# Various proofs of the Cauchy-Schwarz inequality

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**Abstract:** In this paper twelve different proofs are given for the classical Cauchy-Schwarz inequality.
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## 1 Introduction

The Cauchy-Schwarz inequality is an elementary inequality and at the same time a powerful inequality, which can be stated as follows:

**Theorem.** *Let* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *be two sequences of real numbers, then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2, \tag{1}$$

*with equality if and only if the sequences* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *are proportional, i.e., there is a constant* $\lambda$ *such that* $a_k = \lambda b_k$ *for each* $k \in \{1, 2, \ldots, n\}$.

As is known to us, this classical inequality plays an important role in different branches of modern mathematics including Hilbert spaces theory, probability and statistics, classical real and complex analysis, numerical analysis, qualitative theory of differential equations and their applications (see [1-12]). In this paper we show some different proofs of the Cauchy-Schwarz inequality.

## 2 Some different proofs of the Cauchy-Schwarz inequality

**Proof 1.** Expanding out the brackets and collecting together identical terms we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_j^2 - 2\sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} b_j a_j$$

$$= 2\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - 2\left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Because the left-hand side of the equation is a sum of the squares of real numbers it is greater than or equal to zero, thus

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

**Proof 2.** Consider the following quadratic polynomial

$$f\left(x\right)=\left(\sum_{i=1}^{n}a_i^2\right)x^2-2\left(\sum_{i=1}^{n}a_ib_i\right)x+\sum_{i=1}^{n}b_i^2=\sum_{i=1}^{n}\left(a_ix-b_i\right)^2.$$

Since $f(x)\geq 0$ for any $x\in\mathbb{R}$, it follows that the discriminant of $f(x)$ is negative, i.e.,

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2-\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)\leq 0.$$

The inequality (1) is proved.

**Proof 3.** When $\sum\limits_{i=1}^{n}a_i^2=0$ or $\sum\limits_{i=1}^{n}b_i^2=0$, (1) is an identity.
We can now assume that

$$A_n=\sum_{i=1}^{n}a_i^2\neq 0,\quad B_n=\sum_{i=1}^{n}b_i^2\neq 0,\quad x_i=\frac{a_i}{\sqrt{A_n}},\quad y_i=\frac{b_i}{\sqrt{B_n}}\quad(i=1,2,\ldots,n),$$

then

$$\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}y_i^2=1.$$

The inequality (1) is equivalent to

$$x_1y_1+x_2y_2+\cdots x_ny_n\leq 1,$$

that is

$$2(x_1y_1+x_2y_2+\cdots x_ny_n)\leq x_1^2+x_2^2+\cdots+x_n^2+y_1^2+y_2^2+\cdots+y_n^2,$$

or equivalently

$$\left(x_1-y_1\right)^2+\left(x_2-y_2\right)^2+\cdots+\left(x_n-y_n\right)^2\geq 0,$$

which is evidently true. The desired conclusion follows.

**Proof 4.** Let $A=\sqrt{a_1^2+a_2^2+\cdots+a_n^2},\quad B=\sqrt{b_1^2+b_2^2+\cdots+b_n^2}$.
By the arithmetic-geometric means inequality, we have

$$\sum_{i=1}^{n}\frac{a_ib_i}{AB}\leq\sum_{i=1}^{n}\frac{1}{2}\left(\frac{a_i^2}{A^2}+\frac{b_i^2}{B^2}\right)=1,$$

so that

$$\sum_{i=1}^{n}a_ib_i\leq AB=\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\sqrt{b_1^2+b_2^2+\cdots+b_n^2}.$$

Thus

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\leq\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right).$$

**Proof 5.** Let $A_n=a_1^2+a_2^2+\cdots+a_n^2,\ B_n=a_1b_1+a_2b_2+\cdots+a_nb_n,\ C_n=b_1^2+b_2^2+\cdots+b_n^2$.
It follows from the arithmetic-geometric means inequality that

$$\frac{A_nC_n}{B_n^2}+1=\sum_{i=1}^{n}\frac{a_i^2C_n}{B_n^2}+\sum_{i=1}^{n}\frac{b_i^2}{C_n}=\sum_{i=1}^{n}\left(\frac{a_i^2C_n}{B_n^2}+\frac{b_i^2}{C_n}\right)\geq 2\sum_{i=1}^{n}\frac{a_ib_i}{B_n}=2,$$

therefore

$$A_n C_n \geq B_n^2,$$

that is

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2.$$

**Proof 6.** Below, we prove the Cauchy-Schwarz inequality by mathematical induction.
Beginning the induction at 1, the $n = 1$ case is trivial.
Note that

$$(a_1 b_1 + a_2 b_2)^2 = a_1^2 b_1^2 + 2a_1 b_1 a_2 b_2 + a_2^2 b_2^2 \leq a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_2^2 = (a_1^2 + a_2^2)(b_1^2 + b_2^2),$$

which implies that the inequality (1) holds for $n = 2$.
Assume that the inequality (1) holds for an arbitrary integer $k$, i.e.,

$$\left(\sum_{i=1}^{k} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right).$$

Using the induction hypothesis, one has

$$\begin{aligned}\sqrt{\sum_{i=1}^{k+1} a_i^2} \cdot \sqrt{\sum_{i=1}^{k+1} b_i^2} &= \sqrt{\sum_{i=1}^{k} a_i^2 + a_{k+1}^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2 + b_{k+1}^2} \\ &\geq \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2} + |a_{k+1} b_{k+1}| \\ &\geq \sum_{i=1}^{k} |a_i b_i| + |a_{k+1} b_{k+1}| = \sum_{i=1}^{k+1} |a_i b_i|.\end{aligned}$$

It means that the inequality (1) holds for $n = k + 1$, we thus conclude that the inequality (1) holds for all natural numbers $n$. This completes the proof of inequality (1).

**Proof 7.** Let

$$\begin{aligned} A &= \{a_1 b_1, \cdots a_1 b_n, a_2 b_1, \cdots, a_2 b_n, \cdots, a_n b_1, \cdots a_n b_n\} \\ B &= \{a_1 b_1, \cdots a_1 b_n, a_2 b_1, \cdots, a_2 b_n, \cdots, a_n b_1, \cdots a_n b_n\} \end{aligned}$$

$$\begin{aligned} C &= \{a_1 b_1, \cdots a_1 b_n, a_2 b_1, \cdots, a_2 b_n, \cdots, a_n b_1, \cdots a_n b_n\} \\ D &= \{a_1 b_1, \cdots a_n b_1, a_1 b_2, \cdots, a_n b_2, \cdots, a_1 b_n, \cdots a_n b_n\} \end{aligned}$$

It is easy to observe that the set $A$ and $B$ are similarly sorted, while the set $C$ and $D$ are mixed sorted.
Applying the rearrangement inequality, we have

$$(a_1 b_1)(a_1 b_1) + \cdots + (a_1 b_n)(a_1 b_n) + (a_2 b_1)(a_2 b_1) + \cdots + (a_2 b_n)(a_2 b_n) + \cdots + (a_n b_1)(a_n b_1) + \cdots + (a_n b_n)(a_n b_n)$$

$$\geq (a_1 b_1)(a_1 b_1) + \cdots + (a_1 b_n)(a_n b_1) + (a_2 b_1)(a_1 b_2) + \cdots + (a_2 b_n)(a_n b_2) + \cdots + (a_n b_1)(a_1 b_n) + \cdots + (a_n b_n)(a_n b_n),$$

which can be simplified to the inequality

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

as desired.

**Proof 8.** By the arithmetic-geometric means inequality, one has for $\lambda > 0$,

$$|a_i b_i| \le \frac{1}{2}\left(\lambda a_i^2 + \frac{b_i^2}{\lambda}\right).$$

Choosing $\lambda = \sqrt{\sum\limits_{i=1}^{n} b_i^2 \Big/ \sum\limits_{i=1}^{n} a_i^2}$ in the above inequality gives

$$|a_i b_i| \le \left[\sqrt{\frac{\sum\limits_{i=1}^{n} b_i^2}{\sum\limits_{i=1}^{n} a_i^2}}\, a_i^2 + \sqrt{\frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} b_i^2}}\, b_i^2\right].$$

Hence

$$\sum_{i=1}^{n} |a_i b_i| \le \frac{1}{2}\left[\sqrt{\frac{\sum\limits_{i=1}^{n} b_i^2}{\sum\limits_{i=1}^{n} a_i^2}} \sum_{i=1}^{n} a_i^2 + \sqrt{\frac{\sum\limits_{i=1}^{n} a_i^2}{\sum\limits_{i=1}^{n} b_i^2}} \sum_{i=1}^{n} b_i^2\right],$$

or equivalently

$$\sum_{i=1}^{n} |a_i b_i| \le \frac{1}{2}\left(\sqrt{\sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}\right) = \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}.$$

The desired conclusion follows.

**Proof 9.** Construct the vectors $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$. Then for arbitrary real numbers $t$, one has the following identities for scalar product:

$$(\alpha + t\beta)\cdot(\alpha + t\beta) = \alpha\cdot\alpha + 2(\alpha\cdot\beta)\,t + (\beta\cdot\beta)t^2$$

$$\Longleftrightarrow |\alpha|^2 + 2(\alpha\cdot\beta)\,t + |\beta|^2\,t^2 = |\alpha + t\beta|^2 \ge 0.$$

Thus

$$(\alpha\cdot\beta)^2 - |\alpha|^2\,|\beta|^2 \le 0.$$

Using the expressions

$$\alpha\cdot\beta = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \quad |\alpha|^2 = \sum_{i=1}^{n} a_i^2, \quad |\beta|^2 = \sum_{i=1}^{n} b_i^2,$$

we obtain

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 - \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \le 0.$$

**Proof 10.** Construct the vectors $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$.
From the formula for scalar product:

$$\alpha\cdot\beta = |\alpha|\,|\beta|\cos(\alpha, \beta),$$

we deduce that

$$\alpha\cdot\beta \le |\alpha|\,|\beta|.$$

Using the expressions

$$\alpha \cdot \beta = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n, \quad |\alpha|^2 = \sum_{i=1}^{n} a_i^2, \quad |\beta|^2 = \sum_{i=1}^{n} b_i^2,$$

we get the desired inequality (1).

**Proof 11.** Since the function $f(x) = x^2$ is convex on $(-\infty, +\infty)$, it follows from the Jensen's inequality that

$$(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)^2 \le p_1 x_1^2 + p_2 x_2^2 + \cdots + p_n x_n^2, \tag{2}$$

where $x_i \in \mathbb{R}$, $p_i > 0$ $(i = 1, 2, \ldots, n)$, $p_1 + p_2 + \cdots + p_n = 1$.

**Case I.** If $b_i \ne 0$ for $i = 1, 2, \ldots, n$, we apply $x_i = a_i / b_i$ and $p_i = b_i^2/(b_1^2 + b_2^2 + \cdots + b_n^2)$ to the inequality (2) to obtain that

$$\left( \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{b_1^2 + b_2^2 + \cdots + b_n^2} \right)^2 \le \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{b_1^2 + b_2^2 + \cdots + b_n^2},$$

that is

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

**Case II.** If there exists $b_{i_1} = b_{i_2} = \cdots = b_{i_k} = 0$, one has

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 = \left( \sum_{i \ne i_1, \ldots, i_k, 1 \le i \le n} a_i b_i \right)^2$$

$$\le \left( \sum_{i \ne i_1, \ldots, i_k, 1 \le i \le n} a_i^2 \right) \left( \sum_{i \ne i_1, \ldots, i_k, 1 \le i \le n} b_i^2 \right) \le \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right).$$

This completes the proof of inequality (1).

**Proof 12.** Define a sequence $\{S_n\}$ by

$$S_n = (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 - \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

Then

$$S_{n+1} - S_n = (a_1 b_1 + a_2 b_2 + \cdots + a_{n+1} b_{n+1})^2 - \left(a_1^2 + a_2^2 + \cdots + a_{n+1}^2\right)\left(b_1^2 + b_2^2 + \cdots + b_{n+1}^2\right)$$
$$-(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 + \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right),$$

which can be simplified to

$$S_{n+1} - S_n = -\left[ (a_1 b_{n+1} - b_1 a_{n+1})^2 + (a_2 b_{n+1} - b_2 a_{n+1})^2 + \cdots + (a_n b_{n+1} - b_n a_{n+1})^2 \right],$$

so

$$S_{n+1} \le S_n \quad (n \in \mathbb{N}).$$

We thus have

$$S_n \le S_{n-1} \le \cdots \le S_1 = 0,$$

which implies the inequality (1).

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