APPROXIMATION OF THE LAMBERT W FUNCTION AND HYPERPOWER FUNCTION

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Abstract. In this note, we get some explicit approximations for the Lambert W function $W(x)$, defined by $W(x)e^{W(x)} = x$ for $x \geq -e^{-1}$. Also, we get upper and lower bounds for the hyperpower function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$.

1. Introduction

The Lambert W function $W(x)$, is defined by $W(x)e^{W(x)} = x$ for $x \geq -e^{-1}$. For $-e^{-1} \leq x < 0$, there are two possible values of $W(x)$, which we take such values that aren't less than -1. The history of the function goes back to J. H. Lambert (1728-1777). One can find in [2] more detailed definition of W as a complex variable function, historical background and various applications of it in Mathematics and Physics. Expansion

$$W(x) = \log x - \log\log x + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km}\frac{(\log\log x)^m}{(\log x)^{k+m}},$$

holds true for large values of x, with $c_{km} = \frac{(-1)^k}{m!}S[k+m, k+1]$ where $S[k+m, k+1]$ is Stirling cycle number [2]. The series in above expansion being to be absolutely convergent and it can be rearranged into the form

$$W(x) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2-2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} + O\left(\left(\frac{L_2}{L_1}\right)^4\right),$$

where $L_1 = \log x$ and $L_2 = \log\log x$. Note that by log we mean logarithm in the base e. Since Lambert W function appears in some problems in Mathematics, Physics and Engineering, having some explicit approximations of it is very useful. In [5] it is shown that

$$\log x - \log\log x < W(x) < \log x, \tag{1.1}$$

which the left hand side holds true for $x > 41.19$ and the right hand side holds true for $x > e$. Aim of present note is to get some better bounds.

2. Better Approximations of the Lambert W Function

It is easy to see that $W(-e^{-1}) = -1$, $W(0) = 0$ and $W(e) = 1$. Also, for $x > 0$, since $W(x)e^{W(x)} = x > 0$ and $e^{W(x)} > 0$, we have $W(x) > 0$. About derivation, an easy calculation yields that

$$\frac{d}{dx}W(x) = \frac{W(x)}{x(1+W(x))}.$$

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So, $x\frac{d}{dx}W(x) > 0$ holds true for $x > 0$ and consequently $W(x)$ is strictly increasing for $x > 0$ (and also for $-e^{-1} \leq x \leq 0$, but not by this reason).

Theorem 2.1. *For every $x \geq e$, we have*

$$\log x - \log\log x \leq W(x) \leq \log x - \frac{1}{2}\log\log x, \tag{2.1}$$

with equality holding only for $x = e$. The coefficients -1 and $-\frac{1}{2}$ of $\log\log x$ both are best possible for the range $x \geq e$.

Proof. For constant $0 < p \leq 2$ consider the function

$$f(x) = \log x - \frac{1}{p}\log\log x - W(x),$$

for $x \geq e$. Easily

$$\frac{d}{dx}f(x) = \frac{p\log x - 1 - W(x)}{px(1 + W(x))\log x},$$

and if $p = 2$, then

$$\frac{d}{dx}f(x) = \frac{(\log x - W(x)) + (\log x - 1)}{2x(1 + W(x))\log x}.$$

Considering right hand side of (1.1) implies $\frac{d}{dx}f(x) > 0$ for $x > e$ and consequently $f(x) > f(e) = 0$, and this gives right hand side of (2.1). Trivially, equality holds for only $x = e$. If $0 < p < 2$, then $\frac{d}{dx}f(e) = \frac{p-2}{2ep} < 0$, and this yields that the coefficient $-\frac{1}{2}$ of $\log\log x$ in the right hand side of (2.1) is best possible for the range $x \geq e$.
For the another side, note that $\log W(x) = \log x - W(x)$ and the inequality $\log W(x) \leq \log\log x$ holds for $x \geq e$, because of the right hand side of (1.1). Thus, $\log x - W(x) \leq \log\log x$ holds for $x \geq e$ with equality only for $x = e$. Sharpness of (2.1) with coefficient -1 for $\log\log x$ comes from the relation $\lim_{x\to\infty}(W(x) - \log x + \log\log x) = 0$. This completes the proof. □

Now, we try to obtain some upper bounds for the function $W(x)$ with main term $\log x - \log\log x$. To do this we need the following lemma.

Lemma 2.2. *For every $t \in \mathbb{R}$ and $y > 0$, we have*

$$(t - \log y)e^t + y \geq e^t,$$

with equality for $t = \log y$.

Proof. Letting

$$f(t) = (t - \log y)e^t + y - e^t,$$

we have

$$\frac{d}{dt}f(t) = (t - \log y)e^t$$

and

$$\frac{d^2}{dt^2}f(t) = (t + 1 - \log y)e^t.$$

Now, we observe that

$$f(\log y) = \frac{d}{dt}f(\log y) = 0,$$

and

$$\frac{d^2}{dt^2} f(\log y) = y > 0.$$

This means the function $f(t)$ takes its minimum value equal to 0 at $t = \log y$, only. This gives the result of lemma. □

Theorem 2.3. *For $y > \frac{1}{e}$ and $x > -\frac{1}{e}$ we have*

$$W(x) \leq \log\left(\frac{x+y}{1+\log y}\right), \tag{2.2}$$

with equality only for $x = y \log y$.

Proof. Using the result of above lemma with $t = W(x)$, we get

$$(W(x) - \log y)e^{W(x)} - (e^{W(x)} - y) \geq 0,$$

which considering $W(x)e^{W(x)} = x$ gives $(1 + \log y)e^{W(x)} \leq x + y$ and this is desired inequality for $y > \frac{1}{e}$ and $x > -\frac{1}{e}$. The equality holds when $W(x) = \log y$, i.e. $x = y \log y$. □

Corollary 2.4. *For $x \geq e$ we have*

$$\log x - \log\log x \leq W(x) \leq \log x - \log\log x + \log(1 + e^{-1}), \tag{2.3}$$

where equality holds in left hand side for $x = e$ and in left hand side for $x = e^{e+1}$.

Proof. Consider (2.2) with $y = \frac{x}{e}$, and the left hand side of (2.1). □

Remark 2.5. Taking $y = x$ in (2.2) we get $W(x) \leq \log x - \log\left(\frac{1+\log x}{2}\right)$, which is sharper than right hand side of (2.1).

Theorem 2.6. *For $x > 1$ we have*

$$W(x) \geq \frac{\log x}{1 + \log x}(\log x - \log\log x + 1), \tag{2.4}$$

with equality only for $x = e$.

Proof. For $t > 0$ and $x > 1$, let

$$f(t) = \frac{t - \log x}{\log x} - (\log t - \log\log x),$$

We have

$$\frac{d}{dt} f(t) = \frac{1}{\log x} - \frac{1}{t},$$

and

$$\frac{d^2}{dt^2} f(t) = \frac{1}{t^2} > 0,$$

Now, we observe that $\frac{d}{dt} f(\log x) = 0$ and so

$$\min_{t>0} f(t) = f(\log x) = 0.$$

Thus, for $t > 0$ and $x > 1$ we have $f(t) \geq 0$ with equality at $t = \log x$. Putting $t = W(x)$ and simplifying, we get the result, with equality at $W(x) = \log x$ or equivalently at $x = e$. □

Corollary 2.7. *For $x > 1$ we have*

$$W(x) \leq (\log x)^{\frac{\log x}{1+\log x}}.$$

Proof. This refinement of the right hand side of (1.1), can be obtained simplifying (2.4) with $W(x) = \log x - \log W(x)$. ☐

Bounds which we have obtained up to now have the form $W(x) = \log x - \log\log x + O(1)$. Now, we give bounds with error term $O(\frac{\log\log x}{\log x})$ instead $O(1)$, with explicit constants for error term.

Theorem 2.8. *For every $x \geq e$ we have*

$$\log x - \log\log x + \frac{1}{2}\frac{\log\log x}{\log x} \leq W(x) \leq \log x - \log\log x + \frac{e}{e-1}\frac{\log\log x}{\log x}, \tag{2.5}$$

with equality only for $x = e$.

Proof. Taking logarithm from the right hand side of (2.1), we have

$$\log W(x) \leq \log\left(\log x - \frac{1}{2}\log\log x\right) = \log\log x + \log\left(1 - \frac{\log\log x}{2\log x}\right).$$

Using $\log W(x) = \log x - W(x)$, we get

$$W(x) \geq \log x - \log\log x - \log\left(1 - \frac{\log\log x}{2\log x}\right),$$

which considering $-\log(1-t) \geq t$ for $0 \leq t < 1$ (see [1]) with $t = \frac{\log\log x}{2\log x}$, implies the left hand side of (2.5). To prove another side, we take logarithm from the left hand side of (2.1) to get

$$\log W(x) \geq \log(\log x - \log\log x) = \log\log x + \log\left(1 - \frac{\log\log x}{\log x}\right).$$

Again, using $\log W(x) = \log x - W(x)$, we get

$$W(x) \leq \log x - \log\log x - \log\left(1 - \frac{\log\log x}{\log x}\right).$$

Now we use the inequality $-\log(1-t) \leq \frac{t}{1-t}$ for $0 \leq t < 1$ (see [1]) with $t = \frac{\log\log x}{\log x}$, to get

$$-\log\left(1 - \frac{\log\log x}{\log x}\right) \leq \frac{\log\log x}{\log x}\left(1 - \frac{\log\log x}{\log x}\right)^{-1} \leq \frac{1}{m}\frac{\log\log x}{\log x},$$

where $m = \min_{x \geq e}\left(1 - \frac{\log\log x}{\log x}\right) = 1 - \frac{1}{e}$. So, we have $-\log\left(1 - \frac{\log\log x}{\log x}\right) \leq \frac{e}{e-1}\frac{\log\log x}{\log x}$, which gives desired bounds. This completes the proof. ☐

3. Studying the hyperpower function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$

Consider the hyperpower function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$. One can define this function as the limit of the sequence $\{h_n(x)\}_{n\in\mathbb{N}}$ with $h_1(x) = x$ and $h_{n+1}(x) = x^{h_n(x)}$. It is proven that this sequence converge if and only if $e^{-e} \leq x \leq e^{\frac{1}{e}}$ (see [4] and references therein). This function satisfies the relation $h(x) = x^{h(x)}$, which taking logarithm from both sides and a simple calculation yields

$$h(x) = \frac{W(\log(x^{-1}))}{\log(x^{-1})}.$$

In this section we get some explicit upper and lower bounds for this function. To do this we won't use the bounds of Lambert W function, cause of they holds true and are sharp for x large enough. Instead, we do it directly.

Theorem 3.1. *Taking* $\lambda = e - 1 - \log(e-1) = 1.176956974\cdots$, *for* $e^{-e} \leq x \leq e^{\frac{1}{e}}$ *we have*

$$\frac{1+\log(1-\log x)}{1-2\log x} \leq h(x) \leq \frac{\lambda+\log(1-\log x)}{1-2\log x}, \tag{3.1}$$

where equality holds in left hand side for $x = 1$ *and in the right hand side for* $x = e^{\frac{1}{e}}$.

Proof. For $t > 0$ we have $t \geq \log t + 1$, which taking $t = z - \log z$ with $z > 0$, implies $z\big(1-2\log(z^{\frac{1}{z}})\big) \geq \log\big(1-\log(z^{\frac{1}{z}})\big)+1$, and putting $z^{\frac{1}{z}} = x$, or equivalently $z = h(x)$, it yields that $h(x)(1-2\log x) \geq \log(1-\log x)+1$; this is the left hand side (3.1), cause of $1-2\log x$ is positive for $e^{-e} \leq x \leq e^{\frac{1}{e}}$. Note that equality holds for $t = z = x = 1$.

For the right hand side, we define $f(z) = z - \log z$ with $\frac{1}{e} \leq z \leq e$. Easily we see that $1 \leq f(z) \leq e-1$; in fact it takes its minimum value 1 at $z = 1$. Also, consider the function $g(t) = \log t - t + \lambda$ for $1 \leq t \leq e-1$, with $\lambda = e - 1 - \log(e-1)$. Since $\frac{d}{dt}g(t) = \frac{1}{t} - 1$ and $g(e-1) = 0$, we obtain the inequality $\log t - t + \lambda \geq 0$ for $1 \leq t \leq e-1$, and putting $t = z - \log z$ with $\frac{1}{e} \leq z \leq e$ in this inequality, we obtain $\log(1-\log z) + \lambda \geq z\big(1-2\log(z^{\frac{1}{z}})\big)$. Taking $z^{\frac{1}{z}} = x$, or equivalently $z = h(x)$ yields the right hand side (3.1). Note that equality holds for $x = e^{\frac{1}{e}}$ $(z = e, t = e-1)$. This completes the proof. □

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