INEQUALITIES FOR THE NUMERICAL RADIUS IN UNITAL NORMED ALGEBRAS

S.S. DRAGOMIR

ABSTRACT. In this paper, some inequalities between the numerical radius of an element from a unital normed algebra and certain semi-inner products involving that element and the unity are given.

1. INTRODUCTION

Let A be a unital normed algebra over the complex number field \mathbb{C} and let $a \in A$. Recall that the numerical radius of a is given by (see [2, p. 15])

(1.1)
$$v(a) = \sup\{|f(a)|, f \in A', \|f\| \le 1 \text{ and } f(1) = 1\},\$$

where A' denotes the dual space of A, i.e., the Banach space of all continuous linear functionals on A.

It is known that $v(\cdot)$ is a norm on A that is equivalent to the given norm $\|\cdot\|$. More precisely, the following double inequality holds true:

(1.2)
$$\frac{1}{e} \|a\| \le v(a) \le \|a\|$$

for any $a \in A$.

Following [2], we notice that this crucial result appears slightly hidden in Bohnenblust and Karlin [1, Theorem 1] together with the inequality $||x|| \le e\Phi(x)$, which occurs on page 219. A simpler proof was given by Lumer [5], though with the constant $\frac{1}{4}$ in place of $\frac{1}{e}$. For a simple proof of (1.2) that borrows ideas from Lumer and from Glickfeld [6], see [2, p. 34].

A generalisation of (1.2) for powers has been obtained by M.J. Crabb [3] which proved that

(1.3)
$$||a^n|| \le n! \left(\frac{e}{n}\right)^n [v(a)]^n, \quad n = 1, 2, ...$$

for any $a \in A$.

In this paper, some inequalities between the numerical radius of an element and the superior semi-inner product of that element and the unity in the normed algebra A are given via the celebrated representation result of Lumer from [5].

2. Some Subsets in A

Let $D(1) := \{f \in A' | ||f|| \le 1 \text{ and } f(1) = 1\}$. For $\lambda \in \mathbb{C}$ and r > 0, we define the subset of A by

$$\bar{\Delta}(\lambda, r) := \left\{ a \in A | | f(a) - \lambda | \le r \text{ for each } f \in D(1) \right\}.$$

Date: 16 April, 2008.

²⁰⁰⁰ Mathematics Subject Classification. 46H05, 46K05, 47A10.

Key words and phrases. Unital normed algebras, Numerical radius, Semi-inner products.

The following result holds.

Proposition 1. Let $\lambda \in \mathbb{C}$ and r > 0. Then $\overline{\Delta}(\lambda, r)$ is a closed convex subset of A and

(2.1)
$$\overline{B}(\lambda, r) \subseteq \overline{\Delta}(\lambda, r),$$

where $\overline{B}(\lambda, r) := \{a \in A | ||a - \lambda|| \le r\}.$

Now, for $\gamma, \Gamma \in \mathbb{C}$, define the set

$$\bar{U}(\gamma,\Gamma) := \left\{ a \in A | \operatorname{Re}\left[(\Gamma - f(a)) \left(\overline{f(a)} - \overline{\gamma} \right) \right] \ge 0 \text{ for each } f \in D(1) \right\}.$$

The following representation result may be stated.

Proposition 2. For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have:

(2.2)
$$\bar{U}(\gamma,\Gamma) = \bar{\Delta}\left(\frac{\gamma+\Gamma}{2}, \frac{1}{2}|\Gamma-\gamma|\right).$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z - \frac{\gamma + \Gamma}{2}\right| \le \frac{1}{2} \left|\Gamma - \gamma\right|$$

if and only if

$$\operatorname{Re}\left[\left(\Gamma-z\right)\left(\bar{z}-\bar{\gamma}\right)\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Gamma-\gamma\right|^{2}-\left|z-\frac{\gamma+\Gamma}{2}\right|^{2}=\operatorname{Re}\left[\left(\Gamma-z\right)\left(\bar{z}-\bar{\gamma}\right)\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple conclusion of this fact.

Making use of some obvious properties in \mathbb{C} and for continuous linear functionals, we can state the following corollary as well.

Corollary 1. For any $\gamma, \Gamma \in \mathbb{C}$, we have

(2.3)
$$\bar{U}(\gamma,\Gamma) = \left\{ a \in A \mid \operatorname{Re}\left[f(\Gamma-a)\overline{f(a-\gamma)}\right] \ge 0 \text{ for each } f \in D(1) \right\}$$
$$= \left\{ a \in A \mid (\operatorname{Re}\Gamma - \operatorname{Re}f(a)) (\operatorname{Re}f(a) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im}f(a)) (\operatorname{Im}f(a) - \operatorname{Im}\gamma) \ge 0 \text{ for each } f \in D(1) \right\}.$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following subset of A:

(2.4)
$$\overline{S}(\gamma, \Gamma) := \{a \in A \mid \operatorname{Re}(\Gamma) \ge \operatorname{Re} f(a) \ge \operatorname{Re}(\gamma) \text{ and}$$

 $\operatorname{Im}(\Gamma) \ge \operatorname{Im} f(a) \ge \operatorname{Im}(\gamma) \text{ for each } f \in D(1)\}.$

One can easily observe that $\bar{S}(\gamma, \Gamma)$ is closed, convex and

(2.5)
$$\overline{S}(\gamma,\Gamma) \subseteq \overline{U}(\gamma,\Gamma).$$

 $\mathbf{2}$

3. Semi-Inner Products and Lumer's Theorem

Let $(X, \|\cdot\|)$ be a normed linear space over the real of complex number field K. The mapping $f: X \to \mathbb{R}, f(x) = \frac{1}{2} ||x||^2$ is obviously convex and then there exists the following limits:

$$\begin{split} \langle x, y \rangle_i &= \lim_{t \to 0^-} \frac{\|y + tx\|^2 - \|y\|^2}{2t}, \\ \langle x, y \rangle s &= \lim_{t \to 0^+} \frac{\|y + tx\|^2 - \|y\|^2}{2t}. \end{split}$$

for every two elements $x, y \in X$. The mapping $\langle \cdot, \cdot \rangle_s$ $(\langle \cdot, \cdot \rangle_i)$ will be called the superior semi-inner product (the interior semi-inner product) associated to the norm $\|\cdot\|$.

We list some properties of these semi-inner products that can be easily derived from the definition (see for instance [4]):

- (i) $\langle x, x \rangle_p = ||x||^2$; $\langle ix, x \rangle_p = \langle x, ix \rangle_p = 0, x \in X$; (ii) $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_p$; $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_p$ for $\lambda \ge 0, x, y \in X$; (iii) $\langle \lambda x, y \rangle_p = \lambda \langle x, y \rangle_q$; $\langle x, \lambda y \rangle_p = \lambda \langle x, y \rangle_q$ for $\lambda < 0, x, y \in X$; (iv) $\langle ix, y \rangle_p = -\langle x, iy \rangle_p$; $\langle \alpha x, \beta y \rangle = \alpha \beta \langle x, y \rangle$ if $\alpha \beta \ge 0, x, y \in X$; (v) $\langle -x, y \rangle_p = \langle x, -y \rangle_p = -\langle x, y \rangle_q$, $x, y \in X$; (vi) $|\langle x, y \rangle_p| \le ||x|| ||y||$, $x, y \in X$; (vii) $\langle x_1 + x_2, y \rangle_{s(i)} \le (\ge) \langle x_1, y \rangle_{s(i)} + \langle x_2, y \rangle_{s(i)}$ for $x_1, x_2, y \in X$; (iv) $\langle x_1 + x_2, y \rangle_{s(i)} \le (\ge) \langle x_1, y \rangle_{s(i)} + \langle x_2, y \rangle_{s(i)}$ for $x_1, x_2, y \in X$;
- (ix) $\langle \alpha x + y, x \rangle_p = \alpha ||x||^2 + \langle y, x \rangle_p, \ \alpha \in \mathbb{R}, \ x, y \in X;$
- (x) $\left| \langle y+z,x \rangle_p \langle z,x \rangle_p \right| \le \left\| y \right\| \left\| x \right\|, x,y,z \in X;$
- (xi) The mapping $\langle \cdot, x \rangle_p^{-1}$ is continuous on $(X, \|\cdot\|)$ for each $x \in X$, where $p, q \in$ $\{s, i\}$ and $p \neq q$.

The following result essentially due to Lumer [5] (see [2, p. 17]) can be stated.

Theorem 1. Let A be a unital normed algebra over \mathbb{K} ($\mathbb{K} = \mathbb{C}, \mathbb{R}$). For each $a \in A$,

(3.1)
$$\max \{ \operatorname{Re} \lambda | \lambda \in V(a) | \} = \inf_{\alpha > 0} \frac{1}{\alpha} [\| 1 + \alpha a \| - 1] = \lim_{\alpha \to 0^+} \frac{1}{\alpha} [\| 1 + \alpha a \| - 1],$$

where V(a) is the numerical range of a (see for instance [2, p. 15]).

Remark 1. In terms of semi-inner products, the above identity can be stated as:

(3.2)
$$\max \left\{ \operatorname{Re} f\left(a\right) | f \in D\left(1\right) \right\} = \left\langle a, 1 \right\rangle_{s}$$

The following result that provides more information may be stated.

Theorem 2. For any $a \in A$, we have:

$$(3.3) \qquad \qquad \langle a,1\rangle_{v,s} = \langle a,1\rangle_s$$

where

$$\langle a, b \rangle_{v,s} := \lim_{t \to 0^+} \frac{v^2 (b + ta) - v^2 (b)}{2t}$$

is the superior semi-inner product associated with the numerical radius.

Proof. Since $v(a) \leq ||a||$, we have:

$$\begin{split} \langle a,1\rangle_{v,s} &= \lim_{t\to 0^+} \frac{v^2 \left(1+ta\right) - v^2 \left(1\right)}{2t} = \lim_{t\to 0^+} \frac{v^2 \left(1+ta\right) - 1}{2t} \\ &\leq \lim_{t\to 0^+} \frac{\|1+ta\|^2 - 1}{2t} = \langle a,1\rangle_s \,. \end{split}$$

Now, let $f \in D(1)$. Then, for each $\alpha > 0$,

$$f(a) = \frac{1}{\alpha} \left[f(1 + \alpha a) - f(1) \right] = \frac{1}{\alpha} \left[f(1 + \alpha a) - 1 \right],$$

giving

$$\operatorname{Re} f(a) = \frac{1}{\alpha} \left[\operatorname{Re} f(1 + \alpha a) - f(1) \right] \leq \frac{1}{\alpha} \left[\left| f(1 + \alpha a) \right| - 1 \right]$$
$$\leq \frac{1}{\alpha} \left[v(1 + \alpha a) - 1 \right].$$

Taking the infimum over $\alpha > 0$, we deduce

(3.4)
$$\operatorname{Re} f(a) \leq \inf_{\alpha > 0} \left[\frac{1}{\alpha} \left[v \left(1 + \alpha a \right) - 1 \right] \right] = \lim_{\alpha \to 0^+} \left[\frac{v^2 \left(1 + \alpha a \right) - 1}{2\alpha} \right]$$
$$= \lim_{\alpha \to 0^+} \frac{v \left(1 + \alpha a \right) - 1}{\alpha} = \langle a, 1 \rangle_{v,s}.$$

If we now take the supremum over $f \in D(1)$ in (3.4), we obtain:

 $\sup \left\{ \operatorname{Re} f\left(a\right) | f \in D\left(1\right) \right\} \le \left\langle a, 1 \right\rangle_{v,s}$

which gives, by Lumer's identity that $\langle a,1\rangle_s\leq \langle a,1\rangle_{v,s}\,.$

Corollary 2. We have the inequality

$$(3.5) \qquad \qquad |\langle a,1\rangle_s| \le v(a) \quad (\le ||a||).$$

Proof. Schwarz's inequality for the norm v(.) gives that

$$\left|\left\langle a,1\right\rangle _{v,s}
ight|\leq v\left(a
ight)v\left(1
ight)=v\left(a
ight)$$

and by (3.3), the inequality (3.5) is proved.

4. Reverse Inequalities for the Numerical Radius

Utilising the inequality (3.5) we observe that for any complex number β located in the closed disc centered in 0 and with radius 1 we have $|\langle \beta a, 1 \rangle_s|$ as a lower bound for the numerical radius v(a). Therefore, it is a natural question to ask how far these quantities are from each other under various assumptions for the element a in the unital normed algebra A and the scalar β . A number of results answering this question are incorporated in the following theorems.

Theorem 3. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and r > 0. If $a \in \overline{\Delta}(\lambda, r)$, then

(4.1)
$$v(a) \le \left\langle \frac{\overline{\lambda}}{|\lambda|} a, 1 \right\rangle_s + \frac{1}{2} \cdot \frac{r^2}{|\lambda|}.$$

Proof. Since $a \in \overline{\Delta}(\lambda, 1)$, then $|f(a) - \lambda|^2 \le r^2$, for each $f \in D(1)$, giving that (4.2) $|f(a)|^2 + |\lambda|^2 \le 2 \operatorname{Re}\left[f(\overline{\lambda}a)\right] + r^2$

for each $f \in D(1)$.

Taking the supremum of $f \in D(1)$ in (4.2) and utilising the representation (3.2), we deduce

(4.3)
$$v^{2}(a) + |\lambda|^{2} \leq 2 \left\langle \bar{\lambda}a, 1 \right\rangle_{s} + r^{2}$$

which is an inequality of interest in itself.

On the other hand, we have the elementary inequality

(4.4)
$$2v(a)|\lambda| \le v^2(a) + |\lambda|^2$$

which, together with (4.3) implies the desired result (4.1).

Remark 2. Notice that, by the inclusion (2.1) a sufficient condition for (4.1) to holds is that $a \in \overline{B}(\lambda, r)$.

Corollary 3. Let $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq \pm \gamma$. If $a \in \overline{U}(\gamma, \Gamma)$, then

(4.5)
$$v(a) \le \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_s + \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}.$$

Remark 3. If $M > m \ge 0$ and $a \in \overline{U}(m, M)$, then

(4.6)
$$(0 \le) v(a) - \langle a, 1 \rangle_s \le \frac{1}{4} \cdot \frac{(M-m)^2}{m+M}.$$

Observe that, due to the inclusion (2.5), a sufficient condition for (4.6) to holds is that $M \ge \operatorname{Re} f(a)$, $\operatorname{Im} f(a) \ge m$ for any $f \in D(1)$.

The following result may be stated as well.

Theorem 4. Let $\lambda \in \mathbb{C}$ and r > 0 with $|\lambda| > r$. If $a \in \overline{\Delta}(\lambda, r)$, then

(4.7)
$$v(a) \le \left\langle \frac{\bar{\lambda}}{\sqrt{|\lambda|^2 - r^2}} a, 1 \right\rangle_s$$

and, equivalently,

(4.8)
$$v^{2}(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1 \right\rangle_{s}^{2} + \frac{r^{2}}{|\lambda|^{2}} \cdot v^{2}(a).$$

Proof. Since $|\lambda| > r$, hence by (4.3) we have, on dividing by $\sqrt{|\lambda|^2 - r^2} > 0$, that

(4.9)
$$\frac{v^{2}(a)}{\sqrt{|\lambda|^{2} - r^{2}}} + \sqrt{|\lambda|^{2} - r^{2}} \leq \frac{2}{\sqrt{|\lambda|^{2} - r^{2}}} \left\langle \bar{\lambda}a, 1 \right\rangle_{s}.$$

On the other hand, we also have

$$2v(a) \le \frac{v^2(a)}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2}$$

which, together with (4.9), gives

(4.10)
$$v(a) \le \frac{1}{\sqrt{|\lambda|^2 - r^2}} \left\langle \bar{\lambda}a, 1 \right\rangle_s.$$

Taking the square in (4.10), we have

$$v^{2}(a)\left(\left|\lambda\right|^{2}-r^{2}\right)\leq\left\langle \bar{\lambda}a,1\right\rangle _{s}^{2},$$

which is clearly equivalent to (4.7).

Corollary 4. Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$. If $a \in \overline{U}(\gamma, \Gamma)$, then,

(4.11)
$$v(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}}a, 1 \right\rangle_{s}.$$

Remark 4. If $M \ge m > 0$ and $a \in \overline{U}(m, M)$, then

(4.12)
$$v\left(a\right) \le \frac{M+m}{2\sqrt{mM}}\left\langle a,1\right\rangle_{s},$$

or, equivalently,

$$(0 \le) v(a) - \langle a, 1 \rangle_s \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \langle a, 1 \rangle_s \quad \left(\le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \|a\|\right).$$

The following result may be stated as well.

Theorem 5. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and r > 0 with $|\lambda| > r$. If $a \in \overline{\Delta}(\lambda, r)$, then

(4.13)
$$v^{2}(a) \leq \left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1 \right\rangle_{s}^{2} + 2\left(\left|\lambda\right| - \sqrt{\left|\lambda\right|^{2} - r^{2}}\right) \left\langle \frac{\bar{\lambda}}{|\lambda|}a, 1 \right\rangle_{s}$$

Proof. Since (by (4.2)) Re $[f(\bar{\lambda}a)] > 0$, then dividing by it in (4.2) gives:

$$\frac{\left|f\left(a\right)\right|^{2}}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} + \frac{\left|\lambda\right|^{2}}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} \leq 2 + \frac{r^{2}}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]},$$

which is clearly equivalent to:

$$(4.14) \quad \frac{|f(a)|^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} - \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^2} \\ \leq 2 + \frac{r^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} - \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^2} - \frac{|\lambda|^2}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} =: I.$$

Since

$$(4.15) I = 2 - \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{\left|\lambda\right|^{2}} - \frac{\left(\left|\lambda\right|^{2} - r^{2}\right)}{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]} \\ = 2 - 2\frac{\sqrt{\left|\lambda\right|^{2} - r^{2}}}{\left|\lambda\right|} - \left[\frac{\sqrt{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}}{\left|\lambda\right|} - \frac{\sqrt{\left|\lambda\right|^{2} - r^{2}}}{\sqrt{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}}\right]^{2} \\ \le 2\left(1 - \sqrt{1 - \left(\frac{r}{\left|\lambda\right|}\right)^{2}}\right),$$

hence by (4.14) and (4.15) we have

(4.16)
$$|f(a)|^{2} \leq \frac{\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right]}{|\lambda|^{2}} + 2\left(1 - \sqrt{1 - \left(\frac{r}{|\lambda|}\right)^{2}}\right)\operatorname{Re}\left[f\left(\bar{\lambda}a\right)\right].$$

Taking the supremum in $f \in D(1)$ and utilising Lumer's result, we deduce the desired inequality (4.13).

Corollary 5. Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$. If $a \in \overline{U}(\gamma, \Gamma)$, then,

$$v^{2}(a) \leq \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_{s}^{2} + 2\left(\left| \frac{\gamma + \Gamma}{2} \right| - \sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)} \right) \left\langle \frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} a, 1 \right\rangle_{s}.$$

Remark 5. If $M > m \ge 0$ and $a \in \overline{U}(m, M)$, then

$$(0 \le) v^2(a) - \langle a, 1 \rangle_s^2 \le \left(\sqrt{M} - \sqrt{m}\right)^2 \langle a, 1 \rangle_s \left(\le \left(\sqrt{M} - \sqrt{m}\right)^2 \|a\| \right).$$

Finally, the following result can be stated as well.

Theorem 6. Let $\lambda \in \mathbb{C}$ and r > 0 with $|\lambda| > r$. If $a \in \overline{\Delta}(\lambda, r)$, then

$$(4.17) \quad v(a) \leq \left(\left|\lambda\right| + \sqrt{\left|\lambda\right|^2 - r^2}\right) \left\langle \frac{\bar{\lambda}}{r^2} a, 1 \right\rangle_s + \frac{\left|\lambda\right| \left(\left|\lambda\right| + \sqrt{\left|\lambda\right|^2 - r^2}\right) \left(\left|\lambda\right| - 2\sqrt{\left|\lambda\right|^2 - r^2}\right)}{2r^2}$$

Proof. From the proof of Theorem 3 above, we have

(4.18)
$$|f(a)|^2 + |\lambda|^2 \le 2 \operatorname{Re} \left[f(\bar{\lambda}a) \right] + r^2$$

which is equivalent with

(4.19)
$$|f(a)|^{2} + \left(|\lambda| + \sqrt{|\lambda|^{2} - r^{2}}\right)^{2}$$
$$\leq 2 \operatorname{Re} \left[f\left(\bar{\lambda}a\right)\right] + r^{2} - |\lambda|^{2} + \left(|\lambda| - \sqrt{|\lambda|^{2} - r^{2}}\right)^{2}$$
$$= 2 \operatorname{Re} \left[f\left(\bar{\lambda}a\right)\right] + |\lambda|^{2} - 2 |\lambda| \sqrt{|\lambda|^{2} - r^{2}}.$$

Taking the supremum in (4.19) over $f \in D(1)$ and utilising Lumer's representation theorem, we get:

(4.20)
$$v^{2}(a) + \left(\left|\lambda\right| - \sqrt{\left|\lambda\right|^{2} - r^{2}}\right)^{2} \leq 2\left\langle\bar{\lambda}a, 1\right\rangle_{s} + \left|\lambda\right| \left(\left|\lambda\right| - 2\sqrt{\left|\lambda\right|^{2} - r^{2}}\right).$$

Since $r \neq 0$, then $|\lambda| - \sqrt{|\lambda|^2 - r^2} > 0$, giving

(4.21)
$$2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)v(a) \le v^2(a) + \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)^2.$$

Now, utilising (4.20) and (4.21), we deduce

$$v\left(a\right) \leq \frac{1}{\left|\lambda\right| - \sqrt{\left|\lambda\right|^{2} - r^{2}}} \left\langle \bar{\lambda}a, 1 \right\rangle_{s} + \frac{\left|\lambda\right| \left(\left|\lambda\right| - 2\sqrt{\left|\lambda\right|^{2} - r^{2}}\right)}{2\left(\left|\lambda\right| - \sqrt{\left|\lambda\right|^{2} - r^{2}}\right)},$$

which is clearly equivalent with the desired result (4.17).

Remark 6. If $M > m \ge 0$ and $a \in \overline{U}(m, M)$, then

$$v(a) \leq \frac{M+m}{\left(\sqrt{M}-\sqrt{m}\right)^2} \left[\langle a,1 \rangle_s + \frac{1}{2} \left(\frac{m+M}{2} - 2\sqrt{mM} \right) \right].$$

In particular, if $a \in \overline{U}(0, \delta)$ with $\delta > 0$, then we have the following reverse inequality as well

$$(0 \le) v(a) - \langle a, 1 \rangle_s \le \frac{1}{4}\delta.$$

References

- H.F. BOHNENBLUST and S. KARLIN, Geometrical properties of the unit sphere of Banach algebras, Ann. of Math., 62 (1955), 217-229.
- [2] F.F. BONSALL and J. DUNCAN, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, Cambridge University Press, 1971.
- [3] M.J. CRABB, Numerical range estimates for the norm of iterated operators, *Glasgow Math. J.*, 11 (1970), 85-87.
- S.S. DRAGOMIR, Semi-inner Products and Applications. Nova Science Publishers, Inc., Hauppauge, NY, 2004. x+222 pp.
- [5] G. LUMER, Semi-inner product spaces, Trans. Amer. Math. Soc., 100 (1961), 29-43.
- [6] B.W. GLICKFELD, On an inequality of Banach algebra geometry and semi-inner product space theory, *Illinois J. Math.*, 14 (1970), 76-81.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne, Victoria, 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au *URL*: http://www.staff.vu.edu.au/rgmia/dragomir/