SOME APPLICATIONS OF FEJÉR’S INEQUALITY FOR CONVEX FUNCTIONS (II)

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Abstract. Some applications of Fejér’s inequality for convex functions are explored. Upper and lower bounds for the weighted integral
\[ \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx \]
under various assumptions for \( f \) with applications to the trapezoidal quadrature rule are given. Some inequalities for special means are derived.

1. Introduction

The Hermite-Hadamard integral inequality for convex functions \( f : [a, b] \to \mathbb{R} \)
\[(HH)\quad f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}\]
is well known in the literature and has many applications for special means.

For related results, see for instance the research papers [1], [11], [12], [13], [15], [14], [16], [17], [18], the monograph online [10] and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

**Theorem 1.** Consider the integral \( \int_a^b h(x) w(x) \, dx \), where \( h \) is a convex function in the interval \( (a, b) \) and \( w \) is a positive function in the same interval such that
\[ w(a+t) = w(b-t), \quad 0 \leq t \leq \frac{1}{2} (a+b), \]
i.e., \( y = w(x) \) is a symmetric curve with respect to the straight line which contains the point \( \left( \frac{1}{2} (a+b), 0 \right) \) and is normal to the \( x \)-axis. Under those conditions the following inequalities are valid:
\[
(1.1) \quad h \left( \frac{a+b}{2} \right) \int_a^b w(x) \, dx \leq \int_a^b h(x) w(x) \, dx \leq \frac{h(a) + h(b)}{2} \int_a^b w(x) \, dx.
\]
If \( h \) is concave on \( (a, b) \), then the inequalities reverse in (1.1).

Clearly, for \( w(x) \equiv 1 \) on \( [a, b] \) we get \( \text{HH} \).

We observe that, if we take \( w(x) = \left( x - \frac{a+b}{2} \right)^2 \), \( x \in [a, b] \), then \( w \) satisfies the conditions in Theorem 1
\[
\int_a^b \left( x - \frac{a+b}{2} \right)^2 \, dx = \frac{1}{12} (b-a)^3.
\]

1991 Mathematics Subject Classification. 26D15; 25D10.

**Key words and phrases.** Convex functions, Hermite-Hadamard inequality, Fejér’s Inequality, Special means.
and by (1.1) we have the following inequality

\[(1.2) \quad \frac{1}{12} h \left( \frac{a + b}{2} \right) (b - a)^3 \leq \int_a^b \left( x - \frac{a + b}{2} \right)^2 h(x) dx \leq \frac{h(a) + h(b)}{24} (b - a)^3, \]

that holds for any convex function \( h : [a, b] \to \mathbb{R}. \) If the function \( h \) is concave the inequalities in (1.2) reverse.

In this paper we establish amongst other results some better bounds for the weighted integral

\[ \int_a^b \left( x - \frac{a + b}{2} \right)^2 h(x) dx \]

in the case of convex functions \( h : [a, b] \to \mathbb{R}. \) We also investigate the connection with the trapezoid rule and apply some of the obtained results for special means.

For some recent inequalities concerning the weighted integral

\[ \int_a^b (b - x) (x - a) h(x) dx \]

under various assumptions for the function \( h : [a, b] \to \mathbb{R}, \) see the paper [8].

2. The Results

We start with the following equality that is of interest in itself.

**Lemma 1.** Let \( f : [a, b] \to \mathbb{C} \) be such that the derivative \( f' \) is of bounded variation on \([a, b].\) Then we have the equality

\[(2.1) \quad \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) dx \right] = \int_a^b \left( x - \frac{a + b}{2} \right)^2 df'(x), \]

where the last integral is taken in the Riemann-Stieltjes sense.

**Proof.** Since \( f' (\cdot) \) is of bounded variation and \((\cdot - \frac{a + b}{2})^2 \) is continuous on \([a, b]\) then the Riemann-Stieltjes integral from the right hand side of the equality (2.1) exists and utilizing the integration by parts rule we have

\[(2.2) \quad \int_a^b \left( x - \frac{a + b}{2} \right)^2 df'(x) \]

\[= \left[ \left( x - \frac{a + b}{2} \right)^2 f'(x) \right]_a^b \] \[- 2 \int_a^b \left( x - \frac{a + b}{2} \right) f'(x) dx \]

\[= \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - 2 \int_a^b \left( x - \frac{a + b}{2} \right) f'(x) dx. \]

By the integration by parts rule for the Riemann integral we also have

\[(2.3) \quad \int_a^b \left( x - \frac{a + b}{2} \right) f'(x) dx = \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) dx. \]

Utilising the equality (2.2) divided by 2 and the equality (2.3), we get the desired result (2.1). \( \square \)
Remark 1. If $f'$ is absolutely continuous on $[a, b]$, then the equality (2.1) becomes

\[
\frac{1}{8} (b-a)^2 \left[ f'(b) - f'(a) \right] = \frac{1}{2} \int_a^b \left( x - \frac{a+b}{2} \right)^2 f''(x) \, dx,
\]

where the second integral is taken in the Lebesgue sense. This equality was obtained in a different way in [2].

Corollary 1. If $f$ is a convex function on $[a, b]$, then we have the inequality

\[
\frac{1}{8} (b-a)^2 \left[ f'(b) - f'(a) \right] \geq \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(x) \, dx.
\]

Proof. If $f$ is convex, then the derivative exists except at a countable number of points in $[a, b]$ and is increasing. The lateral derivatives $f'_- (b)$ and $f'_+ (a)$ exist. If one is infinite then the inequality (2.5) holds trivially. If both of them are finite, then the function

\[
g(x) := \begin{cases} 
  f'_+ (a), & x = a \\
  f'_- (x), & x \in (a, b) \\
  f'_- (b), & x = b
\end{cases}
\]

is monotonic nondecreasing on $[a, b]$ and

\[
\frac{1}{8} (b-a)^2 \left[ f'_- (b) - f'_+ (a) \right] = \frac{1}{2} \int_a^b \left( x - \frac{a+b}{2} \right)^2 dg(x).
\]

Since

\[
\int_a^b \left( x - \frac{a+b}{2} \right)^2 dg(x) \geq 0,
\]

then (2.6) produces the desired result (2.5).

Remark 2. The inequality (2.5) has been obtained in a different way in [6].

Theorem 2. With the assumptions of Lemma 1 we have

\[
\frac{1}{8} (b-a)^2 \left[ f'(b) - f'(a) \right] \leq \frac{1}{8} (b-a) \sqrt[4]{b-a}. \cdot (f').
\]

Moreover, if $f'$ is Lipschitzian with the constant $L > 0$, then

\[
\frac{1}{8} (b-a)^2 \left| f'(b) - f'(a) \right| \leq \frac{1}{48} L (b-a)^2.
\]
Proof. It is known that if \( p : [c, d] \to \mathbb{C} \) is a continuous function and \( v : [c, d] \to \mathbb{C} \) is of bounded variation, then the Riemann-Stieltjes integral \( \int_{c}^{d} p(t) \, dv(t) \) exists and the following inequality holds
\[
\left| \int_{c}^{d} p(t) \, dv(t) \right| \leq \max_{t \in [c, d]} |p(t)| \int_{c}^{d} |v'(t)| dt
\]
where \( |v'(t)| \) denotes the total variation of \( v \) on \([c, d]\).

Utilising this property we have
\[
\left| \int_{a}^{b} \left( x - \frac{a + b}{2} \right)^2 df'(x) \right| \leq \sup_{x \in [a, b]} \left( x - \frac{a + b}{2} \right)^2 \int_{a}^{b} |f'(x)| \text{dx} = \frac{1}{4} (b - a)^2 \int_{a}^{b} |f'(x)| \text{dx}
\]
and by the equality (2.1) we get (2.7).

It is well known that if \( p : [a, b] \to \mathbb{C} \) is a Riemann integrable function and \( v : [a, b] \to \mathbb{C} \) is Lipschitzian with the constant \( M > 0 \), i.e.,
\[
|v(s) - v(t)| \leq M |s - t| \quad \text{for any } t, s \in [a, b],
\]
then the Riemann-Stieltjes integral \( \int_{a}^{b} p(t) \, dv(t) \) exists and the following inequality holds
\[
\left| \int_{a}^{b} p(t) \, dv(t) \right| \leq M \int_{a}^{b} |p(t)| \, dt.
\]

Utilizing this property we have
\[
\left| \int_{a}^{b} \left( x - \frac{a + b}{2} \right)^2 df'(x) \right| \leq L \int_{a}^{b} \left( x - \frac{a + b}{2} \right)^2 \, dx = \frac{1}{12} L (b - a)^3
\]
and by the equality (2.1) we get (2.8). \( \square \)

Now, when some convexity property is assumed for the second derivative, then following result holds.

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\) and such that the second derivative \( f'' \) is convex on \((a, b)\). Then

\[
(2.9) \quad \frac{1}{24} f'' \left( \frac{a + b}{2} \right) (b - a)^3 \leq \frac{1}{8} (b - a)^2 \left[ f'(b) - f'(a) \right] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_{a}^{b} f(x) \, dx \right] \leq \frac{f''(a) + f''(b)}{48} (b - a)^3.
\]
Proof. We know from (2.4) that
\begin{equation}
\frac{1}{8} (b-a)^2 \left[ f''(b) - f''(a) \right] - \frac{f(a) + f(b)}{2} (b-a) - \int_a^b f(x) \, dx
\end{equation}
\begin{align*}
&= \frac{1}{2} \int_a^b \left( x - \frac{a+b}{2} \right)^2 f''(x) \, dx.
\end{align*}

Since \( f'' \) is convex on \((a, b)\), then by (1.2) we have
\begin{equation}
\frac{1}{12} f'' \left( \frac{a+b}{2} \right) (b-a)^3 \leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f''(x) \, dx
\end{equation}
\begin{align*}
&\leq \frac{f''(a) + f''(b)}{24} (b-a)^3.
\end{align*}

Utilising (2.10) and (2.11) we deduce the desired result (2.9).

\[\square\]

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\).

If there exists a real number \( m \) such that \( f''(x) \geq m \) for any \( x \in (a, b) \), then
\begin{equation}
\frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{180} m (b-a)^5
\end{equation}
\begin{align*}
&\leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx
\end{align*}
\begin{align*}
&\leq \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{7}{1440} m (b-a)^5.
\end{align*}

If there exists a real number \( M \) such that \( f''(x) \leq M \) for any \( x \in (a, b) \), then
\begin{equation}
\frac{f(a) + f(b)}{24} (b-a)^3 - \frac{7}{1440} M (b-a)^5
\end{equation}
\begin{align*}
&\leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx
\end{align*}
\begin{align*}
&\leq \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{180} M (b-a)^5.
\end{align*}

**Proof.** Define the function \( h_m : [a, b] \to \mathbb{R} \) by
\[ h_m(x) := f(x) - \frac{1}{2} m \left( x - \frac{a+b}{2} \right)^2. \]

This function is twice differentiable and the second derivative is
\[ h''_m(x) = f''(x) - m \geq 0, \quad x \in (a, b) \]
showing that \( h_m \) is convex on \([a, b]\).
If we apply the inequality (1.2) for $h_m$, then we have

\begin{align*}
(2.14) \quad & \frac{1}{12} f\left(\frac{a+b}{2}\right) (b-a)^3 \\
& \leq \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x) \, dx - \frac{1}{2} m \int_a^b \left(x - \frac{a+b}{2}\right)^4 \, dx \\
& \leq \frac{f(a) - \frac{1}{2} m (b-a)^2}{24} + f(b) - \frac{1}{2} m (b-a)^2 (b-a)^3 \\
& = \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{1}{60} m (b-a)^5.
\end{align*}

We also have

\[ \int_a^b \left(x - \frac{a+b}{2}\right)^4 \, dx = \frac{1}{90} (b-a)^5. \]

Then (2.14) becomes

\[ \frac{1}{12} f\left(\frac{a+b}{2}\right) (b-a)^3 + \frac{1}{180} m (b-a)^5 \]

\[ \leq \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x) \, dx \\
\leq \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{1}{96} m (b-a)^5 + \frac{1}{180} m (b-a)^5 \\
= \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{7}{1440} m (b-a)^5
\]

which is equivalent with (2.12).

Now define the function $h_M : [a, b] \to \mathbb{R}$ by

\[ h_M(x) := \frac{1}{2} M \left(x - \frac{a+b}{2}\right)^2 - f(x). \]

This function is twice differentiable and

\[ h_M''(x) := M - f''(x) \geq 0, \ x \in (a, b) \]

showing that $h_M$ is convex on $[a,b]$.

If we apply the inequality (1.2) for $h_M$, then we have

\[ \frac{1}{12} \left[-f\left(\frac{a+b}{2}\right)\right] (b-a)^3 \]

\[ \leq \int_a^b \left(x - \frac{a+b}{2}\right)^2 \left[\frac{1}{2} M \left(x - \frac{a+b}{2}\right)^2 - f(x)\right] \, dx \\
\leq \frac{1}{8} M (b-a)^2 - f(a) + \frac{1}{8} M (b-a)^2 - f(b) \frac{1}{24} (b-a)^3,
\]
which, by multiplication with $-1$, produces

\[
\frac{1}{12} f \left(\frac{a+b}{2}\right) (b-a)^3 \\
\geq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx - \frac{1}{180} M (b-a)^5 \\
\geq \frac{f(a) + f(b) - \frac{1}{4} M (b-a)^2}{24} (b-a)^3 \\
= \frac{f(a) + f(b)}{24} - \frac{1}{96} M (b-a)^5
\]

that is equivalent with

\[
\frac{f(a) + f(b)}{24} - \frac{1}{96} M (b-a)^5 + \frac{1}{180} M (b-a)^5 \\
\leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx \\
\leq \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{180} M (b-a)^5
\]

and the inequality (2.13) is proved. \qed

Corollary 2. Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\). If there exists a \( K > 0 \) such that \( |f''(x)| \leq K \) for any \( x \in (a, b) \), then

\[
\left(2.15\right) \quad \left| \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx - \frac{1}{24} \left[ f \left( \frac{a+b}{2} \right) + f(a) + f(b) \right] (b-a)^3 \right| \\
\leq \frac{1}{192} K (b-a)^5.
\]

Proof. If we write the inequality (2.12) for \( m = -K \) and the inequality (2.13) for \( M = K \), then we have

\[
\left(2.16\right) \quad \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 - \frac{1}{180} K (b-a)^5 \\
\leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx \\
\leq \frac{f(a) + f(b)}{24} (b-a)^3 + \frac{7}{1440} K (b-a)^5,
\]

and

\[
\left(2.17\right) \quad \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{7}{1440} K (b-a)^5 \\
\leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) \, dx \\
\leq \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{180} K (b-a)^5.
\]

If we add the inequality (2.16) with (2.16) and divide the sum by 2 we get the desired result (2.15). \qed
Remark 3. We observe that the case $m > 0$ in the inequality (2.12) produces a better result than (1.2).

For twice differentiable functions we can provide the following perturbed trapezoid quadrature rule

\[
\int_a^b f(x) \, dx \approx \frac{f(a) + f(b)}{2} (b - a) - \frac{1}{8} (b - a)^2 [f'(b) - f'(a)] \\
+ \frac{1}{24} (b - a)^3 \left[ f''\left(\frac{a + b}{2}\right) + \frac{f''(a) + f''(b)}{2} \right].
\]

Denote $E_{P,T}(f; a, b)$ the error in approximating the integral as in (2.18), namely

\[
E_{P,T}(f; a, b) := \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} (b - a) + \frac{1}{8} (b - a)^2 [f'(b) - f'(a)] \\
- \frac{1}{24} (b - a)^3 \left[ f''\left(\frac{a + b}{2}\right) + \frac{f''(a) + f''(b)}{2} \right].
\]

The following result that provides an a priori error bound for functions whose fourth derivatives are bounded, holds.

Proposition 1. Let $f : [a, b] \to \mathbb{R}$ be a four time differentiable function on $(a, b)$. If there exists a $K > 0$ such that $|f^{(4)}(x)| \leq K$ for any $x \in (a, b)$, then

\[
|E_{P,T}(f; a, b)| \leq \frac{1}{384} K (b - a)^5.
\]

Proof. Writing the inequality (2.15) for the second derivative $f''$ we have

\[
\left| \int_a^b \left( x - \frac{a + b}{2} \right)^2 f''(x) \, dx - \frac{1}{24} \left[ f''\left(\frac{a + b}{2}\right) + \frac{f''(a) + f''(b)}{2} \right] (b - a)^3 \right| \\
\leq \frac{1}{192} K (b - a)^5.
\]

Dividing this inequality by 2 and utilizing the representation (2.10) we have

\[
\left| \frac{1}{8} (b - a)^2 [f'(b) - f'(a)] - \left[ \frac{f(a) + f(b)}{2} (b - a) - \int_a^b f(x) \, dx \right] \\
- \frac{1}{48} (b - a)^3 \left[ f''\left(\frac{a + b}{2}\right) + \frac{f''(a) + f''(b)}{2} \right] \right| \\
\leq \frac{1}{384} K (b - a)^5,
\]

and the inequality (2.19) is proved. \qed

3. Applications for Special Means

Let us recall the following means for two positive numbers.

(1) The Arithmetic mean

\[ A = A(a, b) := \frac{a + b}{2}, \ a, b > 0; \]

(2) The Geometric mean

\[ G = G(a, b) := \sqrt{ab}, \ a, b > 0; \]
(3) The Harmonic mean
\[ H = H(a, b) := \frac{2ab}{a+b}, \quad a, b > 0; \]

(4) The Logarithmic mean
\[ L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b; \end{cases} \quad a, b > 0; \]

(5) The Identric mean
\[ I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left( \frac{b^p}{a^q} \right)^{\frac{1}{p-q}} & \text{if } a \neq b \end{cases} \quad a, b > 0; \]

(6) The \( p \)-Logarithmic mean
\[ L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases} \quad a, b > 0. \]

The following inequality is well known in the literature:
\[
H \leq G \leq L \leq I \leq A.
\]

It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_0 = I \) and \( L_{-1} = L \).

Consider the function \( f : [a, b] \subset (0, \infty) \rightarrow (0, \infty) \), \( f(x) = x^p \) for \( p \geq 3 \). We have the fourth derivative of the function given by
\[
f^{(4)}(x) = p(p-1)(p-2)(p-3)x^{p-4},
\]
which shows that the second derivative \( f'' \) is convex on \([a, b]\). Applying the inequality (2.9) we have
\[
\frac{p(p-1)}{24} A^{p-2} (a, b) (b-a)^2 \\
\leq \frac{1}{8} p(p-1)(b-a)^2 L_{p-2} (a, b) - A(a^p, b^p) + L_p (a, b) \\
\leq \frac{1}{24} p(p-1) A(a^{p-2}, b^{p-2}) (b-a)^2
\]
that holds for any \( a, b > 0 \) and \( p \geq 3 \).

Consider the function \( f : [a, b] \subset (0, \infty) \rightarrow (0, \infty) \), \( f(x) = \frac{1}{x} \). Then \( f''(x) = \frac{2}{x^3} \) and \( f^{(4)}(x) = \frac{24}{x^5} \) showing that the second derivative is convex on \([a, b]\). Applying the inequality (2.9) we have
\[
\frac{1}{12} \left( \frac{a+b}{2} \right)^{-3} (b-a)^3 \\
\leq \frac{1}{8} (b-a)^3 \left( \frac{a+b}{a^2b^2} \right) - \left[ \frac{1}{2} + \frac{1}{2} \right] (b-a) - (\ln b - \ln a) \\
\leq \frac{1}{24} + \frac{1}{24} (b-a)^3.
\]
Dividing by \( b - a > 0 \) we have

\[
\frac{1}{12} A^{-3} (a, b) (b - a)^2 \\
\leq \frac{1}{4} (b - a)^2 \frac{A(a, b)}{G^4(a, b)} - H^{-1}(a, b) + L^{-1}(a, b) \\
\leq \frac{1}{12} H^{-1}(a^3, b^3) (b - a)^2 ,
\]

that holds for any \( a, b > 0 \).

Consider the function \( f : [a, b] \subset (0, \infty) \to (0, \infty) \) , \( f(x) = - \ln x \) . Then \( f''(x) = \frac{1}{x} \) and \( f^{(4)}(x) = \frac{4}{x^2} \) showing that the second derivative is convex on \([a, b]\) . Applying the inequality (2.9) we have

\[
\frac{1}{24} \left( \frac{a + b}{2} \right)^{-2} (b - a)^3 \\
\leq \frac{1}{8} (b - a)^2 \left( \frac{b - a}{ab} \right) + \frac{\ln a + \ln b}{2} (b - a) - \int_a^b \ln x dx \\
\leq \frac{\ln a + \ln b}{48} (b - a)^3 .
\]

Dividing by \( b - a > 0 \) we have

\[
\frac{1}{24} \left( \frac{a + b}{2} \right)^{-2} (b - a)^2 \\
\leq \frac{1}{8} (b - a)^2 \frac{1}{ab} + \frac{\ln a + \ln b}{2} - \frac{1}{(b - a)} \int_a^b \ln x dx \\
\leq \frac{\ln a + \ln b}{48} (b - a)^2 .
\]

Observe that

\[
\frac{1}{b - a} \int_a^b \ln x dx = \frac{1}{b - a} \left[ x \ln x \big|_a^b - (b - a) \right] = \\
= \left[ \ln \left( \frac{b^b}{a^a} \right)^{1/(b-a)} - 1 \right] = \ln I(a, b) ,
\]

and

\[
\frac{\ln a + \ln b}{2} = \ln G(a, b) .
\]

Then we get

\[
\frac{1}{24} A^{-2} (a, b) (b - a)^2 \\
\leq \frac{1}{8} (b - a)^2 G^{-2}(a, b) + \ln G(a, b) - \ln I(a, b) \\
\leq \frac{1}{24} H^{-1}(a^2, b^2) (b - a)^2
\]

that holds for any \( a, b > 0 \).

The interested reader may apply the inequality (2.19) to obtain other similar results. However, the details are omitted here.
References


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