A NEW PROOF METHOD OF ANALYTIC INEQUALITY

XIAO-MING ZHANG

Abstract. This paper gives a new proof method of analytic inequality involving \( n \) variables. As its Applications, we proved some well-known inequalities and improved the Carleman-Inequality.

1. Monotonicity on Special Region

Throughout the paper \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{R}_+ \) denotes the set of strictly positive real numbers, \( n \in \mathbb{N} , n \geq 2 \).

In this section, we shall provide a new proof method of analytic inequality involving \( n \) variables.

Theorem 1.1. Given \( a,b \in \mathbb{R} , c \in [a,b] \). Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) have continuous partial derivative,

\[
D_i = \left\{ (x_1, x_2, \cdots, x_{n-1}, c) \mid \min_{1 \leq k \leq n-1} \{x_k\} \geq c, x_i = \max_{1 \leq k \leq n-1} \{x_k\} \neq c \right\} , i = 1, 2, \cdots, n-1.
\]

If \( \frac{\partial f}{\partial x_i} > 0 \) hold for any \( x \in D_i (i = 1, 2, \cdots, n-1) \), then

\[
f (y_1, y_2, \cdots, y_n, c) \geq f (c, c, \cdots, c)
\]

hold for \( y_i \in [c,b] (i = 1, 2, \cdots, n-1) \).

Proof. Without the losing of generality, we let \( n = 3 \) and \( y_1 > y_2 > c \).

For \( x_1 \in [y_2, y_1] \), it has \((x_1, y_2, c) \in D_1 \), then \( \frac{\partial f}{\partial x_1} (x_1, y_2, c) > 0 \). Owing to the continuity of partial derivative and \( \frac{\partial f}{\partial x_1} (x_1, y_2, c) = 0 \), it exists \( \varepsilon \), such that \( y_2 - \varepsilon > c \) and \( \frac{\partial f}{\partial x_1} (x_1, y_2, c) > 0 \) for any \( x_1 \in [y_2 - \varepsilon, y_2] \). Hence, \( f (\cdot, y_2, c) : x_1 \in [y_2 - \varepsilon, y_1] \rightarrow f (x_1, y_2, c) \) is strictly monotone increasing,

\[
f (y_1, y_2, c) > f (y_2, y_2, c) > f (y_2 - \varepsilon, y_2, c).
\]

For \( x_2 \in [y_2 - \varepsilon, y_2] \), \( y_2 - \varepsilon, x_2, c) \in D_2 \), \( \frac{\partial f}{\partial x_2} (x_2, y_2, c) = 0 \). Then

\[
f (y_1, y_2, c) > f (y_2, y_2, c) > f (y_2 - \varepsilon, y_2, c) > f (y_2 - \varepsilon, y_2 - \varepsilon, c).
\]

If \( y_2 - \varepsilon = c \), this completes the proof of the Theorem 1.1. Otherwise, we repeat the above process. It is clear that the first variable and the second variable of function \( f \) are decreasing and no less than \( c \). Let \( s, t \) are their limits, then \( f (y_1, y_2, c) > f (s, t, c) \), where \( s, t \geq c \). If \( s, t = c \), this completes the proof of the Theorem 1.1. Otherwise, we repeat the above process. Let the greatest lower bound of the first variable and the second variable are \( p, q \). It is easy to see \( p = q = c \), and \( f (y_1, y_2, c) > f (c, c, c) \).

Similarly to the above, we know Theorem 1.2 is true.

Theorem 1.2. Given \( a,b \in \mathbb{R} , c \in [a,b] \). Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) have continuous partial derivative,

\[
D_i = \left\{ (x_1, x_2, \cdots, x_{n-1}, c) \mid \max_{1 \leq k \leq n-1} \{x_k\} \leq c, x_i = \min_{1 \leq k \leq n-1} \{x_k\} \neq c \right\} , i = 1, 2, \cdots, n-1.
\]

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If \( \partial f(x)/\partial x_i < 0 \) hold for any \( x \in D_i \) \( (i = 1, 2, \cdots, n - 1) \), then
\[
\begin{aligned}
& f(y_1, y_2, \cdots, y_{n-1}, c) \geq f(c, c, \cdots, c, c)
\end{aligned}
\] hold for \( y_i \in [a, c] \) \((i = 1, 2, \cdots, n - 1)\).

In particular, according to Theorem 1.1 and Theorem 1.2 the following four corollaries hold.

**Corollary 1.1.** Let \( a, b \in \mathbb{R} \), \( f : [a, b]^n \rightarrow \mathbb{R} \) have continuous partial derivative,
\[
D_i = \left\{ x = (x_1, x_2, \cdots, x_n) \mid a \leq \min_{1 \leq k \leq n} \{x_k\} < x_i = \max_{1 \leq k \leq n} \{x_k\} \leq b \right\}, \quad i = 1, 2, \cdots, n
\]
If \( \partial f(x)/\partial x_i > 0 \) hold for any \( x \in D_i \) and any \( i = 1, 2, \cdots, n \), then
\[
\begin{aligned}
& f(x_1, x_2, \cdots, x_n) \geq f(x_{\min}, x_{\min}, \cdots, x_{\min})
\end{aligned}
\] hold for \( x_i \in [a, b] \) \((i = 1, 2, \cdots, n)\), with \( x_{\min} = \min_{1 \leq k \leq n} \{x_k\} \).

**Corollary 1.2.** Supposes \( a, b \in \mathbb{R} \),
\[
D_1 = \left\{ x = (x_1, x_2, \cdots, x_n) \mid a \leq \min_{1 \leq k \leq n} \{x_k\} < x_1 = \max_{1 \leq k \leq n} \{x_k\} \leq b \right\}.
\]
Let \( f : [a, b]^n \rightarrow \mathbb{R} \) be symmetric, all partial differentiations of \( f \) be continuous. If \( \partial f(x)/\partial x_1 > 0 \) hold for \( x = (x_1, x_2, \cdots, x_n) \) \( \in D_1 \), then
\[
\begin{aligned}
& f(x_1, x_2, \cdots, x_n) \geq f(x_{\min}, x_{\min}, \cdots, x_{\min}),
\end{aligned}
\]
with \( x_{\min} = \min_{1 \leq k \leq n} \{x_k\}. \) Equality holds if and only if \( x_1 = x_2 = \cdots = x_n \).

**Corollary 1.3.** Supposes \( a, b \in \mathbb{R} \), \( f : [a, b]^n \rightarrow \mathbb{R} \) have continuous partial derivative,
\[
D_i = \left\{ x = (x_1, x_2, \cdots, x_n) \mid a \leq x_i = \min_{1 \leq k \leq n} \{x_k\} < \max_{1 \leq k \leq n} \{x_k\} \leq b \right\}.
\]
If \( \partial f(x)/\partial x_i < 0 \) hold for any \( x \in D_i \) and any \( i = 1, 2, \cdots, n \), then
\[
\begin{aligned}
& f(x_1, x_2, \cdots, x_n) \geq f(x_{\max}, x_{\max}, \cdots, x_{\max}),
\end{aligned}
\]
with \( x_{\max} = \max_{1 \leq k \leq n} \{x_k\}. \) Equality holds if and only if \( x_1 = x_2 = \cdots = x_n \).

**Corollary 1.4.** Supposes \( a, b \in \mathbb{R} \),
\[
D_n = \left\{ x = (x_1, x_2, \cdots, x_n) \mid a \leq x_n = \min_{1 \leq k \leq n} \{x_k\} < \max_{1 \leq k \leq n} \{x_k\} \leq b \right\}.
\]
Let \( f : [a, b]^n \rightarrow \mathbb{R} \) be symmetric, all partial differentiations of \( f \) be continuous. If \( \partial f(x)/\partial x_n < 0 \) hold for \( x = (x_1, x_2, \cdots, x_n) \) \( \in D_n \), then
\[
\begin{aligned}
& f(x_1, x_2, \cdots, x_n) \geq f(x_{\max}, x_{\max}, \cdots, x_{\max}),
\end{aligned}
\]
with \( x_{\max} = \max_{1 \leq k \leq n} \{x_k\}. \) Equality holds if and only if \( x_1 = x_2 = \cdots = x_n \).
2. Unifying Proof of Some Well-known Inequality

In this section, we denote \( a = (a_1, a_2, \ldots, a_n) \), \( a_{\text{min}} = \min_{1 \leq k \leq n} \{a_k\} \), \( a_{\text{max}} = \max_{1 \leq k \leq n} \{a_k\} \) and

\[
D_i = \{a|a_i = a_{\text{max}} > a_{\text{min}} > 0\}, \quad i = 1, 2, \ldots, n.
\]

**Proposition 2.1.** (Power Mean Inequality) The power mean \( M_r(a) \) of order \( r \) with respect to the positive numbers \( a_1, a_2, \ldots, a_n \) is defined as \( M_r(a) = \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}} \) for \( r \neq 0 \), and \( M_0(a) = \prod_{i=1}^{n} a_i^{\frac{1}{n}} \). Then \( M_r(a) \geq M_s(a) \) if \( r > s \), equality holds if and only if \( a_1 = a_2 = \cdots = a_n \).

Proof. Obviously, \( M_r(a) \) is symmetric with respect to \( a_1, a_2, \ldots, a_n \), \( r \mapsto M_r(a) \) is continuous. Without the losing of generality, we let \( r, s \neq 0 \),

\[
f(a) = \frac{1}{r} \ln \left( \frac{\sum_{i=1}^{n} a_i^r}{n} \right) - \frac{1}{s} \ln \left( \frac{\sum_{i=1}^{n} a_i^s}{n} \right), \quad a \in \mathbb{R}_+^n.
\]

Then

\[
\frac{\partial f(a)}{\partial a_1} = \frac{a_1^{r-1} - a_1^{s-1}}{\sum_{i=1}^{n} a_i^r - \sum_{i=1}^{n} a_i^s} \\
= \frac{\sum_{i=2}^{n} (a_i^{r-1}a_i^s - a_i^{s-1}a_i^r)}{\sum_{i=1}^{n} a_i^r \cdot \sum_{i=1}^{n} a_i^s} \\
= \frac{\sum_{i=2}^{n} a_i^{s-1}a_i^r [a_1/a_i]^{r-s} - 1}{\sum_{i=1}^{n} a_i^r \cdot \sum_{i=1}^{n} a_i^s}.
\]

If \( a \in D_1 \), we get \( \partial f(a)/\partial a_1 > 0 \). According to Corollary 1.2 it has

\[
f(a_1, a_2, \ldots, a_n) \geq f(a_{\text{min}}, a_{\text{min}}, \ldots, a_{\text{min}}),
\]

\[
\left( \frac{\sum_{i=1}^{n} a_i^r}{n} \right)^{1/r} \geq \left( \frac{\sum_{i=1}^{n} a_i^s}{n} \right)^{1/s}, \quad M_r(a) \geq M_s(a).
\]

Equality holds if and only if \( a_1 = a_2 = \cdots = a_n \). \( \square \)

**Proposition 2.2.** (Holder-Inequality) Let \( (x_1, x_2, \ldots, x_n) \), \( (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n \), \( p, q > 1 \), and \( 1/p + 1/q = 1 \). Then

\[
\left( \sum_{k=1}^{n} x_k^p \right)^{1/p} \left( \sum_{k=1}^{n} y_k^q \right)^{1/q} \geq \sum_{k=1}^{n} x_k y_k.
\]

Proof. Let \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n \),

\[
f: a \in \mathbb{R}_+^n \rightarrow \left( \sum_{k=1}^{n} b_k \right)^{1/p} \left( \sum_{k=1}^{n} b_k a_k \right)^{1/q} - \sum_{k=1}^{n} b_k a_k^{1/q}, \quad a \in \mathbb{R}_+^n.
\]

If \( a \in D_1 \),

\[
\frac{\partial f(a)}{\partial a_1} = \frac{1}{q} b_1 \left( \sum_{k=1}^{n} b_k \right)^{1/p} \left( \sum_{k=1}^{n} b_k a_k \right)^{1/q-1} - \frac{1}{q} b_1 a_1^{1/q-1} \\
= \frac{1}{q} b_1 a_1^{-1/p} \left( \sum_{k=1}^{n} b_k a_k \right)^{-1/p} \left[ \left( \sum_{k=1}^{n} b_k \right)^{1/p} a_1^{1/p} - \left( \sum_{k=1}^{n} b_k a_k \right)^{1/p} \right] \\
> \frac{1}{q} b_1 a_1^{-1/p} \left( \sum_{k=1}^{n} b_k a_k \right)^{-1/p} \left[ \left( \sum_{k=1}^{n} b_k \right)^{1/p} a_1^{1/p} - \left( \sum_{k=1}^{n} b_k a_1 \right)^{1/p} \right] \\
= 0.
\]

Similarly, If \( a \in D_i (i = 2, 3, \ldots, n) \), \( \partial f(a)/\partial a_i > 0 \). According to Theorem 1.1

\[
f(a_1, a_2, \ldots, a_n) \geq f(a_{\text{min}}, a_{\text{min}}, \ldots, a_{\text{min}}),
\]
In above inequality, let \( a_k = y_k^p / x_k^p \), \( b_k = x_k^p \), we complete the proof of Proposition 2.2.

**Proposition 2.3.** (Minkowski-Inequality) Let \( (x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n_+ \), \( p > 1 \), then

\[
\left( \sum_{k=1}^{n} x_k^p \right)^{1/p} + \left( \sum_{k=1}^{n} y_k^p \right)^{1/p} \geq \left( \sum_{k=1}^{n} (x_k + y_k)^p \right)^{1/p}.
\]

**Proof.** Let \( b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n_+ \),

\[
f : a \in \mathbb{R}^n_+ \rightarrow \left( \sum_{k=1}^{n} b_k a_k \right)^{1/p} - \left( \sum_{k=1}^{n} b_k \left( a_k^{1/p} + 1 \right)^p \right)^{1/p}, \quad a \in \mathbb{R}^n_+.
\]

If \( a \in D_1 \),

\[
\frac{\partial f (a)}{\partial a_1} = \frac{1}{p} b_1 \left( \sum_{k=1}^{n} b_k a_k \right)^{1/p-1} - \frac{1}{p} b_1 a_1^{1/p-1} \left( a_1^{1/p} + 1 \right)^{p-1} \left( \sum_{k=1}^{n} b_k \left( a_k^{1/p} + 1 \right)^p \right)^{1/p-1}
\]

\[
= \frac{1}{p} b_1 \left( \sum_{k=1}^{n} b_k a_k \right)^{1/p-1} \left( \sum_{k=1}^{n} b_k \left( a_k^{1/p} + 1 \right)^p \right)^{1/p-1}
\]

\[
\cdot \left[ \left( \sum_{k=1}^{n} b_k \left( a_k^{1/p} + 1 \right)^p \right)^{1/p-1} - \left( \sum_{k=1}^{n} b_k \left( a_k^{1/p} + a_k^{1/p} - 1 \right)^p \right)^{1/p-1} \right]
\]

\[
= 0.
\]

Similarly, If \( a \in D_i (i = 2, 3, \cdots, n) \), \( \partial f (a) / \partial a_i > 0 \). According to Theorem 1.1,

\[
f (a_1, a_2, \cdots, a_n) \geq f (a_{\min}, a_{\min}, \cdots, a_{\min}),
\]

\[
\left( \sum_{k=1}^{n} b_k a_k \right)^{1/p} \geq \left( \sum_{k=1}^{n} b_k \left( a_k^{1/p} + 1 \right)^p \right)^{1/p}.
\]

In above inequality, let \( a_k = y_k^p / x_k^p \), \( b_k = x_k^p \), we complete the proof of Proposition 2.3.

**3. A Refinement on the Carleman’s Inequality**

If \( a_n \geq 0 \) \( (n \in \mathbb{N}, n \geq 1) \) with \( 0 < \sum_{n=1}^{\infty} a_n < \infty \), then the famous Carleman’s inequality is

\[
\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k \right)^{1/n} < e \sum_{n=1}^{\infty} a_n,
\]

where the constant factor is the best possible (see [1]).

Recently, Yang et al. [9] gave a strengthened version of (3.1) as follows.

\[
\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k \right)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{1}{2n+2} \right) a_n.
\]

Some other strengthened version of (3.1) were given by [1]–[9]. In the section, we shall obtain another refinement on the Carleman’s inequality in the form of Corollary 3.2.
Lemma 3.1. Let $i \in \mathbb{N}$, $i \geq 1$, then

\begin{equation}
\left(1 - \frac{2}{3i + 7}\right) \frac{1}{i} > \sum_{k=i}^{\infty} \frac{1}{k (k!)^{1/k}}.
\end{equation}

\begin{equation}
\left(1 - \frac{2}{3i + 10}\right) \frac{1}{i + 1} > \frac{1}{((i + 1)!)^{1/(i+1)}}.
\end{equation}

Proof. Let $\psi(i) = e \left(1 - \frac{2}{3i + 7}\right) \frac{1}{i} - \sum_{k=i}^{\infty} \frac{1}{k (k!)^{1/k}}$, then $\psi(i) > \psi(i + 1)$ is equivalent to

\begin{equation}
1 - \frac{2i + 2}{3i + 7} + \frac{2i}{3i + 10} > \frac{i + 1}{e (i!)^{1/i}}.
\end{equation}

If $1 \leq i \leq 16$, after brief computation, we know inequality (3.5) hold. If $i \geq 17$, we get $\sqrt{2\pi i} \geq e^{7/3}$.

\begin{equation}
\sqrt{2\pi i} \geq e^{(2i^2 + 7i + 70)/(9i^2 + 39i + 50)}.
\end{equation}

If $x > 0$, it have $e > (1 + 1/x)^x$. Thus

\begin{equation}
e > \left(1 + \frac{21i^2 + 7i + 70}{(9i^2 + 39i + 50)i}\right)^{((9i^2 + 39i + 50)i)/(2i^2 + 7i + 70)}.
\end{equation}

By virtue of (3.6) and (3.7), we get

\begin{equation}
\sqrt{2\pi i} > \left(1 + \frac{21i^2 + 7i + 70}{(9i^2 + 39i + 50)i}\right)^i, \quad \left(2\pi i\right)^{1/(2i)} > \frac{(i + 1)(3i + 7)(3i + 10)}{i(9i^2 + 39i + 50)}.
\end{equation}

The well-known Stirling-equality is $i! = \sqrt{2\pi i} (i/e)^i \exp(\theta_i/12i)$ with $0 < \theta_i < 1$. We have

\begin{equation}
i! > \sqrt{2\pi i} \left(\frac{i}{e}\right)^i.
\end{equation}

Owing to inequality (3.8) and (3.9), inequality (3.5) hold. Hence, $\left\{\psi(i)\right\}_{i=1}^{\infty}$ is a strictly decreasing sequence. Because $\lim_{i \to +\infty} \psi(i) = 0$, we have $\psi(i) > 0$. Inequality (3.3) is proved.

Meanwhile,

\begin{align*}
\sqrt{2\pi (i + 1)} &> e^{2/3}, \\
\sqrt{2\pi (i + 1)} &> e^{(2i+2)/(3i+8)}, \\
\sqrt{2\pi (i + 1)} &> \left(1 + \frac{2}{3i + 8}\right)^{(3i+8)/(2(2i+2)/(3i+8))}, \\
(2\pi (i + 1))^{1/(2i+2)} &> \frac{3i+10}{3i + 8},
\end{align*}

\begin{equation}
e \left(1 - \frac{2}{3i + 10}\right) \frac{1}{i + 1} > \frac{e}{(i + 1)(2\pi (i + 1))^{1/(2i+2)}}.
\end{equation}

According to $(i + 1)! > \sqrt{2\pi (i + 1)} ((i + 1)/e)^i$, inequality (3.4) hold. \qed
Theorem 3.1. Let \( n \in \mathbb{N}, n \geq 1, a_k > 0 (k = 1, 2, \cdots, n), B_n = \min_{1 \leq k \leq n} \{ka_k\}, \) then
\[
(3.11) \quad e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) a_k - \sum_{k=1}^{n} \left( \prod_{j=1}^{k} a_j \right)^{1/k} \geq B_n \left[ e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{(k!)^{1/k}} \right].
\]

Proof. Let \( b_k = ka_k, k = 1, 2, \cdots, n, \) \( b = (b_1, b_2, \cdots, b_n) \)
\[
D_i = \left\{ b | b_i = \max_{1 \leq k \leq n} \{b_k\} > \min_{1 \leq k \leq n} \{b_k\} > 0 \right\}, \quad i = 1, 2, \cdots, n,
\]
and
\[
f : b \in \mathbb{R}_+^n \to e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) \frac{b_k}{k} - \sum_{k=1}^{n} \left( \frac{1}{k!} \prod_{j=1}^{k} b_j \right)^{1/k}, \quad b \in \mathbb{R}_+^n.
\]
Then inequality (3.11) is equivalent to the following (3.12)
\[
(3.12) \quad e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) \frac{b_k}{k} - \sum_{k=1}^{n} \left( \frac{1}{k!} \prod_{j=1}^{k} b_j \right)^{1/k} \geq B_n \left[ e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{(k!)^{1/k}} \right],
\]
and \( B_n = \min_{1 \leq k \leq n} \{b_k\}. \)

If \( b \in D_i (i = 1, 2, \cdots, n), \)
\[
\frac{\partial f(b)}{\partial b_i} = e \left( 1 - \frac{2}{3i+7} \right) \frac{1}{i} - \sum_{k=i}^{n} \frac{1}{kb_i} \left( \frac{1}{k!} \prod_{j=1}^{k} b_j \right)^{1/k} \]
\[
> e \left( 1 - \frac{2}{3i+7} \right) \frac{1}{i} - \sum_{k=i}^{n} \frac{1}{k(k!)^{1/k}} \]
\[
> e \left( 1 - \frac{2}{3i+7} \right) \frac{1}{i} - \sum_{k=i}^{\infty} \frac{1}{k(k!)^{1/k}}.
\]
According to inequality (3.3), \( \partial f(b)/\partial b_i > 0. \) In view of Theorem 1.1, \( f(b_1, b_2, \cdots, b_n) \geq f(B_n, B_n, \cdots, B_n). \)
This implies inequality (3.12) hold.

\[\square\]

Corollary 3.1. Let \( n \in \mathbb{N}, n \geq 1, a_k > 0 (k = 1, 2, \cdots, n), B_n = \min_{1 \leq k \leq n} \{ka_k\}, \) then
\[
(3.13) \quad e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) a_k - \sum_{k=1}^{n} \left( \prod_{j=1}^{k} a_j \right)^{1/k} \geq B_n \left( \frac{4}{5} e - 1 \right).
\]

Proof. Let \( T(i) = e \sum_{k=1}^{i} \left( 1 - \frac{2}{3k+7} \right) \frac{1}{k} - \sum_{k=1}^{i} \frac{1}{(k!)^{1/k}}, \quad i = 1, 2, \cdots, n. \) Inequality (3.4) implies \( \{T(i)\}_{i=1}^{n} \) is a strictly increasing sequence. According to inequality (3.11), we have
\[
e \sum_{k=1}^{n} \left( 1 - \frac{2}{3k+7} \right) a_k - \sum_{k=1}^{n} \left( \prod_{j=1}^{k} a_j \right)^{1/k} \geq B_n T(n) \geq B_n T(1) = B_n \left( \frac{4}{5} e - 1 \right).
\]

Let \( n \to +\infty, \) we know the following Corollary 3.2 is true.

\[\square\]
Corollary 3.2. If $a_n \geq 0 \ (n \in \mathbb{N}, n \geq 1)$ with $0 < \sum_{n=1}^{\infty} a_n < \infty$, then
\begin{equation}
\sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} a_j\right)^{1/n} \leq e \sum_{n=1}^{\infty} \left(1 - \frac{2}{3n + 7}\right) a_n.
\end{equation}

Remark 3.1. a lot of Application of Theorem 1.1 will appear in other papers.

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(X.-M. Zhang) ZHEJIANG BROADCAST AND TV UNIVERSITY HAINING COLLEGE, HAINING CITY, ZHEJIANG PROVINCE, 314400, P. R. CHINA

E-mail address: zjzxm790126.com