ON SCHUR-GEOMETRICAL CONVEXITY OF FOUR-PARAMETER FAMILY OF MEANS

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Abstract. We prove that the four-parameter family of means

\[ R(u, v; r, s; x, y) = \left[ \frac{E(r, s; x^u, y^v)}{E(r, s; x^r, y^s)} \right]^{1/(u-v)} \]

is Schur-geometrically convex in \( x, y \) if \((u+v)(r+s) \geq 0\) and Schur-geometrically concave otherwise.

In [1] the authors investigate Schur geometrical convexity of extended mean values (called also Stolarsky means, as they were introduced by Kenneth B. Stolarsky in [3])

\[
E(r, s; x, y) = \begin{cases} 
\left( \frac{y^r - x^r}{y^s - x^s} \right)^{1/(s-r)} & \text{if } sr(s-r)(x-y) \neq 0, \\
\left( \frac{1}{r \log y - \log x} \right)^{1/r} & \text{if } r(x-y) \neq 0, s = 0, \\
e^{-1/r} \left( \frac{y^r / x^r}{x^r - y^r} \right)^{1/(y^r - x^r)} & \text{if } r = s, r(x-y) \neq 0, \\
\sqrt{xy} & \text{if } r = s = 0, \\
x & \text{if } x = y
\end{cases}
\]

Their main result is that \( E(r, s; x, y) \) is Schur-geometrically concave in variables \( x, y \) if \( r+s \geq 0 \) and Schur-geometrically concave otherwise. In [2] incomplete analysis of Schur-geometrical convexity of Gini means defined by

\[
G(r, s; x, y) = \begin{cases} 
\left( \frac{x^r + y^r}{x^r + y^r} \right)^{1/(r-s)} & r \neq s \\
\exp \left( \frac{x^r \log x + y^r \log y}{x^r + y^r} \right) & r = s
\end{cases}
\]

is given: \( G(r, s; x, y) \) is Schur-geometrically convex in \( x, y \) if \( r, s \geq 0 \).

Both Stolarsky and Gini means (or, more generally S-means) are members of the four-parameter family of means introduced in [4], see also [5]

\[
R(u, v; r, s; x, y) = \begin{cases} 
\left[ \frac{E(r, s; x^u, y^v)}{E(r, s; x^r, y^s)} \right]^{1/(u-v)} & u \neq v \\
\exp \left( \frac{d}{du} \log E(r, s; x^u, y^v) \right) & u = v.
\end{cases}
\]

Letting \((u, v) = (1, 0)\) we obtain Stolarsky means while \((u, v) = (2, 1)\) and \((u, v) = (3/2, 1/2)\) give Stolarsky and Heronian means respectively.

Let us recall some definitions: we say that \( x = (x_1, x_2) \) is majorized by \( y = (y_1, y_2) \) (and write \( x \prec y \)) if \( \max(x_1, x_2) \leq \max(y_1, y_2) \) and \( x_1 + x_2 = y_1 + y_2 \). For positive \( x_i \) we denote...
log x = (log x₁, log x₂). A real function of two variables f is said to be Schur-geometrically convex if log x < log y implies f(x) ≤ f(y).

The lemma below provides an useful characterization of Schur-convex functions:

**Lemma 1.1.** Let I ⊂ ℝ⁺ be an interval (possibly unbounded). A function f : I × I → ℝ is Schur-geometrically convex if and only if for every positive a the function fₐ(x) = f(x, a/x) is decreasing for x < √a (assuming that (x, a/x) ∈ I × I).

**Proof.** Suppose f is Schur-geometrically convex. For t < s < √a log(s/a/s) < log(t, a/t), hence f(s, a/s) ≤ f(t, a/s).

Conversely, assume log x < log y and set s = min(log x₁, log x₂), t = min(log y₁, log y₂), a = x₁x₂. Then t ≤ s ≤ √a and monotonicity of fₐ implies f(t, a/t) ≥ f(s, a/s). Now symmetry of f implies f(y) ≥ f(x). □

R means are homogeneous of order 1 in x, y so one can easily see that R(u, v; r, s; x, y) is Schur-geometrically convex (Schur-geometrically concave) in x, y if and only if R(u, x; r, s; x, 1/x) decreases (increases) for 0 ≤ x ≤ 1.

**Theorem 1.2.** The R-means are Schur-geometrically convex if (u + v)(r + s) ≥ 0 and Schur-geometrically concave otherwise.

To prove it we shall need the following

**Lemma 1.3.** For t, A, B > 0 let

h(t, A, B) = At coth At − Bt coth Bt.

If s ≠ t and A ≠ b, then

\[ \text{sgn}(h(t, A, B) − h(s, A, B)) = \text{sgn}(t − s)(A − B). \]

**Proof.** The function k(x) = x coth x is even, so k'(0) = 0 and k''(x) = \( \frac{2 \cosh x}{\sinh^2 x} (x − \tanh x) \geq 0 \) coth. This implies that k is increasing for positive x, which is equivalent to (A − B)h(t, A, B) > 0.

Convexity of k means that its divided difference \( \frac{h(t, A, B)}{t(A−B)} \) increases in t. The inequality

\[ 0 \leq \left( \frac{h(t, A, B)}{t(A−B)} \right)' = \frac{t(A − B)h'(t, A, B) − (A − B)h(t, A, B)}{t^2(A−B)^2} \]

implies (A − B)h'(t, A, B) > 0 for every t, so h' and A − B are of the same sign. The Mean Value Theorem gives now

\[ \text{sgn}(h(t, A, B) − h(s, A, B)) = \text{sgn}(t − s)h'(\xi, A, B) = \text{sgn}(t − s)(A − B) \]

which completes the proof. □

**Proof of Theorem** (1.2). By Lemma (1.1) we have to show that R(u, v; r, s; x, 1/x) decreases for 0 < x < 1 if (u + v)(r + s) ≥ 0 and increases otherwise, or, equivalently that T(t) = log R(u, v; r, s; eᵗ, e⁻ᵗ) increases for t > 0 (decreases respectively). We have

\[ T(t) = \frac{\log \left| \sinh urt \right| − \log \left| \sinh ust \right| − \log \left| \sinh vrt \right| + \log \left| \sinh vst \right|}{(u − v)(r − s)} = \frac{\log \sinh |urt| − \log \sinh |ust| − \log \sinh |vrt| + \log \sinh |vst|}{(u − v)(r − s)} \]
and
\[ T'(t) = \frac{|ur|t \coth |ur|t - |us|t \coth |us|t - |vr|t \coth |vs|t}{t(u-v)(r-s)} \]
\[ = \frac{h(|u|, |r|t, |s|t) - h(|v|, |r|t, |s|t)}{t(u-v)(r-s)}. \]

Applying Lemma (1.3) we obtain
\[ \text{sgn} T'(t) = \text{sgn} \frac{|u| - |v|}{u - v} \text{sgn} \frac{|r| - |s|}{r - s} = \text{sgn} (u + v)(r + s), \]

because
\[ \frac{|u| - |v|}{u - v} = \frac{u^2 - v^2}{(u-v)(|u|+|v|)} = \frac{u+v}{|u|+|v|}. \]

\[ \square \]

References


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