ON INVARIANCE EQUATION FOR MEANS OF POWER GROWTH

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Abstract. We discuss properties of the solutions of the invariance equation

\[ M(N(x,y), K(x,y)) = M(x,y) \]

for homogeneous, symmetric means \( M, N, K \) of power growth.

By a mean we understand a function \( M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) satisfying for all \( x, y \in \mathbb{R}_+ \) the conditions

\[ \min(x,y) \leq M(x,y) \leq \max(x,y). \]

A mean is symmetric if for all \( x, y \) holds \( M(x, y) = M(y, x) \) and homogeneous if \( M(tx, ty) = tM(x, y) \) for all positive \( t \). Given two means \( N, K \), finding another mean \( M \) satisfying for all \( x, y \) the equation

\[ M(N(x,y), K(x,y)) = M(x,y) \]

is called the invariance problem, and the equation (1) is called invariance equation. There is a vast literature on the subject. The book "Pi and the AGM" ([3]) gives many examples and discusses probably the best known case of the arithmetic-geometric mean, while the historical overview and information on recent developments can be found in [1] and in [2].

The solution to the invariance equation is known to exist in most cases, so it is quite natural to ask whether the solution shares some properties of means \( N \) and \( K \). Sometimes the answer is immediate: if both \( K \) and \( N \) are symmetric, then obviously \( M \) is symmetric too. Sometimes it is not obvious and surprising: if \( K \) and \( N \) are homogeneous, then \( M \) need not be homogeneous.

Ádám Besenyei proved in [1] that in the class of Heinz means

\[ H_p(x,y) = \frac{x^p y^{1-p} + y^p x^{1-p}}{2}, \quad 0 \leq p \leq \frac{1}{2} \]

the invariance equation

\[ H_p(H_q(x,y), H_r(x,y)) = H_p(x,y) \]

has only trivial solutions \( p = q = r \). The aim of this note is to extend this result to a much broader class of means.

**Definition 1.** We say that a homogeneous, symmetric mean \( M \) is of power growth if there exist a real number \( \text{ord}(M) \) and a positive number \( C_M \) such that

\[ \lim_{x \to 0} \frac{M(x,1)}{x^{\text{ord}(M)}} = C_M. \]

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We shall call \( \text{ord}(M) \) the order of \( M \).

Observe that for \( 0 < x < 1 \) the inequality \( x \leq M(x, 1) \leq 1 \) yields \( x^{1-m} \leq \frac{M(x, 1)}{x^m} \leq x^{-m} \). For \( m < 0 \) the right-hand side tends to 0 and for \( m > 1 \) the left-hand side tends to infinity. Thus we conclude that \( 0 \leq \text{ord}(M) \leq 1 \).

**Theorem 1.** Let \( M, N, K \) be symmetric, homogeneous means on \( \mathbb{R}^2_+ \) of power growth. Assume additionally \( \text{ord}(N) \geq \text{ord}(K) \) and

\[
C_N^{\text{ord}(M)} C_K^{1 - \text{ord}(M)} \neq 1 \quad \text{or} \quad \text{ord}(M)(1 - \text{ord}(N) + \text{ord}(K)) \neq \text{ord}(K).
\]

If

\[
M(N(x, y), K(x, y)) = M(x, y),
\]

then \( \text{ord}(M) = \text{ord}(N) = \text{ord}(K) \).

**Proof.** Denote the orders of \( M, N, K \) by \( m, n, k \) respectively. Suppose first that \( n > k \). We have

\[
\frac{M(x, 1)}{x^m} = \frac{M(N(x, 1), K(x, 1))}{x^m} = x^{-m} K(x, 1) M\left(\frac{N(x, 1)}{K(x, 1)}, 1\right)
\]

\[
= x^{-m} K(x, 1) \left(\frac{N(x, 1)}{K(x, 1)}\right)^m \frac{M\left(\frac{N(x, 1)}{K(x, 1)}, 1\right)}{\left(\frac{N(x, 1)}{K(x, 1)}\right)^m}
\]

\[
= x^{-m + mn + (1-m)k} \left(\frac{N(x, 1)}{x^n}\right)^m \left(\frac{K(x, 1)}{x^k}\right)^{1-m} M\left(\frac{N(x, 1)}{K(x, 1)}, 1\right)^m.
\]

Since \( \frac{N(x, 1)}{K(x, 1)} \) tends to 0 as \( x \) tends to 0, we obtain a contradiction: the limit of the left-hand side equals \( C_M \) while the right-hand side tends to 0 or infinity (in case \( \text{ord}(M)(1 - \text{ord}(N) + \text{ord}(K)) \neq \text{ord}(K) \)), or to \( C_M C_K^{1-m} C_M \neq C_M \).

Therefore we conclude that \( n = k \). But this implies

\[
M\left(\frac{N(x, 1)}{x^n}, \frac{K(x, 1)}{x^n}\right) = M\left(\frac{N(x, 1)}{x^n}, K(x, 1)\right) = \frac{M(x, 1)}{x^n}
\]

and since the left-hand side remains bounded and separated from 0 for small \( x \) we conclude that \( n = m \).

\( \square \)

It is worth observing that the Heinz means are linked to the arithmetic mean by the formula \( H_\alpha(x, y) = A(x^\alpha y^{1-\alpha}, x^{1-\alpha} y^\alpha) \). Clearly, we can apply the same method to an arbitrary homogeneous, symmetric mean \( M \) thus obtaining a one-parameter family of means interpolating between \( M \) and the geometric mean. The following theorem deals with one-parameter families created this way.

**Theorem 2.** Let \( M \) be a symmetric, homogeneous mean of order \( \text{ord}(M) \neq \frac{1}{2} \) with \( C_M \neq 1 \) and let

\[
M_\alpha(x, y) = M(x^\alpha y^{1-\alpha}, x^{1-\alpha} y^\alpha) \quad \text{for } 0 \leq \alpha \leq \frac{1}{2}.
\]

Then the invariance equation

\[
M_\alpha(M_\beta(x, y), M_\gamma(x, y)) = M_\alpha(x, y)
\]

admits only trivial solutions \( \alpha = \beta = \gamma \).
Proof. The identity
\[ M_\alpha(x, 1) = M(x^\alpha, x^{1-\alpha}) = x^\alpha M(1, x^{1-2\alpha}) = x^{\alpha + \text{ord}(M)(1 - 2\alpha)} \frac{M(1, x^{1 - 2\alpha})}{x^{\text{ord}(M)(1 - 2\alpha)}} , \]
implies that
\[ C_{M_\alpha} = \begin{cases} C_M & \alpha < \frac{1}{2}, \\ 1 & \alpha = \frac{1}{2}, \text{ and } \text{ord}(M_\alpha) = \alpha + \text{ord}(M)(1 - 2\alpha) \end{cases} \]
thus the means in the family are of different order and the result would follow from Theorem 1 once we verify the condition (4). To this end assume \( \beta \leq \gamma \).
Consider two cases:

Case 1: \( \text{ord}(M) < \frac{1}{2} \)
The function \( \delta \to \text{ord}(M_\delta) \) increases from \( \text{ord}(M) \) to \( \frac{1}{2} \), so \( \text{ord}(M_\delta) \leq \text{ord}(M_\gamma) \) and \( C_{M_\beta}^{1 - \text{ord}(M_\beta)} C_{M_\gamma}^{\text{ord}(M_\gamma)} = 1 \) is possible only if \( C_{M_\beta} = C_{M_\gamma} = 1 \) (which is equivalent to \( \beta = \gamma = \frac{1}{2} \)) or \( C_{M_\beta} = 1 \) and \( 1 - \text{ord}(M_\beta) = 0 \). The first case gives immediately \( \alpha = \frac{1}{2} \), while the second case is impossible, as \( 1 - \text{ord}(M_\alpha) \geq \frac{1}{2} \).

Case 2: \( \text{ord}(M) > \frac{1}{2} \)
Now \( \text{ord}(M_\delta) \) decreases from \( \text{ord}(M) \) to \( \frac{1}{2} \), so \( \text{ord}(M_\beta) \geq \text{ord}(M_\gamma) \) and the equality \( C_{M_\beta}^{\text{ord}(M_\beta)} C_{M_\gamma}^{1 - \text{ord}(M_\gamma)} = 1 \) can hold only if \( C_{M_\beta} = C_{M_\gamma} = 1 \) or \( C_{M_\beta} = 1 \) and \( \text{ord}(M_\beta) = 0 \). Again, the first case leads to \( \alpha = \frac{1}{2} \), while the second case is impossible, as \( \text{ord}(M_\alpha) \geq \frac{1}{2} \).

Applying Theorem 2 to the arithmetic mean we obtain the result of Besenyei.

Corollary 1 ([1], Theorem 4). In the class of Heinz means (2) the identity (3) holds if and only if \( p = q = r \).

As an application of Theorem 1 consider the following families of means:

\[ Q_s(x, y) = G \frac{2}{x} (x, y) E^{\frac{1}{s-2}} (s - 1, 1; x, y) \]
and

\[ H^1_s(x, y) = G \frac{2}{x} (x, y) E^{\frac{1}{s-2}} (1 - 1/s, 1/s; x, y), \]
where \( G \) is the geometric mean, \( E(p, q; x, y) = \left( \frac{q x^p - y^p}{p x^p - y^p} \right)^{1/(p - q)} \) is the Stolarsky mean and \( s \geq 2 \). Note that

\[ Q_n(x, y) = \left( \frac{x^{n-1} y + x^{n-2} y^2 + \cdots + x y^{n-1}}{n - 1} \right)^{1/n} \]
and

\[ H^1_n(x, y) = \frac{x^{n-1} y^{\frac{1}{n}} + \cdots + x^{\frac{1}{n}} y^{n-1}}{n - 1}. \]

We see that \( \text{ord}(Q_s) = \text{ord}(H^1_s) = 1 - 1/s, C_{Q_s} = (s - 1)^{-1/s} \) and \( C_{H^1_s} = (s - 1)^{-1} \). By Theorem 1 the invariance equations admit only trivial solutions in the two families. (The assumption (4) does not hold if \( N = K = Q_2 \) or \( N = K = H^1_2 \), but in this case triviality of the solution of the invariance equation follows immediately).
References


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