# INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL NORMED ALGEBRAS 

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#### Abstract

In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.


## 1. Introduction

Let $A$ be a unital normed algebra over the complex number field $\mathbb{C}$ and let $a \in A$. Recall that the numerical radius of $a$ is given by (see [2, p. 15])

$$
\begin{equation*}
v(a)=\sup \left\{|f(a)|, f \in A^{\prime},\|f\| \leq 1 \text { and } f(1)=1\right\} \tag{1.1}
\end{equation*}
$$

where $A^{\prime}$ denotes the dual space of $A$, i.e., the Banach space of all continuous linear functionals on $A$.

It is known that $v(\cdot)$ is a norm on $A$ that is equivalent to the given norm $\|\cdot\|$. More precisely, the following double inequality holds true:

$$
\begin{equation*}
\frac{1}{e}\|a\| \leq v(a) \leq\|a\| \tag{1.2}
\end{equation*}
$$

for any $a \in A$.
Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality $\|x\| \leq e \Phi(x)$, which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant $\frac{1}{4}$ in place of $\frac{1}{e}$. For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M.J. Crabb [3] which proved that

$$
\begin{equation*}
\left\|a^{n}\right\| \leq n!\left(\frac{e}{n}\right)^{n}[v(a)]^{n}, \quad n=1,2, \ldots \tag{1.3}
\end{equation*}
$$

for any $a \in A$.
In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra $A$ are given via the celebrated representation result of Lumer from [5].

## 2. Some Subsets in $A$

Let $D(1):=\left\{f \in A^{\prime} \mid\|f\| \leq 1\right.$ and $\left.f(1)=1\right\}$. For $\lambda \in \mathbb{C}$ and $r>0$, we define the subset of $A$ by

$$
\bar{\Delta}(\lambda, r):=\{a \in A| | f(a)-\lambda \mid \leq r \text { for each } f \in D(1)\}
$$

[^0]The following result holds.
Proposition 1. Let $\lambda \in \mathbb{C}$ and $r>0$. Then $\bar{\Delta}(\lambda, r)$ is a closed convex subset of $A$ and

$$
\begin{equation*}
\bar{B}(\lambda, r) \subseteq \bar{\Delta}(\lambda, r), \tag{2.1}
\end{equation*}
$$

where $\bar{B}(\lambda, r):=\{a \in A \mid\|a-\lambda\| \leq r\}$.
Now, for $\gamma, \Gamma \in \mathbb{C}$, define the set

$$
\bar{U}(\gamma, \Gamma):=\{a \in A \mid \operatorname{Re}[(\Gamma-f(a))(\overline{f(a)}-\bar{\gamma})] \geq 0 \text { for each } f \in D(1)\} .
$$

The following representation result may be stated.
Proposition 2. For any $\gamma, \Gamma \in \mathbb{C}, \gamma \neq \Gamma$, we have:

$$
\begin{equation*}
\bar{U}(\gamma, \Gamma)=\bar{\Delta}\left(\frac{\gamma+\Gamma}{2}, \frac{1}{2}|\Gamma-\gamma|\right) . \tag{2.2}
\end{equation*}
$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$
\left|z-\frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma|
$$

if and only if

$$
\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})] \geq 0
$$

This follows by the equality

$$
\frac{1}{4}|\Gamma-\gamma|^{2}-\left|z-\frac{\gamma+\Gamma}{2}\right|^{2}=\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})]
$$

that holds for any $z \in \mathbb{C}$.
The equality (2.2) is thus a simple conclusion of this fact.
Making use of some obvious properties in $\mathbb{C}$ and for continuous linear functionals, we can state the following corollary as well.

Corollary 1. For any $\gamma, \Gamma \in \mathbb{C}$, we have

$$
\begin{align*}
\bar{U}(\gamma, \Gamma)= & \{a \in A \mid \operatorname{Re}[f(\Gamma-a) \overline{f(a-\gamma)}] \geq 0 \text { for each } f \in D(1)\}  \tag{2.3}\\
= & \{a \in A \mid(\operatorname{Re} \Gamma-\operatorname{Re} f(a))(\operatorname{Re} f(a)-\operatorname{Re} \gamma) \\
& +(\operatorname{Im} \Gamma-\operatorname{Im} f(a))(\operatorname{Im} f(a)-\operatorname{Im} \gamma) \geq 0 \text { for each } f \in D(1)\}
\end{align*}
$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following subset of $A$ :

$$
\begin{align*}
& \bar{S}(\gamma, \Gamma):=\{a \in A \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(a) \geq \operatorname{Re}(\gamma) \text { and }  \tag{2.4}\\
& \quad \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(a) \geq \operatorname{Im}(\gamma) \text { for each } f \in D(1)\} .
\end{align*}
$$

One can easily observe that $\bar{S}(\gamma, \Gamma)$ is closed, convex and

$$
\begin{equation*}
\bar{S}(\gamma, \Gamma) \subseteq \bar{U}(\gamma, \Gamma) \tag{2.5}
\end{equation*}
$$

## 3. Semi-Inner Products and Lumer's Theorem

Let $(X,\|\cdot\|)$ be a normed linear space over the real of complex number field $\mathbb{K}$. The mapping $f: X \rightarrow \mathbb{R}, f(x)=\frac{1}{2}\|x\|^{2}$ is obviously convex and then there exists the following limits:

$$
\begin{aligned}
\langle x, y\rangle_{i} & =\lim _{t \rightarrow 0^{-}} \frac{\|y+t x\|^{2}-\|y\|^{2}}{2 t} \\
\langle x, y\rangle s & =\lim _{t \rightarrow 0^{+}} \frac{\|y+t x\|^{2}-\|y\|^{2}}{2 t}
\end{aligned}
$$

for every two elements $x, y \in X$. The mapping $\langle\cdot, \cdot\rangle_{s}\left(\langle\cdot, \cdot\rangle_{i}\right)$ will be called the superior semi-inner product (the interior semi-inner product) associated to the norm $\|\cdot\|$.

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance [4]):
(i) $\langle x, x\rangle_{p}=\|x\|^{2} ;\langle i x, x\rangle_{p}=\langle x, i x\rangle_{p}=0, x \in X$;
(ii) $\langle\lambda x, y\rangle_{p}=\lambda\langle x, y\rangle_{p} ;\langle x, \lambda y\rangle_{p}=\lambda\langle x, y\rangle_{p}$ for $\lambda \geq 0, x, y \in X$;
(iii) $\langle\lambda x, y\rangle_{p}=\lambda\langle x, y\rangle_{q} ;\langle x, \lambda y\rangle_{p}=\lambda\langle x, y\rangle_{q}$ for $\lambda<0, x, y \in X$;
(iv) $\langle i x, y\rangle_{p}=-\langle x, i y\rangle_{p} ;\langle\alpha x, \beta y\rangle=\alpha \beta\langle x, y\rangle$ if $\alpha \beta \geq 0, x, y \in X$;
(v) $\langle-x, y\rangle_{p}=\langle x,-y\rangle_{p}=-\langle x, y\rangle_{q}, x, y \in X$;
(vi) $\left|\langle x, y\rangle_{p}\right| \leq\|x\|\|y\|, x, y \in X$;
(vii) $\left\langle x_{1}+x_{2}, y\right\rangle_{s(i)} \leq(\geq)\left\langle x_{1}, y\right\rangle_{s(i)}+\left\langle x_{2}, y\right\rangle_{s(i)}$ for $x_{1}, x_{2}, y \in X$;
(ix) $\langle\alpha x+y, x\rangle_{p}=\alpha\|x\|^{2}+\langle y, x\rangle_{p}, \alpha \in \mathbb{R}, x, y \in X$;
(x) $\left|\langle y+z, x\rangle_{p}-\langle z, x\rangle_{p}\right| \leq\|y\|\|x\|, x, y, z \in X$;
(xi) The mapping $\langle\cdot, x\rangle_{p}$ is continuous on $(X,\|\cdot\|)$ for each $x \in X$, where $p, q \in$ $\{s, i\}$ and $p \neq q$.
The following result essentially due to Lumer [5] (see [2, p. 17]) can be stated.
Theorem 1. Let $A$ be a unital normed algebra over $\mathbb{K}(\mathbb{K}=\mathbb{C}, \mathbb{R})$. For each $a \in A$,

$$
\begin{equation*}
\max \{\operatorname{Re} \lambda|\lambda \in V(a)|\}=\inf _{\alpha>0} \frac{1}{\alpha}[\|1+\alpha a\|-1]=\lim _{\alpha \rightarrow 0^{+}} \frac{1}{\alpha}[\|1+\alpha a\|-1] \tag{3.1}
\end{equation*}
$$

where $V(a)$ is the numerical range of $a$ (see for instance [2, p. 15]).
Remark 1. In terms of semi-inner products, the above identity can be stated as:

$$
\begin{equation*}
\max \{\operatorname{Re} f(a) \mid f \in D(1)\}=\langle a, 1\rangle_{s} \tag{3.2}
\end{equation*}
$$

The following result that provides more information may be stated.
Theorem 2. For any $a \in A$, we have:

$$
\begin{equation*}
\langle a, 1\rangle_{v, s}=\langle a, 1\rangle_{s} \tag{3.3}
\end{equation*}
$$

where

$$
\langle a, b\rangle_{v, s}:=\lim _{t \rightarrow 0^{+}} \frac{v^{2}(b+t a)-v^{2}(b)}{2 t}
$$

is the superior semi-inner product associated with the numerical radius.

Proof. Since $v(a) \leq\|a\|$, we have:

$$
\begin{aligned}
\langle a, 1\rangle_{v, s} & =\lim _{t \rightarrow 0^{+}} \frac{v^{2}(1+t a)-v^{2}(1)}{2 t}=\lim _{t \rightarrow 0^{+}} \frac{v^{2}(1+t a)-1}{2 t} \\
& \leq \lim _{t \rightarrow 0^{+}} \frac{\|1+t a\|^{2}-1}{2 t}=\langle a, 1\rangle_{s} .
\end{aligned}
$$

Now, let $f \in D(1)$. Then, for each $\alpha>0$,

$$
f(a)=\frac{1}{\alpha}[f(1+\alpha a)-f(1)]=\frac{1}{\alpha}[f(1+\alpha a)-1],
$$

giving

$$
\begin{aligned}
\operatorname{Re} f(a) & =\frac{1}{\alpha}[\operatorname{Re} f(1+\alpha a)-f(1)] \leq \frac{1}{\alpha}[|f(1+\alpha a)|-1] \\
& \leq \frac{1}{\alpha}[v(1+\alpha a)-1]
\end{aligned}
$$

Taking the infimum over $\alpha>0$, we deduce

$$
\begin{align*}
\operatorname{Re} f(a) & \leq \inf _{\alpha>0}\left[\frac{1}{\alpha}[v(1+\alpha a)-1]\right]=\lim _{\alpha \rightarrow 0^{+}}\left[\frac{v^{2}(1+\alpha a)-1}{2 \alpha}\right]  \tag{3.4}\\
& =\lim _{\alpha \rightarrow 0^{+}} \frac{v(1+\alpha a)-1}{\alpha}=\langle a, 1\rangle_{v, s}
\end{align*}
$$

If we now take the supremum over $f \in D(1)$ in (3.4), we obtain:

$$
\sup \{\operatorname{Re} f(a) \mid f \in D(1)\} \leq\langle a, 1\rangle_{v, s}
$$

which gives, by Lumer's identity that $\langle a, 1\rangle_{s} \leq\langle a, 1\rangle_{v, s}$.
Corollary 2. We have the inequality

$$
\begin{equation*}
\left|\langle a, 1\rangle_{s}\right| \leq v(a) \quad(\leq\|a\|) \tag{3.5}
\end{equation*}
$$

Proof. Schwarz's inequality for the norm $v($.$) gives that$

$$
\left|\langle a, 1\rangle_{v, s}\right| \leq v(a) v(1)=v(a),
$$

and by (3.3), the inequality (3.5) is proved.

## 4. Reverse Inequalities for the Numerical Radius

Utilising the inequality (3.5) we observe that for any complex number $\beta$ located in the closed disc centered in 0 and with radius 1 we have $\left|\langle\beta a, 1\rangle_{s}\right|$ as a lower bound for the numerical radius $v(a)$. Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element $a$ in the unital normed algebra $A$ and the scalar $\beta$. A number of results answering this question are incorporated in the following theorems.
Theorem 3. Let $\lambda \in \mathbb{C} \backslash\{0\}$ and $r>0$. If $a \in \bar{\Delta}(\lambda, r)$, then

$$
\begin{equation*}
v(a) \leq\left\langle\frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_{s}+\frac{1}{2} \cdot \frac{r^{2}}{|\lambda|} \tag{4.1}
\end{equation*}
$$

Proof. Since $a \in \bar{\Delta}(\lambda, 1)$, then $|f(a)-\lambda|^{2} \leq r^{2}$, for each $f \in D(1)$, giving that

$$
\begin{equation*}
|f(a)|^{2}+|\lambda|^{2} \leq 2 \operatorname{Re}[f(\bar{\lambda} a)]+r^{2} \tag{4.2}
\end{equation*}
$$

for each $f \in D(1)$.
Taking the supremum of $f \in D(1)$ in (4.2) and utilising the representation (3.2), we deduce

$$
\begin{equation*}
v^{2}(a)+|\lambda|^{2} \leq 2\langle\bar{\lambda} a, 1\rangle_{s}+r^{2} \tag{4.3}
\end{equation*}
$$

which is an inequality of interest in itself.
On the other hand, we have the elementary inequality

$$
\begin{equation*}
2 v(a)|\lambda| \leq v^{2}(a)+|\lambda|^{2} \tag{4.4}
\end{equation*}
$$

which, together with (4.3) implies the desired result (4.1).
Remark 2. Notice that, by the inclusion (2.1) a sufficient condition for (4.1) to holds is that $a \in \bar{B}(\lambda, r)$.
Corollary 3. Let $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \pm \gamma$. If $a \in \bar{U}(\gamma, \Gamma)$, then

$$
\begin{equation*}
v(a) \leq\left\langle\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|} a, 1\right\rangle_{s}+\frac{1}{4} \cdot \frac{|\Gamma-\gamma|^{2}}{|\Gamma+\gamma|} \tag{4.5}
\end{equation*}
$$

Remark 3. If $M>m \geq 0$ and $a \in \bar{U}(m, M)$, then

$$
\begin{equation*}
(0 \leq) v(a)-\langle a, 1\rangle_{s} \leq \frac{1}{4} \cdot \frac{(M-m)^{2}}{m+M} \tag{4.6}
\end{equation*}
$$

Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to holds is that $M \geq \operatorname{Re} f(a), \operatorname{Im} f(a) \geq m$ for any $f \in D(1)$.

The following result may be stated as well.
Theorem 4. Let $\lambda \in \mathbb{C}$ and $r>0$ with $|\lambda|>r$. If $a \in \bar{\Delta}(\lambda, r)$, then

$$
\begin{equation*}
v(a) \leq\left\langle\frac{\bar{\lambda}}{\sqrt{|\lambda|^{2}-r^{2}}} a, 1\right\rangle_{s} \tag{4.7}
\end{equation*}
$$

and, equivalently,

$$
\begin{equation*}
v^{2}(a) \leq\left\langle\frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_{s}^{2}+\frac{r^{2}}{|\lambda|^{2}} \cdot v^{2}(a) \tag{4.8}
\end{equation*}
$$

Proof. Since $|\lambda|>r$, hence by (4.3) we have, on dividing by $\sqrt{|\lambda|^{2}-r^{2}}>0$, that

$$
\begin{equation*}
\frac{v^{2}(a)}{\sqrt{|\lambda|^{2}-r^{2}}}+\sqrt{|\lambda|^{2}-r^{2}} \leq \frac{2}{\sqrt{|\lambda|^{2}-r^{2}}}\langle\bar{\lambda} a, 1\rangle_{s} \tag{4.9}
\end{equation*}
$$

On the other hand, we also have

$$
2 v(a) \leq \frac{v^{2}(a)}{\sqrt{|\lambda|^{2}-r^{2}}}+\sqrt{|\lambda|^{2}-r^{2}}
$$

which, together with (4.9), gives

$$
\begin{equation*}
v(a) \leq \frac{1}{\sqrt{|\lambda|^{2}-r^{2}}}\langle\bar{\lambda} a, 1\rangle_{s} \tag{4.10}
\end{equation*}
$$

Taking the square in (4.10), we have

$$
v^{2}(a)\left(|\lambda|^{2}-r^{2}\right) \leq\langle\bar{\lambda} a, 1\rangle_{s}^{2}
$$

which is clearly equivalent to (4.7).
Corollary 4. Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma \bar{\gamma})>0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,

$$
\begin{equation*}
v(a) \leq\left\langle\frac{\bar{\Gamma}+\bar{\gamma}}{2 \sqrt{\operatorname{Re}(\Gamma \bar{\gamma})}} a, 1\right\rangle_{s} . \tag{4.11}
\end{equation*}
$$

Remark 4. If $M \geq m>0$ and $a \in \bar{U}(m, M)$, then

$$
\begin{equation*}
v(a) \leq \frac{M+m}{2 \sqrt{m M}}\langle a, 1\rangle_{s} \tag{4.12}
\end{equation*}
$$

or, equivalently,

$$
(0 \leq) v(a)-\langle a, 1\rangle_{s} \leq \frac{(\sqrt{M}-\sqrt{m})^{2}}{2 \sqrt{m M}}\langle a, 1\rangle_{s} \quad\left(\leq \frac{(\sqrt{M}-\sqrt{m})^{2}}{2 \sqrt{m M}}\|a\|\right)
$$

The following result may be stated as well.
Theorem 5. Let $\lambda \in \mathbb{C} \backslash\{0\}$ and $r>0$ with $|\lambda|>r$. If $a \in \bar{\Delta}(\lambda, r)$, then

$$
\begin{equation*}
v^{2}(a) \leq\left\langle\frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_{s}^{2}+2\left(|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}\right)\left\langle\frac{\bar{\lambda}}{|\lambda|} a, 1\right\rangle_{s} . \tag{4.13}
\end{equation*}
$$

Proof. Since (by (4.2)) Re $[f(\bar{\lambda} a)]>0$, then dividing by it in (4.2) gives:

$$
\frac{|f(a)|^{2}}{\operatorname{Re}[f(\bar{\lambda} a)]}+\frac{|\lambda|^{2}}{\operatorname{Re}[f(\bar{\lambda} a)]} \leq 2+\frac{r^{2}}{\operatorname{Re}[f(\bar{\lambda} a)]}
$$

which is clearly equivalent to:

$$
\begin{align*}
& \frac{|f(a)|^{2}}{\operatorname{Re}[f(\bar{\lambda} a)]}-\frac{\operatorname{Re}[f(\bar{\lambda} a)]}{|\lambda|^{2}}  \tag{4.14}\\
& \leq 2+\frac{r^{2}}{\operatorname{Re}[f(\bar{\lambda} a)]}-\frac{\operatorname{Re}[f(\bar{\lambda} a)]}{|\lambda|^{2}}-\frac{|\lambda|^{2}}{\operatorname{Re}[f(\bar{\lambda} a)]}=: I .
\end{align*}
$$

Since

$$
\begin{align*}
I & =2-\frac{\operatorname{Re}[f(\bar{\lambda} a)]}{|\lambda|^{2}}-\frac{\left(|\lambda|^{2}-r^{2}\right)}{\operatorname{Re}[f(\bar{\lambda} a)]}  \tag{4.15}\\
& =2-2 \frac{\sqrt{|\lambda|^{2}-r^{2}}}{|\lambda|}-\left[\frac{\sqrt{\operatorname{Re}[f(\bar{\lambda} a)]}}{|\lambda|}-\frac{\sqrt{|\lambda|^{2}-r^{2}}}{\sqrt{\operatorname{Re}[f(\bar{\lambda} a)]}}\right]^{2} \\
& \leq 2\left(1-\sqrt{1-\left(\frac{r}{|\lambda|}\right)^{2}}\right)
\end{align*}
$$

hence by (4.14) and (4.15) we have

$$
\begin{equation*}
|f(a)|^{2} \leq \frac{\operatorname{Re}[f(\bar{\lambda} a)]}{|\lambda|^{2}}+2\left(1-\sqrt{1-\left(\frac{r}{|\lambda|}\right)^{2}}\right) \operatorname{Re}[f(\bar{\lambda} a)] \tag{4.16}
\end{equation*}
$$

Taking the supremum in $f \in D(1)$ and utilising Lumer's result, we deduce the desired inequality (4.13).

Corollary 5. Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma \bar{\gamma})>0$. If $a \in \bar{U}(\gamma, \Gamma)$, then,

$$
v^{2}(a) \leq\left\langle\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|} a, 1\right\rangle_{s}^{2}+2\left(\left|\frac{\gamma+\Gamma}{2}\right|-\sqrt{\operatorname{Re}(\Gamma \bar{\gamma})}\right)\left\langle\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|} a, 1\right\rangle_{s} .
$$

Remark 5. If $M>m \geq 0$ and $a \in \bar{U}(m, M)$, then

$$
(0 \leq) v^{2}(a)-\langle a, 1\rangle_{s}^{2} \leq(\sqrt{M}-\sqrt{m})^{2}\langle a, 1\rangle_{s}\left(\leq(\sqrt{M}-\sqrt{m})^{2}\|a\|\right)
$$

Finally, the following result can be stated as well.
Theorem 6. Let $\lambda \in \mathbb{C}$ and $r>0$ with $|\lambda|>r$. If $a \in \bar{\Delta}(\lambda, r)$, then

$$
\begin{align*}
v(a) \leq\left(|\lambda|+\sqrt{|\lambda|^{2}-r^{2}}\right) & \left\langle\frac{\bar{\lambda}}{r^{2}} a, 1\right\rangle_{s}  \tag{4.17}\\
+ & \frac{|\lambda|\left(|\lambda|+\sqrt{|\lambda|^{2}-r^{2}}\right)\left(|\lambda|-2 \sqrt{|\lambda|^{2}-r^{2}}\right)}{2 r^{2}}
\end{align*}
$$

Proof. From the proof of Theorem 3 above, we have

$$
\begin{equation*}
|f(a)|^{2}+|\lambda|^{2} \leq 2 \operatorname{Re}[f(\bar{\lambda} a)]+r^{2} \tag{4.18}
\end{equation*}
$$

which is equivalent with

$$
\begin{align*}
& |f(a)|^{2}+\left(|\lambda|+\sqrt{|\lambda|^{2}-r^{2}}\right)^{2}  \tag{4.19}\\
& \leq 2 \operatorname{Re}[f(\bar{\lambda} a)]+r^{2}-|\lambda|^{2}+\left(|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}\right)^{2} \\
& =2 \operatorname{Re}[f(\bar{\lambda} a)]+|\lambda|^{2}-2|\lambda| \sqrt{|\lambda|^{2}-r^{2}}
\end{align*}
$$

Taking the supremum in (4.19) over $f \in D(1)$ and utilising Lumer's representation theorem, we get:

$$
\begin{equation*}
v^{2}(a)+\left(|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}\right)^{2} \leq 2\langle\bar{\lambda} a, 1\rangle_{s}+|\lambda|\left(|\lambda|-2 \sqrt{|\lambda|^{2}-r^{2}}\right) \tag{4.20}
\end{equation*}
$$

Since $r \neq 0$, then $|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}>0$, giving

$$
\begin{equation*}
2\left(|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}\right) v(a) \leq v^{2}(a)+\left(|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}\right)^{2} \tag{4.21}
\end{equation*}
$$

Now, utilising (4.20) and (4.21), we deduce

$$
v(a) \leq \frac{1}{|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}}\langle\bar{\lambda} a, 1\rangle_{s}+\frac{|\lambda|\left(|\lambda|-2 \sqrt{|\lambda|^{2}-r^{2}}\right)}{2\left(|\lambda|-\sqrt{|\lambda|^{2}-r^{2}}\right)}
$$

which is clearly equivalent with the desired result (4.17).
Remark 6. If $M>m \geq 0$ and $a \in \bar{U}(m, M)$, then

$$
v(a) \leq \frac{M+m}{(\sqrt{M}-\sqrt{m})^{2}}\left[\langle a, 1\rangle_{s}+\frac{1}{2}\left(\frac{m+M}{2}-2 \sqrt{m M}\right)\right]
$$

In particular, if $a \in \bar{U}(0, \delta)$ with $\delta>0$, then we have the following reverse inequality as well

$$
(0 \leq) v(a)-\langle a, 1\rangle_{s} \leq \frac{1}{4} \delta
$$

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