INEQUALITIES FOR THE GAMMA FUNCTION

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ABSTRACT. A complete monotonicity property and some inequalities for the gamma function are given. These results refine the classical Stirling approximation and its many recent improvements.

1. INTRODUCTION

The gamma Γ and psi ψ (or digamma) functions are defined by

$$\Gamma(z) = \int_0^\infty u^{z-1}e^{-u}du \ \ , \ \ \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

for all complex numbers z with $\mathbf{Re}z > 0$, respectively. In this paper we restrict z to a positive real number x. The gamma function is a natural extension of the factorial from integers n to real (and complex) numbers z. It was first defined and studied by Leonard Euler(1707-1783), and it is of fundamental importance to many areas of science, like probability theory, mathematical physics, number theory and special functions. It also appears in the study of many important series and integrals. For its basic properties and some historical remarks, we refer to Srinivasan's paper [23]. and Chapter 1 of [9]. In the literature the derivatives ψ', ψ'', ψ''',.... are known as polygamma functions. The polygamma functions have the following integral representations:

$$(-1)^{n-1}\psi^{(n)}(u) = \int_0^\infty \frac{t^n e^{-ut}}{1-e^{-t}}dt \tag{1.1}$$

for n=1,2,3,... . See these and other properties of these functions the first chapter of [9] and [24]. Recently numerous interesting inequalities for the gamma function have been proved by many mathematicians, we refer to [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 17, 20, 22], and the references therein. In this paper we aim at presenting a new complete monotonicity property and several new upper and lower bounds for the gamma function. We want to recall that a function f is completely monotonic in an interval I if f has derivatives of all orders in I which alternate in sign, that is $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and n=0,1,2,3,.... If this inequality is strict for all $x \in I$ and all non-negative integers n, then f is said to be strictly completely monotonic. Completely monotonic functions have important applications in different branches of science, for example, they have applications in probability theory [14, 18, 21], potential theory[13], physics[16] and numerical analysis [19]. In particular, completely monotonic functions involving $\log(\Gamma(x))$ are important because these functions produce bounds for the polygamma functions.

2000 *Mathematics Subject Classification.* Primary: 33B15; Secondary: 26D07.
Key words and phrases. Gamma function, Stirling formula, Burnside's formula, digamma function, polygamma functions, complete monotonicity, harmonic numbers and inequalities.

The Hausdorff-Bernstein-Widder theorem characterizes completely monotonic functions. This theorem states that f is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a non-negative measure on $[0,\infty)$ such that this integral converges for all $x > 0$, see [25, Theorem 12b, p.161]. We collect our basic theorems in Section 2. In Section 3, we prove some Stirling-type formulas which refine and improve the classical Stirling approximation

$$\Gamma(n+1) = n! \sim n^n e^{-n}\sqrt{2\pi n} = \alpha_n,$$

and Burnside's formula [15]

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} = \beta_n.$$

Numerical computations indicate that the approximation [10]

$$n! \sim \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}}$$

is better than both of the above approximations in terms of the numbers α_n and β_n. Some of our results give much more accurate values for $n!$ than all of the these approximations. Our results refine the classical Stirling approximation and its many recent improvements. Our main results are the following theorems:

Theorem 1.1. *Let x be a positive real number. Then the function defined by*

$$F(x) = x\log x - x + \frac{1}{2}\log(2\pi) - \log(\Gamma(x+1)) + \frac{1}{2}\log(x+1/2) - \frac{1}{6(x+3/8)}$$

is strictly completely monotonic in $(0,\infty)$.

Corollary 1.2. *Let x be a positive real number. Then the following inequalities hold*

$$\alpha x^x e^{-x}\sqrt{x+1/2}\exp\left\{-\frac{1}{6(x+3/8)}\right\} < \Gamma(x+1) \tag{1.2}$$
$$< \beta x^x e^{-x}\sqrt{x+1/2}\exp\left\{-\frac{1}{6(x+3/8)}\right\}$$

with the best possible constants $\alpha = \sqrt{2}e^{4/9} = 2.20564...$ and $\beta = \sqrt{2\pi} =$ 2.50662....

It is well known that for any positive integer n $\psi(n+1) = H_n + \gamma$, where γ is the Euler's constant and $H_n = \sum_{k=1}^n \frac{1}{k}$ is the n^{th} harmonic number. Using this fact and monotonic increase of F', we get $\gamma - \frac{210}{363} = F'(1) \le F'(n) < \lim_{n\to\infty} F'(n) = 0$. This produces the following new bounds for the harmonic numbers H_n.

Corollary 1.3. *Let n be a positive integer. Then it holds that*

$$\gamma + \log n + \frac{1}{2(n+1/2)} + \frac{1}{6(n+3/8)^2} \le H_n$$
$$< \frac{70}{121} + \log n + \frac{1}{2(n+1/2)} + \frac{1}{6(n+3/8)^2}.$$

Theorem 1.4. *Let x be a positive real number. Then the following inequalities hold:*

$$\alpha^{-\alpha}e^{-x}(x+\alpha)^{x+\alpha} \le \Gamma(x+1) \le \beta^{-\beta}e^{-x}(x+\beta)^{x+\beta},$$

where

$$\alpha = 1/2 = 0.5 \ \ and \ \ \beta = e^{-\gamma} = 0.56146\ldots . \tag{1.3}$$

Theorem 1.5. *For any positive real number x the following double inequality holds:*

$$a\left(\frac{x+1/2}{e}\right)^{x+1/2} \le \Gamma(x+1) < b\left(\frac{x+1/2}{e}\right)^{x+1/2},$$

where

$$a = \sqrt{2e} = 2.33164\ldots \quad and \quad b = \sqrt{2\pi} = 2.50662\ldots \tag{1.4}$$

are best possible constants.

Theorem 1.6. *For all positive real numbers $x \ge 1$ we have*

$$x^x e^{-x}\sqrt{2\pi(x+a)} < \Gamma(x+1) < x^x e^{-x}\sqrt{2\pi(n+b)}, \tag{1.5}$$

with the best possible constants $a = 1/6 = 0.1666666\ldots$ and $b = \frac{e^2}{2\pi} - 1 =$ $0.176005\ldots$.

Numerical computations indicate that the approximations

$$n! \sim n^n e^{-n}\sqrt{2\pi(n+1/6)} = \gamma_n,$$

and

$$n! \sim n^n e^{-n}\sqrt{\pi(2n+1)}\exp\left\{-\frac{1}{6(n+3/8)}\right\} = \delta_n$$

are more accurate than the above formulae involving α_n and β_n. For numerical comparison of the above inequalities see the tables at the end of the paper. An other one of our results provides a converse to the well known inequality $n! \le \sqrt{2\pi}(\frac{n+1/2}{e})^{n+1/2}$ and we prove that

$$(2e/3)^{3/2}\left(\frac{n+1/2}{e}\right)^{n+1/2} \le n! \le \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2},$$

where the constants $(2e/3)^{3/2}$ and $\sqrt{2\pi}$ are best possible. In order to prove our main results we need the following lemmas.

Lemma 1.7. *Let x be a positive real number . Then we have*

$$\log(x+\alpha) < \psi(x+1) \le \log(x+\beta), \tag{1.6}$$

where $\alpha = 1/2$ and $\beta = e^{-\gamma}$ are the best possible constants. If $x \ge 1$, then

$$\log(x+1/2) < \psi(x+1) \le \log(x+e^{1-\gamma}-1) \tag{1.7}$$

holds and the constants $1/2$ and $e^{1-\gamma}-1$are best possible. Here ψ is the digamma function.

Proof. For $x > 0$, we define $g(x) = e^{\psi(x+1)} - x$. By differentiation we get $g'(x) = \psi'(x+1)e^{\psi(x+1)} - 1$, so that by [12, Lemma 1.2] g is strictly decreasing for $x > 0$. In [12] it was proved that $\lim_{x\to\infty}[x - e^{\psi(x)}] = 1/2$. This tells us that $\lim_{x\to\infty} g(x) = 1/2$. Hence we can write for $x > 0$ $1/2 = \lim_{x\to\infty} g(x) < g(x) \le g(0) = e^{-\gamma}$ from which the proof of (1.6) follows. The proof of (1.7) also follows from the inequalities:

$$1/2 = \lim_{x\to\infty} g(x) < g(x) \le g(1) = e^{1-\gamma} - 1.$$

□

Lemma 1.8. *For all integers $k \ge 7$ the following inequality holds.*

$$12^k > (2k+1)4^k + \frac{k8^k}{2} + \frac{32k(k-1)9^{k-2}}{3}. \tag{1.8}$$

Proof. We apply mathematical induction to k. It is clear that (1.8) holds for $k = 7$. We assume that it holds for $k = n \ge 7$. In order to complete the proof we need to prove that it also holds for $k = n + 1$. Since Since (1.8) holds for $k = n$ we have

$$\begin{aligned}12^{n+1} =& 12.12^n > 12\left((2n+1)4^n + \frac{n8^n}{2} + \frac{32n(n-1)9^{n-2}}{3}\right)\\ &= \left\{(2n+3)4^{n+1} + \frac{(n+1)8^{n+1}}{2} + \frac{32n(n+1)9^{n-1}}{3}\right\}\\ &+ \left\{n4^{n+2} + (n-2)2^{3n+1} + 32n(n-7)9^{n-2}\right\}.\end{aligned}$$

The induction proof is complete because we have shown that

$$12^{n+1} > 2n+3)4^{n+1} + \frac{(n+1)8^{n+1}}{2} + \frac{32n(n+1)9^{n-1}}{3}$$

for $k = n \ge 7$. □

Lemma 1.9. *For all real number $x \ge 1$ we have:*

$$\sqrt{\pi}(x/e)^x(8x^3+4x^2+x+1/100)^{\frac{1}{6}} < \Gamma(x+1) < \sqrt{\pi}(x/e)^x(8x^3+4x^2+x+1/30)^{\frac{1}{6}}.$$

See [6].

2. Proofs of Our Theorems

Proof of Theorem 1.1. By differentiation we obtain for $x > 0$

$$F'(x) = \log x - \psi(x) - \frac{1}{x} + \frac{1}{2(x+1/2)} + \frac{1}{6(x+3/8)^2},$$

and

$$F''(x) = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{2(x+1/2)^2} - \frac{1}{3(x+3/8)^3} - \psi'(x).$$

Using the integral representations $\frac{1}{x} = \int_0^\infty e^{-xt}dt$, $\frac{1}{x^2} = \int_0^\infty te^{-xt}dt$ and $\frac{1}{x^3} = \frac{1}{2}\int_0^\infty t^2e^{-xt}dt$ and (1.1) we can write

$$F''(x) = \int_0^\infty \frac{e^{-3t/2}\phi(t)e^{-xt}}{1-e^{-t}}dt, \tag{2.1}$$

where $\phi(t) = 3t + 6e^{3t/2} - 6e^{t/2} - 6te^{t/2} - 3te^t - t^2e^{9t/8} + t^2e^{t/8}$. If we expand the exponential functions as a power series about $t = 0$, we find

$$\phi(t) = 6\sum_{n=5}^{\infty} \frac{a_n(t/8)^n}{n!},$$

where

$$a_n = 12^n - (2n+1)4^n - \frac{n8^n}{2} - \frac{32n(n-1)9^n}{243} + \frac{32n(n-1)}{3}.$$

By Lemma 1.8 this yields that $a_n \geq 0$ for $n \geq 5$. Hence, we have $\phi(t) \geq 0$ for $t > 0$. This immediately gives $F''(x) > 0$ for $x > 0$ by (2.1). Applying (2.1) gives that F'' is strictly completely monotonic in $(0, \infty)$. Using the classical Stirling formula we find that $\lim\limits_{x\to\infty} F(x) = \lim\limits_{x\to\infty} F'(x) = 0$. This implies that $F(x) > 0$ and $F'(x) < 0$ for $x > 0$. Thus we conclude that F is strictly completely monotonic in $(0, \infty)$. □

Proof of Corollary 1.2. Since

$$F(x) = x\log x - x + \frac{1}{2}\log(2\pi) - \log(\Gamma(x+1)) + \frac{1}{2}\log(x+1/2) - \frac{1}{6(x+3/8)}$$

is strictly completely monotonic in $(0, \infty)$ by Theorem 1.1, it is strictly decreasing in the same interval. So, we obtain that

$$0 = \lim_{x\to\infty} F(x) < F(x) \leq F(0) = \frac{1}{2}\log(2\pi) - \frac{1}{2}\log 2 - \frac{4}{9}.$$

Rearranging these inequalities we prove Corollary 1.2. □

Proof of Theorem 1.4. For $x > 0$ and $c > 0$, we let

$$f_c(x) = \log(\Gamma(x+1)) - (x+c)\log(x+c) + x + c - \frac{1}{2}\log(2\pi). \tag{2.2}$$

Differentiation gives $f_c'(x) = \psi(x+1) - \log(x+c)$. By Lemma 1.7 we obtain $f_\alpha'(x) > 0$ and $f_\beta'(x) < 0$ for all $x > 0$. Namely, f_α is strictly increasing and f_β is strictly decreasing in $(0, \infty)$. Now the proof is obtained from $f_\alpha(x) \geq f_\alpha(0) = -\alpha\log\alpha + \alpha - \frac{1}{2}\log(2\pi)$ and $f_\beta(x) \leq f_\beta(0) = -\beta\log\beta + \beta - \frac{1}{2}\log(2\pi)$, where α and β are as defined in (1.3). □

Proof of Theorem 1.5. Define for $x > 0$

$$h(x) = \log(\Gamma(x+1)) - (x+1/2)\log(x+1/2) + x + 1/2. \tag{2.3}$$

If we differentiate we get $h'(x) = \psi(x+1) - \log(x+1/2)$. Applying Lemma 1.7 we arrive at that h is strictly increasing in $(0, \infty)$. Using Stirling formula we can see that $h(\infty) = \lim\limits_{x\to\infty} h(x) = \sqrt{2\pi}$. This leads to

$$h(0) = \frac{1}{2}\log 2 + \frac{1}{2} \leq h(x) < h(\infty) = \sqrt{2\pi}.$$

A simple computation finishes the proof of Theorem 1.5. □

Proof of Theorem 1.6. We define

$$g(x)=\frac{(\Gamma(x+1))^2}{2\pi x^{2x}e^{-2x}}-x, x>1.$$

We shall show that g is strictly decreasing in $(1,\infty)$. By differentiation we find that

$$g'(x)=\frac{(\Gamma(x+1))^2}{\pi x^{2x}e^{-2x}}(1/x-(\log x-\psi(x))-1.$$

By [7, Theorem 8] we have

$$\log x-\psi(x)>\frac{1}{2x}+\frac{1}{12x^2}-\frac{1}{120x^4}+\frac{1}{252x^6}.$$

If we use Lemma 1.9 and this inequality we obtain

$$g'(x)<(8x^3+4x^2+x+1/30)^{1/3}\left(\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}\right)-1.$$

Hence, in order to prove that $g'(x)<0$ for $x>1$, it is sufficient to see that

$$(8x^3+4x^2+x+1/30)(1260x^5-210x^4+21x^2-10)^3-(2520x^6)^3<0$$

holds for $x>1$. Since

$$\begin{aligned}
&(8x^3+4x^2+x+1/30)(1260x^5-210x^4+21x^2-10)^3-(2520x^6)^3=\\
&-\frac{13083757429}{30}-\frac{32324710974}{5}(x-1)-\frac{91879895093}{2}(x-1)^2\\
&-208012393091(x-1)^3-\frac{1342666565583}{2}(x-1)^4\\
&-\frac{8168534687124}{5}(x-1)^5-\frac{30900300345243}{10}(x-1)^6\\
&-4614947865171(x-1)^7-5478102552951(x-1)^8\\
&-5166967298976(x-1)^9-3848019707070(x-1)^{10}\\
&-2233636484940(x-1)^{11}-988874277300(x-1)^{12}\\
&-322352654400(x-1)^{13}-72895183200(x-1)^{14}\\
&-10209326400(x-1)^{15}-666792000(x-1)^{16},
\end{aligned}$$

this says that $g'(x)<0$ for all $x>1$. In order to complete the proof we only need to prove that $\lim_{x\to\infty} g(x)=1/6$. By Lemma 1.9 we have

$$\frac{(8x^3+4x^2+x+1/100)^{1/3}}{2}-x<g(x)<\frac{(8x^3+4x^2+x+1/30)^{1/3}}{2}-x.$$

It is an easy exercise to see that the limits of both of the bounds here tend to $1/6$ as x tends to ∞. This tells us that $\lim_{x\to\infty} g(x)=1/6$. Hence, using this fact and monotonic decrease of g, we get for any real number $x>1$

$$\frac{1}{6}=\lim_{x\to\infty}g(x)<g(x)=\frac{(\Gamma(x+1))^2}{2\pi x^{2x}e^{-2x}}-x\le g(1)=\frac{e^2}{2\pi}-1.$$

Rearranging these inequalities we complete the proof of Theorem 1.6.

3. Stirling-type Formulas

Corollary 3.1. *Let n be a positive integer. Then*

$$(a+1)^{-(a+1)}e^{1-n}(n+a)^{n+a} \leq n! \leq (b+1)^{-(b+1)}e^{1-n}(n+b)^{n+b},$$

where $a = 1/2$ and $b = e^{1-\gamma} - 1$ are best possible constants.

Proof. Let f_c be as defined by (2.2). Since f_a is strictly increasing and f_b is decreasing in $(0, \infty)$, we can write $f_a(n) \geq f_a(1)$ and $f_b(n) \leq f_b(1)$ for all $n \geq 1$. If we make use the definition of f_a, we complete the proof. $\square$

Corollary 3.2. *Let n be a positive integer. Then*

$$a\left(\frac{n+1/2}{e}\right)^{n+1/2} \leq n! < b\left(\frac{n+1/2}{e}\right)^{n+1/2},$$

where $a = (2e/3)^{3/2}$ and $b = \sqrt{2\pi}$ are best possible constants.

Proof. Let h be as defined in (2.3). Since h is strictly increasing in $(0, \infty)$, the proof follows from the fact

$$-\frac{3}{2}\log(3/2) + \frac{3}{2} = h(1) \leq h(n) < \lim_{n\to\infty} h(n) = \sqrt{2\pi}.$$

$\square$

Corollary 3.3. *For all positive integers n we have*

$$n^n e^{-n}\sqrt{2\pi(n+a)} < n! < n^n e^{-n}\sqrt{2\pi(n+b)},$$

with the best possible constants $a = 1/6 = 0.1666666...$ and $b = \frac{e^2}{2\pi} - 1 =$ $0.176005...$.

Proof. It immediately follows from (1.5) by replacing x by n. $\square$

Corollary 3.4. *Let n be a positive integer. Then the following inequalities are valid:*

$$\alpha n^n\sqrt{n+1/2}\exp\left\{-n - \frac{1}{6(n+3/8)}\right\} \leq n!$$
$$< \beta n^n\sqrt{n+1/2}\exp\left\{-n - \frac{1}{6(n+3/8)}\right\},$$

where $\alpha = \sqrt{\frac{2}{3}}e^{37/33} = 2.50548...$ and $\beta = \sqrt{2\pi} = 2.50663....$

Proof. It follows from monotonic decrease of F and the relations $\lim_{n\to\infty} F(n) = 0$ and $F(1) = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(3/2) - \frac{4}{33}$. $\square$

n	$n!$	α_n	β_n	δ_n	γ_n
1	1	0.922137	1.022751	1.00046	0.996022
2	2	1.919	2.03331	2.00011	1.99736
3	6	5.83621	6.07152	6.00009	5.99614
4	24	23.5062	24.2226	24.0001	23.9909
5	120	118.019	120.911	120.00031	119.97
6	720	710.078	724.624	720.001	719.873
7	5040	4980.4	5068.05	5040.004	5039.34
8	40320	39902.4	405187.97	40320.02	40315.9
9	362880	359536.87	364474.04	362880.13	362851
10	3628800	3.598710^6	$3.64322\ 10^6$	$3.6288\ 10^6$	$3.62856\ 10^6$

TABLE 1. The values of α_n, β_n, γ_n and δ_n at the leading terms.

n	$n!$	$\|\alpha_n - n!\|$	$\|\beta_n - n!\|$	$\|\delta_n - n!\|$	$\|\gamma_n - n!\|$
1	1	0.077863	0.0275077	0.0004590	0.003978
2	2	0.080995	0.0333107	0.0001112	0.002636
3	6	0.16379	0.0715196	0.0000881	0.003864
4	24	0.493825	0.222618	0.0001345	0.009110
5	120	1.98083	0.910795	0.0003176	0.029969
6	720	9.92182	4.62384	0.0010326	0.127171
7	5040	59.6042	28.0489	0.0043129	0.662521
8	40320	417.605	197.973	0.0221008	4.099718
9	362880	3343.13	1594.04	0.13451	29.3532
10	3628800	30104.4	14421	0.949379	239.175

TABLE 2. A comparison between the leading terms of α_n, β_n, γ_n and δ_n.

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