq F(q^\beta)$, i.e. $F(p^\alpha q^\beta) \geq \max\{F(p^\alpha), F(q^\beta)\}$.  It is immediate that the same proof applies to $F\left(\prod p^\alpha\right) \geq \max\{F(p^\alpha)\}$, where $p^\alpha$ are distinct prime powers.
**Theorem 2.** Let \( F(p^\alpha) = m_1, F(q^\beta) = m_2 \). By definition (1) of function \( F \) it follows that \( p^\alpha|F(m_1) \) and \( q^\beta|F(m_2) \). Let \( \text{l.c.m.}\{m_1, m_2\} = g \). Since \( m_1|g \), one has \( f(m_1)|f(g) \) by (\(*\)). Similarly, since \( m_2|g \), one can write \( f(m_2)|f(g) \). These imply \( p^\alpha|f(m_1)|f(g) \) and \( q^\beta|f(m_2)|f(g) \), yielding \( p^\alpha q^\beta|f(g) \). By definition (1) this gives \( g \geq F(p^\alpha q^\beta) \), i.e. \( \text{l.c.m.}\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta) \), proving the theorem for \( r = 2 \). The general case follows exactly by the same lines.

**Theorem 3.** By taking into account of (5), one needs only to show that the reverse inequality is true. For simplicity, let us consider again \( r = 2 \). Let \( F(p^\alpha) = m, F(q^\beta) = n \) with \( m \leq n \). By definition (1) one has \( p^\alpha|f(m), q^\beta|f(n) \). Now, by assumption (\( **\)) we can write \( f(m)|f(n) \), so \( p^\alpha|f(n), q^\beta|f(n) \). Therefore, one has \( p^\alpha|f(n), q^\beta|f(n) \). This in turn implies \( p^\alpha q^\beta|f(n) \), so \( n \geq F(p^\alpha q^\beta) \); i.e. \( \max\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta) \). The general case follows exactly the same lines. Thus, we have proved essentially, that \( F(p^\alpha q^\beta) = \max\{F(p^\alpha), F(q^\beta)\} \), or more generally

\[
F \left( \prod_{i=1}^{r} p_i^{\alpha_i} \right) = \max\{F(p_i^{\alpha_i}) : i = 1, r\}. \tag{10}
\]

Now, relation (7) is an immediate consequence of (10), for by writing

\[
m = \prod_{i=1}^{r} p_i^{\alpha_i}, \quad n = \prod_{j=1}^{s} q_j^{\beta_j}, \quad \text{with } (p_i, q_j) = 1,
\]

it follows that

\[
F(mn) = \max\{F(p_i^{\alpha_i}), F(q_j^{\beta_j}) : i = 1, r; j = 1, s\}
\]

\[
= \max\{\max\{E(p_i^{\alpha_i}) : i = 1, r\}, \max\{E(q_j^{\beta_j}) : j = 1, s\}\}
\]
\[ = \max\{F(m), F(n)\}, \]

by equality (10).

**Theorem 4.** By (i) and definition (1) we get

\[ F(n) \leq n. \quad (11) \]

Now, by (i), one has \( p | f(p) \) for any prime \( p \), but by (ii), \( p \) is the least such number. This implies that

\[ F(p) = p \text{ for any prime } p. \quad (12) \]

Now, let \( k \) be an even perfect number. By the Euclid-Euler theorem (see e.g. [7]) \( k \) may be written as \( k = 2^{n-1}(2^n - 1) \), where \( p = 2^n - 1 \) is a prime (“Mersenne prime”). Since (**) holds true, by Theorem 3 we can write

\[ F(k) = F(2^{n-1}(2^n - 1)) = \max\{F(2^{n-1}), F(2^n - 1)\}. \]

Since \( F(2^n - 1) = 2^n - 1 \) (by (12)), and \( F(2^{n-1}) \leq 2^{n-1} \) (by (11)), from \( 2^{n-1} < 2^n - 1 \) for \( n \geq 2 \), we get \( F(k) = 2^n - 1 = \frac{k}{2^s} \), where \( s = n - 1 \) and \( 2^s | k \). This finishes the proof of Theorem 4.

**References**


