Reversing the Arithmetic mean – Geometric mean inequality

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Abstract. In this paper, we discuss some inequalities which are obtained by adding a non-negative expression to one of the sides of the AM-GM inequality. In this way, we can reverse the AM-GM inequality.

1. Introduction

The AM-GM inequality is one of the best inequalities and it is very useful for proving inequalities. We will recall the AM-GM inequality as follows

Theorem 1. For all non-negative real numbers $a_1, a_2, ..., a_n$, we have

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$, **(1)** where n is a natural number greater than 1.

Equality holds if and only if $a_1 = a_2 = ... = a_n$.

Now, we will observe a question: " How can we reverse AM-GM inequality ?". The first thing that springs to mind is adding a non-negative expression to the right-hand-side of the AM-GM inequality. It is natural that the obtained inequality becomes equality if and only if AM-GM inequality becomes equality, that is, if we add an expression $M := f\left(a_1, a_2, ..., a_n\right)$ (M is a function of n variables $a_1, a_2, ..., a_n$) to the right-hand-side of AM-GM inequality, then M must satisfy the two following conditions simultaneously

i) $M := f\left(a_1, a_2, ..., a_n\right) \geq 0$ for non-negative numbers $a_1, a_2, ..., a_n$.

ii) $M := f\left(a_1, a_2, ..., a_n\right) = 0$ if and only if $a_1 = a_2 = ... = a_n$.

So, we can choose many expressions such as:

$M_1 = k_1 \cdot \sum_{1 \leq i < j \leq n} | a_i - a_j |$; $M_2 = k_2 \cdot \sum_{i=1}^{n} | a_i - a_{i+1} |$; $M_3 = k_3 \cdot \max_{1 \leq i < j \leq n} \left\{ | a_i - a_j | \right\}$ or

$M_4 = k_4 \cdot \left(\sum_{1 \leq i < j \leq n} \left(a_i - a_j \right)^2 \right)^{1/2}$, etc…

(with the condition $a_{n+1} \equiv a_1$).

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2. Two inequalities

Proposition 2. Given a natural number $n \geq 1$. Then, the smallest real number k such that for all non-negative numbers $a_1, a_2, ..., a_n$ we have the following inequality

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt[n]{a_1 a_2 \cdots a_n} + k \cdot \sum_{1 \leq i < j \leq n} |a_i - a_j| \quad , \tag{2}$$

is $\frac{1}{n}$.

Proof. Suppose that (2) is satisfied for all $a_1, a_2, ..., a_n \geq 0$. Try $a_1 = a_2 = ... = a_{n-1} > 0$ and $a_n = 0$, from (2) we deduce $k \geq \frac{1}{n}$. Now, we will prove that (2) is correct when $k = \frac{1}{n}$, i.e., $\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt[n]{a_1 a_2 \cdots a_n} + \frac{1}{n} \cdot \sum_{1 \leq i < j \leq n} |a_i - a_j|$ for all non-negative real numbers $a_1, a_2, ..., a_n$.

Without losing generality, we assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Thus, the inequality we need to prove is equivalent to

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt[n]{a_1 a_2 \cdots a_n} + \frac{1}{n} \cdot \Big((n-1)a_1 + (n-3)a_2 + (n-5)a_3 + \cdots +$$
$$+ \left(n - 2.\left[\frac{n}{2}\right] + 1\right) a_{\left[\frac{n}{2}\right]} - \left(n - 2.\left[\frac{n}{2}\right] - 1\right) a_{\left[\frac{n}{2}\right]+1} - ... - (n-1)a_n \Big)$$

or $\left(n - 2 \cdot \left[\frac{n}{2}\right]\right) a_{\left[\frac{n}{2}\right]+1} + \left(n - 2 \cdot \left[\frac{n}{2}\right] - 2\right) a_{\left[\frac{n}{2}\right]+2} + \cdots + n a_n \leq$

$$\leq \left(n - 2 \cdot \left[\frac{n}{2}\right]\right) a_{\left[\frac{n}{2}\right]} + \cdots + (n-2)a_1 + n \cdot \sqrt[n]{a_1 a_2 \cdots a_n}$$

where $[x]$ is the largest integer which is smaller than x. It is easy to see that the last inequality is correct since $a_1 \geq a_2 \geq \cdots \geq a_n$. The proof is completed.

Corollary 3. For all non-negative real numbers $a_1, a_2, ..., a_n$, we always have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt[n]{a_1 a_2 \cdots a_n} + \frac{\binom{n}{2}}{n} \cdot \max_{1 \leq i < j \leq n} \left\{ |a_i - a_j| \right\}, \tag{3}$$

where $\binom{n}{2} = \frac{n!}{2!(n-2)!}$.

Proof. For $1 \leq i, j \leq n$, we have $\binom{n}{2}$ terms $|a_i - a_j|$ and note that $|a_i - a_j|$ is not greater than $\max_{1 \leq i < j \leq n} \left\{ |a_i - a_j| \right\}$, (3) follows immediately from (2).

Corollary 4. $$\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt[n]{a_1a_2\cdots a_n} + \left(\frac{n-1}{n}\right)\cdot\sum_{i=1}^{n-1}|a_i-a_{i+1}| \quad , \qquad \textbf{(4)}$$
for all real numbers $a_1, a_2, ..., a_n \ge 0$.

Proof. Using the inequalities $|a_i-a_j| \le |a_i-a_{i+1}|+|a_{i+1}-a_{i+2}|+\cdots+|a_{j-1}-a_j|$ for all pairs (i,j) satisfying $1 \le i, j \le n$, we can easily deduce that

$\sum_{1\le i<j\le n}|a_i-a_j| \le (n-1)\cdot\sum_{i=1}^{n-1}|a_i-a_{i+1}|$. Thus, from (2) we deduce (4).

Proposition 5. Given a natural number n greater than 1, the smallest real number k such that for all non-negative numbers $a_1, a_2, ..., a_n$ we have the inequality

$$\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt[n]{a_1a_2\cdots a_n} + k\cdot\max_{1\le i<j\le n}\{|a_i-a_j|\} \quad , \qquad \textbf{(5)}$$

is $\frac{n-1}{n}$.

Proof. Since (5) is satisfied for all real numbers $a_1, a_2, ..., a_n \ge 0$ trying $a_1 = a_2 = \cdots = a_{n-1} > 0$ and $a_n = 0$, from (5) we deduce $k \ge \frac{n-1}{n}$.

We will prove that (5) is correct when $k = \frac{n-1}{n}$, that is,

$\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt[n]{a_1a_2\cdots a_n} + \left(\frac{n-1}{n}\right)\cdot\max_{1\le i<j\le n}\{|a_i-a_j|\}$ for all non-negative real numbers $a_1, a_2, ..., a_n$.

Without losing generality, we assume that $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. Then,

$\max_{1\le i<j\le n}\{|a_i-a_j|\} = a_1 - a_n$. Hence, the inequality which needs be proved is equivalent to

$a_1+a_2+\cdots+a_n \le n\cdot\sqrt[n]{a_1a_2\cdots a_n} + (n-1)(a_1-a_n)$

or $\quad a_2+a_3+\cdots+a_{n-1}+na_n \le (n-2)a_1 + n\cdot\sqrt[n]{a_1a_2\cdots a_n}$ since $a_1 \ge a_2 \ge \cdots \ge a_n$,

We have $a_1+a_2+\cdots+a_{n-1} \le (n-2)a_1$ and $a_n \le \sqrt[n]{a_1a_2\cdots a_n}$.

Therefore, $a_2+a_3+\cdots+a_{n-1}+na_n \le n\cdot\sqrt[n]{a_1a_2\cdots a_n} + (n-2)a_1$. This completes the proof.

Corollary 6. For all non-negative real numbers $a_1, a_2, ..., a_n$ we have

$$\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt[n]{a_1a_2\cdots a_n} + \left(\frac{n-1}{n}\right)\cdot\sum_{i=1}^{n}|a_i-a_{i+1}|, \quad \textbf{(6)}$$

with the condition $a_{n+1} \equiv a_1$.

Proof. Suppose that $\max_{1\le i,j\le n}\{|a_i-a_j|\} = a_1 - a_n$.

Using the inequality $|a_1-a_n| \le |a_1-a_2|+|a_2-a_3|+\cdots+|a_{n-1}-a_n|$, from (5), we get

$$\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt[n]{a_1a_2\cdots a_n} + \left(\frac{n-1}{n}\right)\cdot\sum_{i=1}^{n-1}|a_i-a_{i+1}|.$$

On the other hand, we have $\frac{a_1+a_2+\cdots+a_n}{n} \le \sqrt[n]{a_1a_2\cdots a_n} + \left(\frac{n-1}{n}\right)\cdot|a_1-a_n|$.

Adding two above inequalities side by side, we get

$$2\cdot\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\le 2\cdot\sqrt[n]{a_1a_2\cdots a_n}+\left(\frac{n-1}{n}\right)\cdot\sum_{i=1}^{n}|a_i-a_{i+1}|$$

or $$\frac{a_1+a_2+\cdots+a_n}{n}\le\sqrt[n]{a_1a_2\cdots a_n}+\left(\frac{n-1}{2n}\right)\cdot\sum_{i=1}^{n}|a_i-a_{i+1}| \quad .$$

Corollary 7. For all non-negative real numbers $a_1,a_2,...,a_n$ we have

$$\frac{a_1+a_2+\cdots+a_n}{n}\le\sqrt[n]{a_1a_2\cdots a_n}+\left(\frac{n-1}{2\sqrt{n}}\right)\cdot\left(\sum_{i=1}^{n}\left(a_i-a_{i+1}\right)^2\right)^{1/2} \quad , \qquad \textbf{(7)}$$

with the condition $a_{n+1}\equiv a_1$.

Proof. By using the Cauchy–Schwarz inequality $\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\sum_{i=1}^{n}a_i^2\cdot\sum_{i=1}^{n}b_i^2$,we have $\sum_{i=1}^{n}|a_i-a_{i+1}|\le\sqrt{n}\cdot\left(\sum_{i=1}^{n}\left(a_i-a_{i+1}\right)^2\right)^{1/2}$. So, from this and (6) we immediately have (7).

3. Another proof of (2) in the case $n=3$

In this section, we will show a new proof of (2) for the case $n=3$.

Lemma 8. For all non-negative real numbers a,b, we have

$$\left(\frac{a+b}{2}\right)^3+\frac{3}{8}\cdot|a^3-b^3|\ \ge\frac{a^3+b^3}{2}. \qquad \textbf{(8)}$$

Proof. Without losing generality, we assume that $a\ge b\ge 0$. Then, (8) is equivalent to the following inequality $\left(\frac{a+b}{2}\right)^3+\frac{3}{8}\cdot(a^3-b^3)\ge\frac{a^3+b^3}{2}$ or $3ab\left(a+b\right)\ge 6b^3$. This is correct because $a\ge b\ge 0$. The lemma is proved.

Problem 9. For all non-negative real numbers a_1,a_2,a_3 we have

$$\frac{a_1+a_2+a_3}{3}\le\sqrt[3]{a_1a_2a_3}+\frac{1}{3}\cdot\left(|a_1-a_2|+|a_2-a_3|+|a_3-a_1|\right) . \qquad \textbf{(9)}.$$

Proof. We introduce the following notations: $b_1^3=a_1,\ b_2^3=a_2,\ b_3^3=a_3$, then $b_1,b_2,b_3\ge 0$ and (9) becomes $\frac{b_1^3+b_2^3+b_3^3}{3}\le b_1b_2b_3+\frac{1}{3}\cdot\left(|b_1^3-b_2^3|+|b_2^3-b_3^3|+|b_3^3-b_1^3|\right)$.

Using the lemma for three pairs of numbers $\left(b_1,b_2\right),\left(b_2,b_3\right)$ and $\left(b_3,b_1\right)$, we have

$$\left(\frac{b_1+b_2}{2}\right)^3+\frac{3}{8}\cdot|b_1^3-b_2^3|\ \ge\frac{b_1^3+b_2^3}{2},$$

$$\left(\frac{b_2+b_3}{2}\right)^3+\frac{3}{8}\cdot|b_2^3-b_3^3|\ge\frac{b_2^3+b_3^3}{2},$$

$$\left(\frac{b_3+b_1}{2}\right)^3+\frac{3}{8}\cdot|b_3^3-b_1^3|\geq\frac{b_3^3+b_1^3}{2},$$

Adding above inequalities side by side, we get

$$b_1^3+b_2^3+b_3^3\leq\left(\frac{b_1+b_2}{2}\right)^3+\left(\frac{b_2+b_3}{2}\right)^3+\left(\frac{b_3+b_1}{2}\right)^3+\frac{3}{8}\cdot\left(|b_1^3-b_2^3|+|b_2^3-b_3^3|+|b_3^3-b_1^3|\right)$$

or $2\cdot\sum_{i=1}^{3}b_i^3\leq\sum_{i=1}^{3}b_ib_{i+1}\cdot(b_i+b_{i+1})+\sum_{i=1}^{3}|b_i-b_{i+1}|$ (with the condition $b_{n+1}\equiv b_1$ when $n=3$)

Thus, $\sum_{i=1}^{3}b_i^3+\sum_{i=1}^{3}b_i^3+3b_1b_2b_3\leq 3b_1b_2b_3+\sum_{i=1}^{3}|b_i-b_{i+1}|+\sum_{i=1}^{3}b_ib_{i+1}(b_i+b_{i+1})$ (*)

According to Schur's inequality, we have $\sum_{i=1}^{3}b_i^3+3b_1b_2b_3\geq\sum_{i=1}^{3}b_ib_{i+1}(b_i+b_{i+1})$

Hence, from (*) we deduce $\sum_{i=1}^{3}b_i^3\leq 3b_1b_2b_3+\sum_{i=1}^{3}|b_i^3-b_{i+1}^3|$

$$\text{or }\frac{1}{3}\cdot\sum_{i=1}^{3}b_i^3\leq b_1b_2b_3+\frac{1}{3}\cdot\sum_{i=1}^{3}|b_i^3-b_{i+1}^3|.$$

4. An open problem.

The following inequality is a problem of USA-TST (2000)

Problem 10. Prove that $\frac{a+b+c}{3}-\sqrt[3]{abc}\leq\max\left\{\left(\sqrt{a}-\sqrt{b}\right)^2,\left(\sqrt{b}-\sqrt{c}\right)^2,\left(\sqrt{c}-\sqrt{a}\right)^2\right\}$ **(10)**

where a,b,c are non-negative real numbers.

The inequality (10) is a beautiful inequality and there are some ways to prove it (eg: see[2]). A natural question arises, can the power 2 of (10) be replaced by another power? Now, we will observe the following problem.

Lemma 11. Let $f(x)=x-np\sqrt[n]{x}+q$, where $p>0,x\in[a,b],n\in N,n\geq 2$ then

$$f(x)\leq\max\{f(a),f(b)\}.$$

Proof. The lemma is proved easily by using the property of convex function. It is left as an exercise for the reader.

Problem 12. Prove that for all $a_1,a_2,...,a_n\geq 0$ we have the following inequality

$$\frac{a_1+a_2+...+a_n}{n}\leq\sqrt[n]{a_1a_2...a_n}+\left(\frac{n-1}{n}\right)\max_{1\leq i<j\leq n}\left\{\left(\sqrt{a_i}-\sqrt{a_j}\right)^2\right\}\qquad\textbf{(11)}$$

where $n\geq 2,n\in N$.

Proof. Without losing generality, we assume that $a_1\geq a_2\geq...\geq a_n$. So

$$\max_{1\leq i<j\leq n}\left\{\left(\sqrt{a_i}-\sqrt{a_j}\right)^2\right\}=\left(\sqrt{a_1}-\sqrt{a_n}\right)^2=a_1-2\sqrt{a_1a_n}+a_n.$$

Thus (11) is equivalent to the following inequality

$$a_1+a_2+...+a_n\leq n.\sqrt[n]{a_1a_2...a_n}+(n-1)\left(a_1-2\sqrt{a_1a_n}+a_n\right)$$

or $a_2+a_3+...+a_{n-1}-n.\sqrt[n]{a_1a_2...a_{n-1}a_n}-(n-2)a_1-(n-2)a_n+2(n-1)\sqrt{a_1a_n}\leq 0.$ Put

$F(a_1,a_2,...,a_n)$

$:=a_2+a_3+...+a_{n-1}-n.\sqrt[n]{a_1a_2...a_{n-1}a_n}-(n-2)a_1-(n-2)a_n+2(n-1)\sqrt{a_1a_n}.$

It is easy to see that the form of $F(a_1,a_2,...,a_n)$ is the same as the form of $f(x)$ of lemma 11. Note that $a_i \in [a_n,a_1], \forall i=\overline{2,n-1}$, so using the lemma 11 for $a_i, i\in\{2,3,...,n-1\}$ we have $F(a_1,a_2,...,a_n) \le \max\{F(a_1,a_2,...,a_{i-1},a_1,a_{i+1},...,a_n), F(a_1,a_2,...,a_{i-1},a_n,a_{i+1},...,a_n)\}$.
Using the above property for $i=2,3,...,n-1$, we get
$F(a_1,a_2,...,a_n) \le \max\limits_{t_i\in\{a_1,a_n\}} F(t_1,t_2,...,t_n)$ where $t_i \in\{a_1,a_n\}, \forall i=1,2....,n.$
Suppose that there exist m numbers t_i that equal a_1, $(0\le m\le n)$ so
$F(t_1,t_2,...,t_n) = (m-n+2)a_1 - ma_n - n.\sqrt[n]{a_1^{m+1}a_n^{n-m-1}} + 2(n-1)\sqrt{a_1a_n}$.

We will prove that $F(t_1,t_2,...,t_n)\le 0$. Indeed, if $\dfrac{n-m-1}{n} > \dfrac{n}{2}$, we put

$$G(a_1) = F(t_1,t_2,...,t_n) = (m-n+2)a_1 - ma_n - n.\sqrt[n]{a_1^{m+1}a_n^{n-m-1}} + 2(n-1)\sqrt{a_1a_n},$$

$$G'(a_1) = m-n+2-(m+1).\sqrt[n]{\frac{a_n^{n-m-1}}{a_1^{n-m-1}}} + (n-1)\sqrt{\frac{a_n}{a_1}}$$

$$= (m+1)\left(\sqrt{\frac{a_n}{a_1}} - \sqrt[n]{\frac{a_n^{n-m-1}}{a_1^{n-m-1}}}\right) + (n-m-2)\left(\sqrt{\frac{a_n}{a_1}}-1\right)$$

Since $\dfrac{a_n}{a_1}\le 1$, $\sqrt{\dfrac{a_n}{a_1}} \le \sqrt[n]{\dfrac{a_n^{n-m-1}}{a_1^{n-m-1}}}$ and $\sqrt{\dfrac{a_n}{a_1}}\le 1$. Hence $G'(a_1)\le 0$ .From this, we deduce that $G(a_1)\le G(a_n)=0.$ If $\dfrac{n-m-1}{n}\le\dfrac{1}{2}$, we put

$$H(a_n) = F(t_1,t_2,...,t_n) = (m-n+2)a_1 - ma_n - n.\sqrt[n]{a_1^{m+1}a_n^{n-m-1}} + 2(n-1)\sqrt{a_1a_n}$$

Similarly, we also deduce that $H(a_n)\le 0$.
The proof is completed.

Problem 13. Find all real numbers k such that we have the inequality

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \le \max\left\{|\sqrt{a}-\sqrt{b}|^k, |\sqrt{b}-\sqrt{c}|^k, |\sqrt{c}-\sqrt{a}|^k\right\} \quad \textbf{(12)}$$

for all non-negative real numbers a,b,c.

Solution. The answer for this is $k=2$. Indeed, try $b=c=0$ and $a>1$, from (11) we deduce that $\dfrac{a}{3}\le a^{k/2}$. Hence, $1-\dfrac{k}{2}\le\dfrac{\ln 3}{\ln a}$ for every number $a>1$. Let $a\to\infty$, we get $k\ge 2$.
When $k=2$, (10) is correct (problem 10). We will prove that (11) is incorrect with $k>2$.
When $k>2$, try $a=\dfrac{4}{n^2}$, $b=c=\dfrac{1}{n^2}$ (where n is positive real number), we get $\dfrac{2-\sqrt[3]{4}}{n^2}\le\max\left\{\dfrac{1}{n^k},0,\dfrac{1}{n^k}\right\}=\dfrac{1}{n^k}$. Hence, $n^{k-2}\le\dfrac{1}{2-\sqrt[3]{4}}$. This is not true as n tends to infinity.
Therefore, there is only real number $k\,(k=2)$ satisfying (11).

It is clear that when $k=2$, the degree of left-hand-side of (11). Almost classical inequalities are homogeneous inequalities of the first degree.

We conclude this paper with an open problem: for all non-negative real numbers $a_1, a_2, ..., a_n$ (where n is a natural greater than 1), suppose that there exists a function $f := f\left(a_1, a_2, ..., a_n\right)$ satisfying the following three conditions simultaneously

i) $f\left(a_1, a_2, ..., a_n\right) \geq 0$ for all non-negative real numbers $a_1, a_2, ..., a_n$.

ii) $f\left(a_1, a_2, ..., a_n\right) = 0$ if and only if $a_1 = a_2 = ... = a_n$.

iii) $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt[n]{a_1 a_2 ... a_n} + f\left(a_1, a_2, ..., a_n\right)$ for all real numbers $a_1, a_2, ..., a_n \geq 0$.

Prove or disprove that $f\left(a_1, a_2, ..., a_n\right)$ is a homogeneous function of the first degree.

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