THE TWO-SIDED INEQUALITIES FOR THE EULER-MASCHERONI CONSTANT

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ABSTRACT. Let \( \gamma = 0.577215 \ldots \) be the Euler-Mascheroni constant, and let \( R_n = \sum_{k=1}^{n} \frac{1}{k} - \log (n + \frac{1}{2}) \). We prove that for all integers \( n \geq 1 \),
\[
\frac{7}{960(n+a)^4} \leq \gamma - R_n + \frac{1}{24(n+\frac{1}{2})^2} < \frac{7}{960(n+b)^4}
\]
with the best possible constants
\[
a = \frac{1}{\sqrt{\frac{960}{7} \left( \log (\frac{3}{2}) + \gamma - \frac{53}{54} \right)}} - 1 = 0.57027 \ldots \quad \text{and} \quad b = \frac{1}{2}.
\]
This refines the result of D. W. DeTemple, who proved that the two-sided inequalities hold with \( a = 1 \) and \( b = 0 \). Also, the monotonicity properties of functions related to the psi function are obtained.

1. INTRODUCTION

Euler’s constant \( \gamma = 0.57721566490153286 \ldots \) was first introduced by Leonhard Euler (1707-1783) in 1734 as
\[
\gamma = \lim_{n \to \infty} D_n, \quad \text{where} \quad D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.
\]
It is also known as the Euler-Mascheroni constant. According to Glaisher [11], the use of the symbol \( \gamma \) is probably due to the geometer Lorenzo Mascheroni (1750-1800) who used it in 1790 while Euler used the letter C. The constant \( \gamma \) is deeply related to the gamma function \( \Gamma(x) \) thanks to the Weierstrass formula
\[
\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{p=1}^{\infty} \left( 1 + \frac{x}{p} \right) e^{-x/p}.
\]
The Euler-Mascheroni constant plays an important role in Analysis (Gamma function, Bessel functions, exponential-integral, ...) and occurs frequently in Number Theory (order of magnitude of arithmetical functions for instance [12]).

Direct use of formula (1) to compute the Euler-Mascheroni constant is of poor interest since the convergence is very slow. The Euler-Maclaurin summation can be used to have a complete asymptotic expansion of the harmonic numbers \( H_n = \)
\[ \sum_{k=1}^{n} \frac{1}{k}, \quad H_n - \log n = \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}}, \]

where the \( B_{2k} \) are the Bernoulli numbers defined by

\[ \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k. \quad (2) \]

First four Bernoulli numbers with even indices are

\[ B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad (3) \]

and then

\[ \gamma = H_n - \log n - \frac{1}{2n} - \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \ldots. \]

Several bounds for \( D_n - \gamma \) have been given in the literature. We recall some of them:

\[ \frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)} \quad \text{for} \quad n \geq 2 \quad (18); \]
\[ \frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad \text{for} \quad n \geq 1 \quad (15, 21); \]
\[ \frac{1}{n} - \gamma \leq D_n - \gamma < \frac{1}{2n} \quad \text{for} \quad n \geq 1 \quad (4); \]
\[ \frac{1}{2n + \frac{2}{3}} < D_n - \gamma < \frac{1}{2n + \frac{1}{3}} \quad \text{for} \quad n \geq 1 \quad (19, 20); \]
\[ \frac{1}{2n + \frac{2\gamma - 1}{1-\gamma}} \leq D_n - \gamma < \frac{1}{2n + \frac{1}{3}} \quad \text{for} \quad n \geq 1 \quad (3, 6, 19, 20). \]

See also [16, 17].

The convergence of the sequence \( D_n \) to \( \gamma \) is very slow. In 1993, D. W. DeTemple [9] studied a modified sequence which converges faster and proved

\[ \frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \quad (4) \]
\[ \frac{7}{960(n+1)^4} < \gamma - R_n + \frac{1}{24(n+\frac{1}{2})^2} < \frac{7}{960n^4}, \quad (5) \]

where

\[ R_n = \sum_{k=1}^{n} \frac{1}{k} - \log \left( n + \frac{1}{2} \right). \]

Now let

\[ H(n) = n^2(R_n - \gamma), \quad n \geq 1. \]

Since

\[ \psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}, \]

where \( \psi = \Gamma'/\Gamma \) is the psi function, we see that

\[ H(n) = (R_n - \gamma)n^2 = \left[ \psi(n+1) - \log \left( n + \frac{1}{2} \right) \right] n^2. \]
Some computer experiments led M. Vuorinen to conjecture that \( H(n) \) increases on the interval \([1, \infty)\) from \( H(1) = -\gamma + 1 - \log(3/2) = 0.0173 \ldots \) to \( 1/24 = 0.0416 \ldots \). E. A. Karatsuba [13] proved that for all integers \( n \geq 1 \), \( H(n) < H(n+1) \), by clever use of Stirling formula and Fourier series. Some computer experiments also seem to indicate that \( [(n + 1)/n]^2 H(n) \) is a decreasing convex function [5]. The author [7] verified that for all integers \( n \geq 1 \), \( H(n) \) and \( [(n + 1)/n]^2 H(n) \) are both strictly increasing concave sequences, while \( [(n + 1)/n]^2 H(n) \) is strictly decreasing log-convex sequence.

By the asymptotic formula [2, p. 550]

\[
\psi(x) = \log \left( x - \frac{1}{2} \right) + \frac{1}{24(x - 1/2)^2} + O(x^{-4}) \quad \text{as} \quad x \to \infty, \tag{6}
\]

we conclude that

\[
\lim_{n \to \infty} H(n) = \lim_{n \to \infty} [(n + 1)/n]^2 H(n) = \lim_{n \to \infty} [(n + 1)/n]^2 H(n) = \frac{1}{24}. \tag{7}
\]

From the increasingness of \( H(n) \), decreasingness of \( [(n + 1)/n]^2 H(n) \) and (7), we obtain the inequality (4). From the increasingness of \( [(n + 1)/n]^2 H(n) \), decreasingness of \( [(n + 1)/n]^2 H(n) \) and (7), we get that

\[
\frac{1}{24(n + 1)^2} < R_n - \gamma < \frac{1}{24(n + \frac{1}{2})^2}, \quad n \geq 1. \tag{8}
\]

Obviously, the upper in (8) is sharper than one in (4). We remark that the second inequality in (8) comes out from what D. W. DeTemple [9] wrote on page 470 of the article. Also, A. Sintămărian [16] gave results for a generalization of Euler’s constant and taken \( a = 1 \) in [16, Theorem 3.1, part (iii)] we obtain the second inequality in (8).

Recently, the author [8] proved that for all integers \( n \geq 1 \), then

\[
\frac{1}{24(n + a)^2} \leq R_n - \gamma < \frac{1}{24(n + b)^2} \tag{9}
\]

with the best possible constants

\[
a = \frac{1}{\sqrt{24[-\gamma + 1 - \log(3/2)]}} = 1 = 0.55106 \ldots \quad \text{and} \quad b = \frac{1}{2}.
\]

The inequality (5) can be written as

\[
\frac{7}{960(n + 1)^4} < \log \left( n + \frac{1}{2} \right) - \psi(n + 1) + \frac{1}{24 \left( n + \frac{1}{2} \right)^2} < \frac{7}{960n^4}. \tag{10}
\]

Motivated by the inequality (10), we establish the following results.

**Theorem 1.** Let \( a \geq 0 \) be a real number and \( J_a(x) \) be defined by

\[
J_a(x) = (x + a)^4 \left[ \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24 \left( x + \frac{1}{2} \right)^2} \right]. \tag{11}
\]

Then, the functions \( J_{1/2} \) on \((-1/2, \infty)\) and \( J_0 \) on \((0, \infty)\) are strictly increasing.

**Remark 1.** By the asymptotic formula [2, p. 550]

\[
\psi(x) = \log \left( x - \frac{1}{2} \right) + \frac{1}{24(x - 1/2)^2} - \frac{7}{960(x - \frac{1}{2})^4} + O(x^{-6}) \quad \text{as} \quad x \to \infty,
\]
we conclude that
\[
\lim_{x \to \infty} J_0(x) = \lim_{x \to \infty} J_{1/2}(x) = \lim_{x \to \infty} J_1(x) = \frac{7}{960}.
\] (12)
From the increasingness of \(J_0(x)\), decreasingness of \(J_1(x)\) and (12), we obtain the inequality (5). From the increasingness of \(J_{1/2}(x)\), decreasingness of \(J_1(x)\) and (12), we get that
\[
\frac{7}{960(n + 1)^4} < \gamma - R_n + \frac{1}{24(n + \frac{1}{2})^2} \leq \frac{7}{960(n + \frac{1}{2})^4}, \quad n \geq 1.
\] (13)
Obviously, the upper in (13) is sharper than one in (5).

Recall that a function \(f\) is said to be completely monotonic on an interval \(I\) if \(f\) has derivatives of all orders on \(I\) and
\[
(-1)^n f^{(n)}(x) \geq 0
\] for \(x \in I\) and \(n \geq 0\). Dubourdien [10] pointed out that if a non-constant function \(f\) is completely monotonic, then strict inequality holds in (14). Recall that a function \(f\) is said to be a Bernstein function on an interval \(I\) if \(f > 0\) and \(f'\) is completely monotonic on \(I\).

By Theorem 1, we pose the following conjecture.

**Corollary 1.** Let \(J_n\) be defined by (11). Then, the functions \(J_{1/2}\) on \((-1/2, \infty)\) and \(J_0\) on \((0, \infty)\) are Bernstein function, while the function \(J_1\) is completely monotonic on \((0, \infty)\).

In view of the inequality (13) it is natural to ask: What is the smallest number \(a\) and what is the largest number \(b\) such that the inequality
\[
\frac{7}{960(n + a)^4} \leq \gamma - R_n + \frac{1}{24(n + \frac{1}{2})^2} \leq \frac{7}{960(n + b)^4},
\]
holds for all integers \(n \geq 1\)? The following Theorem 2 answers this question.

**Theorem 2.** For all integers \(n \geq 1\), then
\[
\frac{7}{960(n + a)^4} \leq \gamma - R_n + \frac{1}{24(n + \frac{1}{2})^2} < \frac{7}{960(n + b)^4},
\] (15)
with the best possible constants
\[
a = \sqrt{\frac{960}{7}} \left[ \log \left( \frac{3}{2} \right) + \gamma - \frac{53}{54} \right] - 1 = 0.57027 \ldots \quad \text{and} \quad b = \frac{1}{2}.
\]

2. Proofs of Theorems 1 and 2

**Proof of Theorem 1.** Define for \(x > 0\),
\[
f(x) = x^4 \left[ \log x - \psi \left( x + \frac{1}{2} \right) + \frac{1}{24x^2} \right].
\]
Using the representations [1, p. 259]
\[
\psi(x) = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt,
\] (16)
\[
\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt
\] (17)
and
\[ \frac{1}{x^2} = \int_0^\infty te^{-xt} dt, \]  
we imply
\[ f(x) = x^4 \int_0^\infty \mu(t)e^{-xt} dt, \]  
where
\[ \mu(t) = -\frac{1}{t} + \frac{1}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} + \frac{1}{24} t, \quad t > 0. \]

Easy computations reveal that
\[ f'(x) = 4 \int_0^\infty \mu(t)e^{-xt} dt - x \int_0^\infty \mu(t)e^{-xt} dt \]
\[ = 4 \int_0^\infty \mu(t)e^{-xt} dt - \int_0^\infty [\mu(t) + t\mu'(t)]e^{-xt} dt \]
\[ = \int_0^\infty [3\mu(t) - t\mu'(t)]e^{-xt} dt. \]

It is easy to see that for \( t > 0, \)
\[ 3\mu(t) - t\mu'(t) > 0 \iff -4 + \frac{3}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} + \frac{t}{12} + \frac{t(e^{\frac{t}{2}} + e^{-\frac{t}{2}})}{2(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^2} > 0 \]
\[ \iff -\frac{2}{u} + \frac{3}{2\sinh u} + \frac{u}{6} + \frac{u \cosh u}{2(\sinh u)^2} > 0 \quad \text{(where} \quad u = \frac{t}{2}) \]
\[ \iff -12(\sinh u)^2 + 9u \sinh u + u^2(\sinh u)^2 + 3u^2 \cosh u > 0. \]

Define for \( u > 0, \)
\[ g(u) = -12(\sinh u)^2 + 9u \sinh u + u^2(\sinh u)^2 + 3u^2 \cosh u. \]

Then,
\[ g(u) = -6[\cosh(2u) - 1] + 9u \sinh u + \frac{u^2[\cosh(2u) - 1]}{2} + 3u^2 \cosh u \]
\[ = \sum_{n=4}^{\infty} \frac{2n(2n-1)(2^{n-3} + 3) + 18n - 3 \cdot 2^{n+1} u^{2n}}{(2n)!} u^{2n} > 0, \quad u > 0. \]

Hence, \( f'(x) > 0 \) for \( x > 0. \) Clearly, the function
\[ J_{1/2}(x) = \left( x + \frac{1}{2} \right)^4 \left[ \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24} \left( x + \frac{1}{2} \right)^2 \right] \]
is strictly increasing on \((-\frac{1}{2}, \infty).\) It is easy to see that the function
\[ J_0(x) = \left( \frac{x}{x + \frac{1}{2}} \right)^4 J_{1/2}(x) \]
is strictly increasing on \((0, \infty).\) The proof is complete. \(\square\)
In order to prove our Theorem 2 we need to the following results [2]: For $x > \frac{1}{2}, N = 0, 1, 2, \ldots,$

$$
\log \left( x - \frac{1}{2} \right) - \sum_{k=1}^{2N} \frac{B_{2k}(1/2)}{2k(x - \frac{1}{2})^{2k}} < \psi(x) < \log \left( x - \frac{1}{2} \right) - \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2)}{2k(x - \frac{1}{2})^{2k}}
$$

(21)

and

$$
\frac{(n - 1)!}{(x - \frac{1}{2})^n} + \sum_{k=1}^{2N+1} \frac{B_{2k}(1/2) (n + 2k - 1)!}{(2k)! (x - \frac{1}{2})^{n+2k}} < (-1)^{n+1} \psi^{(n)}(x)
$$

$$
< \frac{(n - 1)!}{(x - \frac{1}{2})^n} + \sum_{k=1}^{2N} \frac{B_{2k}(1/2) (n + 2k - 1)!}{(2k)! (x - \frac{1}{2})^{n+2k}}, \quad n = 1, 2, \ldots,
$$

(22)

where

$$
B_k(1/2) = - \left( 1 - \frac{1}{2k-1} \right) B_k, \quad k = 0, 1, 2, \ldots,
$$

$B_k$ are Bernoulli numbers defined by (2). By (3) we get

$$
B_2(1/2) = -\frac{1}{12}, \quad B_4(1/2) = \frac{7}{240}, \quad B_6(1/2) = -\frac{31}{1344}, \quad B_8(1/2) = \frac{127}{3840}.
$$

From (21), we obtain for $x > \frac{1}{2},$

$$
\psi(x) - \log \left( x - \frac{1}{2} \right) < \frac{1}{24(x - \frac{1}{2})^2} - \frac{7}{960(x - \frac{1}{2})^4} + \frac{31}{8064(x - \frac{1}{2})^6}.
$$

(23)

From (22), we obtain for $x > \frac{1}{2},$

$$
\frac{1}{12(x - \frac{1}{2})^3} - \frac{7}{240(x - \frac{1}{2})^5} + \frac{31}{1344(x - \frac{1}{2})^7} - \frac{127}{3840(x - \frac{1}{2})^9} < \frac{1}{x - \frac{1}{2}} - \psi'(x).
$$

(24)

Now we are in position to prove our Theorem 2.

**Proof of Theorem 2.** The inequality (15) can be written as

$$
a \geq \frac{1}{\sqrt{\frac{960}{7} \left[ \log \left( n + \frac{1}{2} \right) - \psi(n + 1) + \frac{1}{24(n + \frac{1}{2})^2} \right]}} - n > b.
$$

In order to prove (15) we define

$$
f(x) = \frac{1}{\sqrt{\frac{960}{7} \left[ \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24(x + \frac{1}{2})^2} \right]}} - x.
$$
Differentiation yields
\[ \frac{960}{7} \left( \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24(x + \frac{1}{2})^2} \right) \]^{5/4} f'(x)
\[ = \frac{240}{7} \left( - \frac{1}{x + \frac{1}{2}} - \psi'(x + 1) + \frac{1}{12(x + \frac{1}{2})^3} \right)
\[ - \left[ \frac{960}{7} \left( \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24(x + \frac{1}{2})^2} \right) \right]^{5/4} \]
\[ < \frac{240}{7} \left[ - \frac{7}{240(x + \frac{1}{2})^6} - \frac{31}{1344(x + \frac{1}{2})^7} + \frac{127}{3840(x + \frac{1}{2})^9} \right]
\[ - \left[ \frac{960}{7} \left( \frac{7}{960(x + \frac{1}{2})^4} - \frac{31}{8064(x + \frac{1}{2})^6} \right) \right]^{5/4} \]
\[ = \frac{1}{u^6} \left[ 1 - \frac{155}{196u^2} + \frac{127}{112u^4} \right] - \left( 1 - \frac{155}{294u^2} \right)^{5/4} \],

where \( u = x + \frac{1}{2} \).

Now we show that there exists a positive real number \( x_0 \) such that \( f'(x_0) < 0 \) for \( x > x_0 \). In order to find \( x_0 \), we consider
\[ 1 - \frac{155}{196u^2} + \frac{127}{112u^4} < \left( 1 - \frac{155}{294u^2} \right)^{5/4}. \] (25)

By Bernoulli’s inequality: Let \( x \geq -1 \), then for \( \alpha < 0 \) or \( \alpha > 1 \), \( (1+x)^\alpha \geq 1 + \alpha x \), the equal sign holds if and only if \( x = 0 \), we have
\[ 1 - \frac{775}{1176u^2} < \left( 1 - \frac{155}{294u^2} \right)^{5/4}, \quad u > 0.726 \ldots \] (26)

The inequality
\[ 1 - \frac{155}{196u^2} + \frac{127}{112u^4} < 1 - \frac{775}{1176u^2} \] (27)
holds for \( u > 31.041 \ldots \), and then, \( f'(x) < 0 \) for \( x > 30.541 \ldots \) Straightforward calculation produces
\[ f(1) = 0.57027 \ldots, f(2) = 0.54774 \ldots, f(3) = 0.53564 \ldots, f(4) = 0.52830 \ldots, \]
\[ f(5) = 0.52341 \ldots, f(6) = 0.52268 \ldots, f(7) = 0.51725 \ldots, f(8) = 0.51519 \ldots, \]
\[ f(9) = 0.51376 \ldots, f(10) = 0.51246 \ldots, f(11) = 0.51139 \ldots, f(12) = 0.51049 \ldots, \]
\[ f(13) = 0.50972 \ldots, f(14) = 0.50905 \ldots, f(15) = 0.50847 \ldots, f(16) = 0.50794 \ldots, \]
\[ f(17) = 0.50751 \ldots, f(18) = 0.50705 \ldots, f(19) = 0.50674 \ldots, f(20) = 0.50615 \ldots, \]
\[ f(21) = 0.50593 \ldots, f(22) = 0.50584 \ldots, f(23) = 0.50559 \ldots, f(24) = 0.50537 \ldots, \]
\[ f(25) = 0.50513 \ldots, f(26) = 0.50496 \ldots, f(27) = 0.50476 \ldots, f(28) = 0.50459 \ldots, \]
\[ f(29) = 0.50444 \ldots, f(30) = 0.50431 \ldots, f(31) = 0.50417 \ldots \]

Thus, the sequence
\[ f(n) = \frac{1}{\sqrt{\frac{960}{7}} \left( \log \left( n + \frac{1}{2} \right) - \psi(n + 1) + \frac{1}{24(n + \frac{1}{2})^2} \right)} - n \quad (n = 1, 2, \ldots) \]
is strictly decreasing. This leads to
\[
\lim_{n \to \infty} f(n) < f(n) \leq f(1) = \frac{1}{\sqrt[4]{960} \left[ \log \left( \frac{3}{2} \right) + \gamma - \frac{53}{54} \right]} - 1 = 0.57027 \ldots \quad (28)
\]
It remains to prove that
\[
\lim_{n \to \infty} f(n) = \frac{1}{2}.
\] (29)
From the asymptotic formula [2, p. 550]
\[
\psi(x) = \log \left( x - \frac{1}{2} \right) + \frac{1}{24(x - \frac{1}{2})^2} - \frac{7}{960(x - \frac{1}{2})^4} + \frac{31}{8064(x - \frac{1}{2})^6} + O(x^{-8})
\]
as \(x \to \infty\), we obtain
\[
f(x) = \frac{1}{\sqrt[4]{960} \left[ \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24(x + \frac{1}{2})^2} \right]} - x
\]
\[
= 1 - x \sqrt[4]{960} \left[ \log \left( x + \frac{1}{2} \right) - \psi(x + 1) + \frac{1}{24(x + \frac{1}{2})^2} \right]
\]
\[
= 1 - x \sqrt[4]{1 - 155 \left( x + \frac{1}{2} \right)^2} + O(x^{-8})
\]
\[
= x + \frac{1}{2} - x \sqrt[4]{1 - \frac{155}{294(x + \frac{1}{2})^2} + O(x^{-4})}
\]
\[
= \frac{1}{2} + O(x^{-1}) \to \frac{1}{2} \quad \text{as} \quad x \to \infty,
\] and then,
\[
\lim_{n \to \infty} f(n) = \frac{1}{2}.
\] (30)
The proof is complete. \(\square\)

References


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