ON THE HERMITE-HADAMARD INEQUALITY AND OTHER INTEGRAL INEQUALITIES INVOLVING TWO FUNCTIONS

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Abstract. In this note we establish some Hermite-Hadamard type inequalities involving two functions. Other integral inequalities for two functions are obtained as well.

1. Introduction

Integral inequalities have played an important role in the development of all branches of Mathematics.

In [15] and [16], Pachpatte established some Hadamard type inequalities involving two convex and log-convex functions, respectively. In [1], Bakula, Özdemir and Pečarić improved Hadamard type inequalities for products of two m-convex and (α, m)-convex functions. In [10], analogous results for s-convex functions were proved by Kırmacı, Bakula, Özdemir and Pečarić. General companion inequalities related to Jensen's inequality for the classes of m-convex and (α, m)-convex functions were presented by Bakula et al., (see [3]).

For several recent results concerning these type of inequalities, see [2, 6, 8, 11, 12, 13, 14] where further references are listed.

The aim of this paper is to establish several new integral inequalities for non-negative and integrable functions that are related to the Hermite-Hadamard result. Other integral inequalities for two functions are also established.

In order to prove some inequalities related to the products of two functions we need the following inequalities. One of inequalities of this type is the following one:

Barnes-Godunova-Levin Inequality (see [17, 18, 19] and references therein) Let f, g be nonnegative concave functions on $[a, b]$. Then for $p, q > 1$ we have

$$\left(\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\,dx\right)^{\frac{1}{q}} \le B(p,q)\int_a^b f(x)g(x)dx, \tag{1.1}$$

where

$$B(p,q) = \frac{6\,(b-a)^{\frac{1}{p}+\frac{1}{q}-1}}{(p+1)^{\frac{1}{p}}\,(q+1)^{\frac{1}{q}}}.$$

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In the special case $q = p$ we have:

$$\left(\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_a^b g^p(x)\,dx\right)^{\frac{1}{p}} \leq B(p,p)\int_a^b f(x)g(x)dx$$

with

$$B(p,p) = \frac{6(b-a)^{\frac{2}{p}-1}}{(p+1)^{\frac{2}{p}}}.$$

To prove our main results we recall some concepts and definitions.

Let $x = (x_1, x_2, ..., x_n)$ and $p = (p_1, p_2, ..., p_n)$ be two positive $n-$tuples, and $r \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, on putting $P_n = \sum_{k=1}^n p_k$, the r^{th} power mean of x with weights p is defined [5] by

$$M_n^{[r]} = \begin{cases} \left(\frac{1}{P_n}\sum_{k=1}^n p_k x_k^r\right)^{\frac{1}{r}}, & r \neq +\infty, 0, -\infty \\ \left(\prod_{k=1}^n x_k^{p_k}\right)^{\frac{1}{P_n}}, & r = 0 \\ \min(x_1, x_2, ..., x_n), & r = -\infty \\ \max(x_1, x_2, ..., x_n), & r = \infty. \end{cases}$$

Note that if $-\infty \leq r < s \leq \infty$, then

$$M_n^{[r]} \leq M_n^{[s]} \tag{1.2}$$

(see, for example [12, p.15]).

The following definition is well known in literature.

For $p \in \mathbb{R}^+$, the p-norm of the function $f : [a,b] \to \mathbb{R}$ is defined as

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}, & 0 < p < \infty \\ \sup|f(x)|, & p = \infty \end{cases}$$

and $L_p([a,b])$ is the set of all functions $f : [a,b] \to \mathbb{R}$ such that $\|f\|_p < \infty$.

One can rewrite the inequality (1.1) as follows:

$$\|f\|_p\,\|g\|_q \leq B(p,q)\int_a^b f(x)g(x)dx.$$

For several recent results concerning p-norms we refer the interested reader to [9].

Also, we need four important inequalities and a remark:

Minkowski integral inequality [4, p.1]**.** Let $p \geq 1$, $0 < \int_a^b f^p(x)dx < \infty$ and $0 < \int_a^b g^p(x)dx < \infty$. Then

$$\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}} \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\,dx\right)^{\frac{1}{p}}. \tag{1.3}$$

Hermite-Hadamard's inequality [12, p.10]**.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on interval I of real numbers and $a, b \in I$ with $a < b$. Then the following Hermite-Hadamard inequality for convex functions holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.4}$$

If the function f is concave, the inequality (1.4) can be written as follows:

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right). \tag{1.5}$$

A reversed Minkowski integral inequality (see [4, p.2]**).** Let f and g be positive functions satisfying

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \qquad (x \in [a,b]).$$

Then, putting $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$, we have

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)\,dx\right)^{\frac{1}{p}} \leq c\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}}. \tag{1.6}$$

One of the most important inequalities of Analysis is Hölder's integral inequality which is stated as follows (for its variant see [12, p.106]):

Hölder integral inequality. Let $p, q \in \mathbb{R} - \{0\}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $f, g : [a,b] \to \mathbb{R}$, $a < b$, be such that $|f(x)|^p, |g(x)|^q$ are integrable on $[a,b]$. If $p, q > 1$, then

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Remark 1. *Observe that whenever f^p is concave on $[a,b]$, the nonnegative function f is also concave on $[a,b]$. Namely,*

$$(f(ta+(1-t)b))^p \geq tf^p(a) + (1-t)f^p(b),$$

i.e.

$$(f(ta+(1-t)b)) \geq (tf^p(a) + (1-t)f^p(b))^{\frac{1}{p}}$$

and $p > 1$, using the power-mean inequality (1.2), we obtain

$$(f(ta+(1-t)b)) \geq tf(a) + (1-t)f(b).$$

For $q > 1$, if g^q is concave on $[a,b]$, the nonnegative function g is concave on $[a,b]$.

2. The Results

Theorem 1. *Let $p, q > 1$ and let $f, g : [a,b] \to \mathbb{R}$ be nonnegative functions, $a < b$ and f^p, g^q are concave functions on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2} \times \frac{g(a)+g(b)}{2} \leq \frac{1}{(b-a)^{\frac{1}{p}+\frac{1}{q}}} B(p,q) \int_a^b f(x)g(x)\,dx \tag{2.1}$$

and if $\frac{1}{p}+\frac{1}{q}=1$*, then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \le f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right). \tag{2.2}$$

Here $B(\cdot,\cdot)$ *is the Barnes-Godunova-Levin constant given by (1.1).*

Proof. Since f^p, g^q are concave functions on $[a,b]$, then from (1.5) and Remark 1, we get

$$\left(\frac{f^p(a)+f^p(b)}{2}\right)^{\frac{1}{p}} \le \frac{1}{(b-a)^{\frac{1}{p}}}\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \le f\left(\frac{a+b}{2}\right)$$

and

$$\left(\frac{g^q(a)+g^q(b)}{2}\right)^{\frac{1}{q}} \le \frac{1}{(b-a)^{\frac{1}{q}}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \le g\left(\frac{a+b}{2}\right).$$

By multiplying the above inequalities, we obtain (2.3) and (2.4)

$$\left(\frac{f^p(a)+f^p(b)}{2}\right)^{\frac{1}{p}}\left(\frac{g^q(a)+g^q(b)}{2}\right)^{\frac{1}{q}} \le \frac{1}{(b-a)^{\frac{1}{p}+\frac{1}{q}}}\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}, \tag{2.3}$$

$$\frac{1}{(b-a)^{\frac{1}{p}+\frac{1}{q}}}\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \le f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right). \tag{2.4}$$

If $p,q>1$, then it easy to show that

$$\left(\frac{f^p(a)+f^p(b)}{2}\right)^{\frac{1}{p}} \ge \frac{f(a)+f(b)}{2} \tag{2.5}$$

and

$$\left(\frac{g^q(a)+g^q(b)}{2}\right)^{\frac{1}{q}} \ge \frac{g(a)+g(b)}{2}. \tag{2.6}$$

Thus, by applying the Barnes-Godunova-Levin inequality to the right-hand side of (2.3) with (2.5) and (2.6), we get (2.1).

Applying the Hölder inequality to the left-hand side of (2.4) with $\frac{1}{p}+\frac{1}{q}=1$, we get (2.2). ∎

Theorem 2. *Let* $p\ge 1$, $0<\int_a^b f^p(x)dx<\infty$ *and* $0<\int_a^b g^p(x)dx<\infty$*, and* $f,g:[a,b]\to\mathbb{R}$ *be positive functions with*

$$0<m\le\frac{f}{g}\le M, \quad \forall x\in[a,b],\ a<b.$$

Then

$$\frac{\|f\|_p^2+\|g\|_p^2}{\|f\|_p\|g\|_p} \ge \left(\frac{1}{s}-2\right) \tag{2.7}$$

where $s=\frac{M}{(M+1)(m+1)}$.

Proof. Since f, g are positive, as in the proof of the inequality (1.6) (see [4, p.2]), we have that

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leq \frac{M}{M+1}\left(\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}}\right)$$

and

$$\left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leq \frac{1}{m+1}\left(\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}}\right).$$

By multiplying the above inequalities, we get

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leq s\left\{\left(\int_a^b (f(x)+g(x))^p\,dx\right)^{\frac{1}{p}}\right\}^2. \tag{2.8}$$

Since $\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} = \|f\|_p$ and $\left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} = \|g\|_p$, by applying the Minkowski integral inequality to the right hand side of (2.8), we obtain inequality (2.7). ∎

Theorem 3. *Let f^p and g^q be as in Theorem 1. Then the following inequality holds:*

$$(f(a)+f(b))^p\,(g(a)+g(b))^q \leq \frac{2^{(p+q)}}{(b-a)^2}\|f\|_p^p\,\|g\|_q^q.$$

Proof. If f^p, g^q are concave on $[a,b]$, then from (1.5) we get

$$\left(\frac{f^p(a)+f^p(b)}{2}\right) \leq \frac{1}{(b-a)}\left(\int_a^b f^p(x)dx\right) \leq f^p\left(\frac{a+b}{2}\right)$$

and

$$\left(\frac{g^q(a)+g^q(b)}{2}\right) \leq \frac{1}{(b-a)}\left(\int_a^b g^q(x)dx\right) \leq g^q\left(\frac{a+b}{2}\right),$$

which imply

$$\begin{aligned}&\frac{(f^p(a)+f^p(b))\,(g^q(a)+g^q(b))}{4}\\ &\qquad\leq \frac{1}{(b-a)^2}\left(\int_a^b f^p(x)dx\right)\left(\int_a^b g^q(x)dx\right).\end{aligned} \tag{2.9}$$

On the other hand, if $p, q \geq 1$, from (1.2) we get

$$\left(\frac{f^p(a)+f^p(b)}{2}\right)^{\frac{1}{p}} \geq 2^{-1}[f(a)+f(b)]$$

and

$$\left(\frac{g^q(a)+g^q(b)}{2}\right)^{\frac{1}{q}} \geq 2^{-1}[g(a)+g(b)],$$

or

$$\frac{f^p(a)+f^p(b)}{2} \geq 2^{-p}[f(a)+f(b)]^p$$

and

$$\frac{g^q(a)+g^q(b)}{2} \geq 2^{-q}[g(a)+g(b)]^q,$$

which imply

$$\begin{aligned}(2.10)\qquad &\frac{\left(f^{p}\left(a\right)+f^{p}\left(b\right)\right)\left(g^{q}\left(a\right)+g^{q}\left(b\right)\right)}{4}\\ &\geq\left(f\left(a\right)+f\left(b\right)\right)^{p}\left(g\left(a\right)+g\left(b\right)\right)^{q}2^{-\left(p+q\right)}.\end{aligned}$$

Combining (2.9) and (2.10), we obtain the desired inequality as:

$$\left(f\left(a\right)+f\left(b\right)\right)^{p}\left(g\left(a\right)+g\left(b\right)\right)^{q}2^{-\left(p+q\right)}\leq\frac{1}{\left(b-a\right)^{2}}\left\Vert f\right\Vert_{p}^{p}\left\Vert g\right\Vert_{q}^{q},$$

i.e.,

$$\left(f\left(a\right)+f\left(b\right)\right)^{p}\left(g\left(a\right)+g\left(b\right)\right)^{q}\leq\frac{2^{\left(p+q\right)}}{\left(b-a\right)^{2}}\left\Vert f\right\Vert_{p}^{p}\left\Vert g\right\Vert_{q}^{q}.$$

∎

To prove the following theorem we need the following Young type inequality (see [6, p.117]):

$$(2.11)\qquad xy\leq\frac{1}{p}x^{p}+\frac{1}{q}y^{q},\ \ x,y\geq0,\ \ \frac{1}{p}+\frac{1}{q}=1.$$

Theorem 4. *Let $f,g:[a,b]\rightarrow\mathbb{R}$ be functions such that f,g and fg are in $L_1[a,b]$, with $f(x), g(x)>1$ and*

$$0<m\leq\frac{f\left(x\right)}{g\left(x\right)}\leq M,\forall x\in\left[a,b\right],\ a,b\in\left[0,\infty\right),\ \ \frac{1}{p}+\frac{1}{q}=1\ (p,q\geq2).$$

Then

$$\begin{aligned}(2.12)\qquad \int_{a}^{b}fg&\leq c_{1}\left[\left(1+\frac{2}{p}\right)\left\Vert f\right\Vert_{p}^{p}+\left(1+\frac{2}{q}\right)\left\Vert g\right\Vert_{q}^{q}\right]\\ &\leq\frac{2M}{\left(M+1\right)\left(m+1\right)}\left[\left\Vert f\right\Vert_{p}^{p}+\left\Vert g\right\Vert_{q}^{q}\right],\end{aligned}$$

where $c_1=\frac{M}{(M+1)(m+1)}$.

Proof. From $0<m\leq\frac{f(x)}{g(x)}\leq M$, $\forall x\in[a,b]$, we have

$$f\left(x\right)\leq\frac{M}{M+1}\left(f\left(x\right)+g\left(x\right)\right)$$

and

$$g\left(x\right)\leq\frac{1}{m+1}\left(f\left(x\right)+g\left(x\right)\right).$$

Since $f(x), g(x)\geq1$, we get

$$f\left(x\right)g\left(x\right)\leq c_{1}\left(f\left(x\right)+g\left(x\right)\right)^{2}$$

or

$$(2.13)\quad \int_{a}^{b}f\left(x\right)g\left(x\right)dx\leq c_{1}\int_{a}^{b}f^{2}\left(x\right)dx+c_{1}\int_{a}^{b}g^{2}\left(x\right)dx+2c_{1}\int_{a}^{b}f\left(x\right)g\left(x\right)dx.$$

From (2.11), we obtain

$$\int_{a}^{b}f\left(x\right)g\left(x\right)dx\leq\frac{1}{p}\int_{a}^{b}f^{p}\left(x\right)dx+\frac{1}{q}\int_{a}^{b}g^{q}\left(x\right)dx.$$

If we rewrite the inequality (2.13), we have

$$\int_a^b f(x)\,g(x)\,dx \le c_1 \int_a^b f^2(x)\,dx + c_1 \int_a^b g^2(x)\,dx + 2c_1 \frac{1}{p} \int_a^b f^p(x)\,dx + 2c_1 \frac{1}{q} \int_a^b g^q(x)\,dx. \tag{2.14}$$

On the other hand , since $f^2 \le f^p,\ g^2 \le g^q$ for $p, q \ge 2$ and

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}, & 0 < p < \infty; \\ \sup |f(x)|, & p = \infty, \end{cases}$$

then we get

$$\begin{aligned} \int_a^b f(x)\,g(x)\,dx &\le c_1 \left(1 + \frac{2}{p}\right) \int_a^b f^p(x)\,dx + c_1 \left(1 + \frac{2}{q}\right) \int_a^b g^q(x)\,dx \\ &= c_1 \left(1 + \frac{2}{p}\right) \|f\|_p^p + c_1 \left(1 + \frac{2}{q}\right) \|g\|_q^q. \end{aligned} \tag{2.15}$$

This completes the proof of the first inequality in (2.12).

The second inequality in (2.12) follows from the facts that

$$1 + \frac{2}{p} \le 2, \quad p \in [2, \infty)$$

and

$$1 + \frac{2}{q} \le 2, \quad q \in [2, \infty).$$

∎

The following theorem follows from Theorem 4:

Theorem 5. *Let f, g be as in Theorem 4. Then the following inequality holds:*

$$0 \le \left(\|f\|_p^p + \|g\|_q^q\right) c_1 + \left(\frac{t^p}{p} \|f\|_p^p + \frac{t^{-q}}{q} \|g\|_q^q\right) (2c_1 - 1) \tag{2.16}$$

for $t > 0$.

Proof. Our starting point here is the identity (see [7, p.57])

$$xy = \inf_{t>0} \left(\frac{t^p}{p} x^p + \frac{t^{-q}}{q} y^q\right) \qquad (x, y \ge 0)$$

which implies

$$fg \le \frac{t^p}{p} f^p + \frac{t^{-q}}{q} g^q \tag{2.17}$$

for $t > 0$.

On the other hand, since the functions f, g are positive, from (2.13) we get

$$\int_a^b f(x)\,g(x)\,dx \le c_1 \int_a^b (f(x) + g(x))^2\,dx$$

and

$$f^2 \le f^p,\ g^2 \le g^q \Rightarrow \int_a^b f^2(x)\,dx \le \int_a^b f^p(x)\,dx$$

and

$$\int_a^b g^2(x)\,dx \le \int_a^b g^q(x)\,dx \qquad \text{for } p,q \ge 2.$$

Hence we obtain

$$0 \le c_1 \int_a^b f^p(x)\,dx + c_1 \int_a^b g^q(x)\,dx + (2c_1 - 1)\int_a^b f(x)\,g(x)\,dx. \tag{2.18}$$

If we use (2.17) in (2.18), we get the desired inequality (2.16). ∎

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