MONOTONICITY PROPERTIES OF THE FUNCTIONS INVOLVING THE PSI FUNCTION

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Abstract. The monotonicity properties of the functions involving the psi function are considered. An inequality for the Euler’s constant is obtained as consequence.

A function \( f \) is said to be completely monotonic on an interval \( I \), if \( f \) has derivatives of all orders on \( I \) and satisfies

\[
(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } x \in I \quad \text{and } n = 0, 1, 2, \ldots.
\]

If the inequality (1) is strict, then \( f \) is said to be strictly completely monotonic on \( I \). It is known (Bernstein’s Theorem) that \( f \) is completely monotonic on \((0, \infty)\) if and only if

\[
f(x) = \int_{0}^{\infty} e^{-xt} \, d\mu(t),
\]

where \( \mu \) is a nonnegative measure on \([0, \infty)\) such that the integral converges for all \( x > 0 \), see [7, p.161]. A detailed collection of the most important properties of completely monotonic functions can be found in [7, Chapter IV].

The classical gamma function

\[
\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt \quad (x > 0)
\]

is one of the most important functions in analysis and its applications. The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [1, pp. 259-260] as

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_{0}^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt,
\]

\[
\psi^{(n)}(x) = (-1)^{n+1} \int_{0}^{\infty} \frac{t^n}{1 - e^{-t}} e^{-xt} \, dt
\]

for \( x > 0 \) and \( n \in \mathbb{N} \), where \( \gamma = 0.57721566490153286 \ldots \) is the Euler-Mascheroni constant defined by

\[
\gamma = \lim_{n \to \infty} D_n, \quad \text{where } D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.
\]

2000 Mathematics Subject Classification. 33B15; 26A48.

Key words and phrases. Completely monotonic function; Bernstein function; Psi function; Euler’s constant; Inequality.

This work was supported by Natural Scientific Research Guidance Planning Project of Education Department of Henan Province (2007110011), by Natural Scientific Research Subsidy Planning Project of Education Department of Henan Province (2008A110007).
\[ \frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}, \quad n = 1, 2, \ldots. \] (4)
In [2, 3, 4, 5], other bounds for \( D_n - \gamma \) were established. Since
\[ \psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \]
the inequality (4) can be written as
\[ \frac{1}{2(n+1)} < \psi(n + 1) - \log n < \frac{1}{2n}, \quad n = 1, 2, \ldots. \] (5)
Define for \( x > 0 \),
\[ f(x) = (x + 1) [\psi(x + 1) - \log x] \quad \text{and} \quad g(x) = x [\psi(x + 1) - \log x]. \]
In this paper, we prove that \( f \) is strictly completely monotonic on \((0, \infty)\), and \( g \) is a so-called Bernstein function on \((0, \infty)\). That is, \( g > 0 \) and \( g' \) is strictly completely monotonic on \((0, \infty)\).
From the representations [1, pp. 258-259]
\[ \psi(x) = \log x - \frac{1}{2x} + O(x^{-2}), \]
\[ \psi(x + 1) = \psi(x) + \frac{1}{x}, \]
we conclude
\[ \lim_{x \to \infty} f(x) = \frac{1}{2} \quad \text{and} \quad \lim_{x \to \infty} g(x) = \frac{1}{2}, \]
the inequalities (4) are immediate consequences of the fact that \( f \) is strictly decreasing on \((0, \infty)\) and \( g \) is strictly increasing on \((0, \infty)\).

**Theorem 1.** The function \( f(x) = (x + 1) [\psi(x + 1) - \log x] \) is strictly completely monotonic on \((0, \infty)\), and \( g(x) = x [\psi(x + 1) - \log x] \) is a Bernstein function on \((0, \infty)\).

**Proof.** Using the representations [1, p. 259]
\[ \psi(x) = \int_{0}^{\infty} \left( \frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt \] (6)
and
\[ \log x = \int_{0}^{\infty} \frac{e^{-t} - e^{-xt}}{t} dt, \] (7)
we imply
\[ f(x) = (x + 1) \int_{0}^{\infty} \delta(t) e^{-(x+1)t} dt, \]
where
\[ \delta(t) = \frac{1}{t} - \frac{1}{e^{t} - 1}, t > 0. \]
Easy computations reveal that the function \( \delta \) is strictly decreasing on \((0, \infty)\) with \( \lim_{x \to 0} \delta(t) = \frac{1}{2} \) and \( \lim_{x \to \infty} \delta(t) = 0 \).
For $x > 0$, $n = 0, 1, 2, \ldots$, we have

$$(-1)^n f^{(n)}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (x + 1)^{(n-k)} \left( \int_0^\infty \delta(t)e^{-(x+1)t}dt \right)^{(k)}$$

$$= (x + 1) \int_0^\infty \delta(t)e^{-(x+1)t}t^n dt - n \int_0^\infty \delta(t)e^{-(x+1)t}t^{n-1}dt$$

$$= \int_0^{n/(x+1)} \delta(t)e^{-(x+1)t}t^{n-1}([x + 1)t - n]dt$$

$$+ \int_{n/(x+1)}^\infty \delta(t)e^{-(x+1)t}t^{n-1}([x + 1)t - n]dt$$

$$> \delta(n/(x+1)) \int_0^{n/(x+1)} e^{-(x+1)t}t^{n-1}([x + 1)t - n]dt$$

$$+ \delta(n/(x+1)) \int_{n/(x+1)}^\infty e^{-(x+1)t}t^{n-1}([x + 1)t - n]dt$$

$$= \delta(n/(x+1)) \int_0^\infty e^{-(x+1)t}t^{n-1}([x + 1)t - n]dt. \quad (8)$$

Since

$$\frac{m!}{(x+s)^{m+1}} = \int_0^\infty t^m e^{-(x+s)t} dt \quad (x > 0; s \geq 0, m = 0, 1, 2, \ldots), \quad (9)$$

we conclude

$$\int_0^\infty e^{-(x+1)t}t^{n-1}([x + 1)t - n] dt = 0,$$

so that (8) implies

$$(-1)^n f^{(n)}(x) > 0 \quad (x > 0, n = 0, 1, 2, \ldots).$$

Hence, the function $f$ is strictly completely monotonic on $(0, \infty)$.

Using the representations (6) and (7), we imply

$$g(x) = x \int_0^\infty \omega(t)e^{-xt}dt, \quad (10)$$

where

$$\omega(t) = \frac{e^t}{t} - \frac{1}{e^t - 1} - 1, t > 0. \quad (11)$$

Easy computations reveal that

$$\frac{t^2(e^t - 1)^2}{e^t} \omega'(t) = \sum_{n=4}^\infty \left[ (n-2)2^{n-1} - 2(n-1) \right] \frac{t^n}{n!} > 0, t > 0, \quad (12)$$
hence, the function $\omega$ is strictly increasing on $(0, \infty)$, and $\omega(t) > \lim_{t \to 0} \omega(t) = \frac{1}{2}$, and then $g(x) > 0$. For $x > 0$, $n = 1, 2, \ldots$, we have

$$
(-1)^n g^{(n)}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left( \int_0^\infty \omega(t)e^{-xt}dt \right)^{(k)}
$$

$$
= x \int_0^\infty \omega(t)e^{-xt}t^n dt - n \int_0^\infty \omega(t)e^{-xt}t^{n-1} dt
$$

$$
= \int_0^{n/x} \omega(t)e^{-xt}t^{n-1}(xt - n) dt
$$

$$
+ \int_{n/x}^\infty \omega(t)e^{-xt}t^{n-1}(xt - n) dt
$$

$$
< \omega(n/x) \int_0^{n/x} e^{-xt}t^{n-1}(xt - n) dt
$$

$$
+ \omega(n/x) \int_{n/x}^\infty e^{-xt}t^{n-1}(xt - n) dt
$$

$$
= \omega(n/x) \int_{n/x}^\infty e^{-xt}t^{n-1}(xt - n) dt = 0.
$$

Hence, $g$ is a Bernstein function on $(0, \infty)$. The proof of Theorem 1 is complete. □

REFERENCES


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