SCHUR-CONVEXITY, SCHUR-GEOMETRIC AND HARMONIC CONVEXITIES OF DUAL FORM OF A CLASS SYMMETRIC FUNCTIONS

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ABSTRACT. By properties of Schur-convex function, Schur-geometrically convex function and Schur-harmonically convex function, Schur-convexity, Schur-geometric and harmonic convexities of the dual form for a class of symmetric functions are simply proved. As an application, several inequalities are obtained, some of which extend the known ones.

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1. INTRODUCTION

Throughout the article, \( \mathbb{R} \) denotes the set of real numbers, \( x = (x_1, x_2, \ldots, x_n) \) denotes \( n \)-tuple \((n\text{-dimensional real vectors})\), the set of vectors can be written as

\[
\mathbb{R}^n = \{ x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n \},
\]

\[
\mathbb{R}_+^n = \{ x = (x_1, \ldots, x_n) : x_i > 0, i = 1, \ldots, n \}.
\]

In particular, the notations \( \mathbb{R} \) and \( \mathbb{R}_+ \) denote \( \mathbb{R}^1 \) and \( \mathbb{R}_+^1 \) respectively. For convenience, we introduce some definitions as follows.

Definition 1. \([1, 2]\) Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \).
(i) \( x \geq y \) means \( x_i \geq y_i \) for all \( i = 1, 2, \ldots, n \).
(ii) Let \( \Omega \subset \mathbb{R}_+^n \), \( \varphi : \Omega \to \mathbb{R} \) is said to be increasing if \( x \geq y \) implies \( \varphi(x) \geq \varphi(y) \). \( \varphi \) is said to be decreasing if and only if \( -\varphi \) is increasing.

Definition 2. \([1, 2]\) Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \).
(i) \( x \) is said to be majorized by \( y \) (in symbols \( x \prec y \)) if \( \sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \) for \( k = 1, 2, \ldots, n - 1 \) and \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \), where \( x[1] \geq \cdots \geq x[n] \) and \( y[1] \geq \cdots \geq y[n] \) are rearrangements of \( x \) and \( y \) in a descending order.
(ii) Let \( \Omega \subset \mathbb{R}_+^n \), \( \varphi : \Omega \to \mathbb{R} \) is said to be a Schur-convex function on \( \Omega \) if \( x \prec y \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a Schur-concave function on \( \Omega \) if and only if \( -\varphi \) is Schur-convex function on \( \Omega \).

Definition 3. \([1, 2]\) Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \).
(i) \( \Omega \subset \mathbb{R}_+^n \) is said to be a convex set if \( x, y \in \Omega, 0 \leq \alpha \leq 1 \) implies \( \alpha x + (1 - \alpha) y = (\alpha x_1 + (1 - \alpha) y_1, \ldots, \alpha x_n + (1 - \alpha) y_n) \in \Omega \).
(ii) Let \( \Omega \subset \mathbb{R}_+^n \) be convex set. A function \( \varphi : \Omega \to \mathbb{R} \) is said to be a convex function on \( \Omega \) if
\[
\varphi(\alpha x + (1 - \alpha) y) \leq \alpha \varphi(x) + (1 - \alpha) \varphi(y)
\]
for all \( x, y \in \Omega \), and all \( \alpha \in [0, 1] \). \( \varphi \) is said to be a concave function on \( \Omega \) if and only if \( -\varphi \) is convex function on \( \Omega \).

(iii) Let \( \Omega \subseteq \mathbb{R}^n \). A function \( \varphi : \Omega \to \mathbb{R} \) is said to be a log-convex function on \( \Omega \) if function \( \ln \varphi \) is convex.

Theorem A. (Schur-Convex Function Decision Theorem)[1, p. 5]: Let \( \Omega \subseteq \mathbb{R}^n \) is symmetric and has a nonempty interior convex set. \( \Omega^0 \) is the interior of \( \Omega \). \( \varphi : \Omega \to \mathbb{R} \) is continuous on \( \Omega \) and differentiable in \( \Omega^0 \). Then \( \varphi \) is the Schur-convex (Schur-concave) function, if and only if \( \varphi \) is symmetric on \( \Omega \) and

\[
(x_1 - x_2) \left( \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\leq 0)
\]

holds for any \( x \in \Omega^0 \).

Definition 4. [3] Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}_+^n \).

(i) \( \Omega \subseteq \mathbb{R}_+^n \) is called a geometrically convex set if \( (x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega \) for all \( x, y \in \Omega \) and \( \alpha, \beta \in [0, 1] \) such that \( \alpha + \beta = 1 \).

(ii) Let \( \Omega \subseteq \mathbb{R}_+^n \). The function \( \varphi : \Omega \to \mathbb{R}_+ \) is said to be Schur-geometrically convex function on \( \Omega \) if \( (\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n) \) on \( \Omega^0 \) implies \( \varphi(x) \leq \varphi(y) \). The function \( \varphi \) is said to be a Schur-geometrically concave function on \( \Omega \) if and only if \( -\varphi \) is Schur-geometrically convex function.

Theorem B. (Schur-Geometrically Convex Function Decision Theorem)[3]: Let \( \Omega \subseteq \mathbb{R}_+^n \) be a symmetric and geometrically convex set with a nonempty interior \( \Omega^0 \). Let \( \varphi : \Omega \to \mathbb{R}_+ \) be continuous on \( \Omega \) and differentiable in \( \Omega^0 \). If \( \varphi \) is symmetric on \( \Omega \) and

\[
(\log x_1 - \log x_2) \left( x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0 \quad (\leq 0) \tag{2}
\]

holds for any \( x = (x_1, \ldots, x_n) \in \Omega^0 \), then \( \varphi \) is a Schur-geometrically convex (Schur-geometrically concave) function.

Definition 5. [4] Let \( \Omega \subseteq \mathbb{R}_+^n \).

(1) A set \( \Omega \) is said to be harmonically convex if \( \frac{x^y}{\lambda x_1^{\lambda - 1} y_1} \in \Omega \) for every \( x, y \in \Omega \) and \( \lambda \in [0, 1] \), where \( x^y = \sum_{i=1}^n x_i y_i \) and \( \frac{1}{\lambda} = (\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}) \).

(2) A function \( \varphi : \Omega \to \mathbb{R}_+ \) is said to be Schur-harmonically convex on \( \Omega \) if \( \frac{1}{x} \prec \frac{1}{y} \) implies \( \varphi(x) \leq \varphi(y) \).

Theorem C. (Schur-Harmonically Convex Function Decision Theorem)[4]: Let \( \Omega \subseteq \mathbb{R}_+^n \) be a symmetric and harmonically convex set with inner points and let \( \varphi : \Omega \to \mathbb{R}_+ \) be a continuously symmetric function which is differentiable on \( \Omega^0 \). Then \( \varphi \) is Schur-harmonically convex function on \( \Omega \) if and only if

\[
(x_1 - x_2) \left( x_1^2 \frac{\partial \varphi(x)}{\partial x_1} - x_2^2 \frac{\partial \varphi(x)}{\partial x_2} \right) \geq 0 \quad (\leq 0), \quad x \in \Omega^0.
\]

Let interval \( I \subseteq \mathbb{R} \) and let \( \varphi : I \to \mathbb{R}_+ \) be a log-convex function. Define the symmetric function \( F_k \) by

\[
F_k(x) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^k f(x_{i_j}), \quad k = 1, \ldots, n.
\]

\[\text{(4)}\]
In 2010, for 1, 2 and \( n - 1 \), I. Roventa [5] proved that \( F_k(x) \) is a Schur-convex function on \( I^n \), but without discuss the case of \( 2 < k < n - 1 \). In 2011, Shu-hong Wang et al. [6] studied completely Schur convexity, Schur geometric and harmonic convexities of \( F_k(x) \) on \( I^n \), using the above decision theorems, i.e. Theorem A, Theorem B and Theorem C respectively to prove the following three theorems.

**Theorem D.** Let \( I \subset \mathbb{R} \) is a symmetric convex set with non-empty interior and let \( f : I \rightarrow \mathbb{R} \) be continuous on \( I \) and differentiable in the interior of \( I \). If \( f \) is a log-convex function, then for any \( k = 1, 2, \ldots, n \), \( F_k(x) \) is a Schur-convex function on \( I^n \).

**Theorem E.** Let \( I \subset \mathbb{R}_+ \) is a symmetric convex set with non-empty interior and let \( f : I \rightarrow \mathbb{R}_+ \) be continuous on \( I \) and differentiable in the interior of \( I \). If \( f \) is an increasing log-convex function, then for any \( k = 1, 2, \ldots, n \), \( F_k(x) \) is a Schur-geometrically convex function on \( I^n \).

**Theorem F.** Let \( I \subset \mathbb{R}_+ \) is a symmetric convex set with non-empty interior and let \( f : I \rightarrow \mathbb{R}_+ \) be continuous on \( I \) and differentiable in the interior of \( I \). If \( f \) is an increasing log-convex function, then for any \( k = 1, 2, \ldots, n \), \( F_k(x) \) is a Schur-harmonically convex function on \( I^n \).

In this paper, we study the dual form of \( F_k(x) \):

\[
F_k^*(x) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{j=1}^{k} f(x_{i_j}), \quad k = 1, \ldots, n. \tag{5}
\]

By properties of Schur-convex function, Schur-geometrically convex function and Schur-harmonically convex function, we obtained the following results:

**Theorem 1.** Let \( I \subset \mathbb{R} \) is a symmetric convex set with non-empty interior and let \( f : I \rightarrow \mathbb{R} \) be continuous on \( I \) and differentiable in the interior of \( I \). If \( f \) is a log-convex function, then for any \( k = 1, 2, \ldots, n \), \( F_k^*(x) \) is a Schur-convex function on \( I^n \).

**Theorem 2.** Let \( I \subset \mathbb{R}_+ \) is a symmetric convex set with non-empty interior and let \( f : I \rightarrow \mathbb{R}_+ \) be continuous on \( I \) and differentiable in the interior of \( I \). If \( f \) is an increasing log-convex function, then for any \( k = 1, 2, \ldots, n \), \( F_k^*(x) \) is a Schur-geometrically convex function on \( I^n \).

**Theorem 3.** Let \( I \subset \mathbb{R}_+ \) is a symmetric convex set with non-empty interior and let \( f : I \rightarrow \mathbb{R}_+ \) be continuous on \( I \) and differentiable in the interior of \( I \). If \( f \) is an increasing log-convex function, then for any \( k = 1, 2, \ldots, n \), \( F_k^*(x) \) is a Schur-harmonically convex function on \( I^n \).

### 2. Lemmas

To prove the above three theorems, we need the following lemmas.

**Lemma 1.** [1, p. 67],[2] If \( \varphi \) is symmetric and convex (concave) on symmetric convex set \( \Omega \), then \( \varphi \) is Schur-convex (Schur-concave) on \( \Omega \).

**Lemma 2.** [1, p. 73],[2] Let \( \Omega \subset \mathbb{R}^n \), \( \varphi : \Omega \rightarrow \mathbb{R}_+ \). Then log \( \varphi \) is Schur-convex (Schur-concave) if and only if \( \varphi \) is Schur-convex (Schur-concave).
Lemma 3. [1, p. 642],[2] Let \( \Omega \subset \mathbb{R}^n \) be open convex set, \( \varphi : \Omega \to \mathbb{R} \). For \( x, y \in \Omega \), defined one variable function \( g(t) = \varphi (tx + (1 - t)y) \) on interval \((0, 1)\). Then \( \varphi \) is convex (concave) on \( \Omega \) if and only if \( g \) is convex (concave) on \([0, 1]\) for all \( x, y \in \Omega \).

Lemma 4. Let \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_m) \) \( \in \mathbb{R}^m \). If \( f \) is a log-convex function, then the functions \( p(t) = \log g(t) \) is convex on \([0, 1]\), where

\[
g(t) = \sum_{j=1}^{m} f(tx_j + (1 - t)y_j).
\]

Proof.

\[
p'(t) = \frac{g'(t)}{g(t)}.
\]

where

\[
g'(t) = \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1 - t)y_j).
\]

\[
p''(t) = \frac{g''(t)g(t) - (g'(t))^2}{g^2(t)},
\]

where

\[
g''(t) = \sum_{j=1}^{m} (x_j - y_j)^2 f''(tx_j + (1 - t)y_j),
\]

by the Cauchy inequality, we have

\[
g''(t)g(t) - (g'(t))^2
\]

\[
= \left( \sum_{j=1}^{m} (x_j - y_j)^2 f''(tx_j + (1 - t)y_j) \right) \left( \sum_{j=1}^{m} f(tx_j + (1 - t)y_j) \right) - \left( \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1 - t)y_j) \right)^2 
\]

\[
\geq \left( \sum_{j=1}^{m} (x_j - y_j) \sqrt{f''(tx_j + (1 - t)y_j)} \cdot \sqrt{f(tx_j + (1 - t)y_j)} \right) \left( \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1 - t)y_j) \right)^2
\]

From the log-convexity of \( f \) it follows that \( (\log f(u))^\prime\prime = \frac{f''(u)f(u) - (f'(u))^2}{f^2(u)} \geq 0 \), hence

\[
\sqrt{f''(tx_j + (1 - t)y_j)} \cdot \sqrt{f(tx_j + (1 - t)y_j)} \geq f'(tx_j + (1 - t)y_j),
\]

and then \( g''(t)g(t) - (g'(t))^2 \geq 0 \), i.e. \( p''(t) \geq 0 \), that is \( p(t) = \log g(t) \) is convex on \([0, 1]\).

The proof of Lemma 4 is completed. \( \square \)
Lemma 5. Let
\[ f(t) = \frac{x^t - 1}{t}. \]
If \( x > 1 \), then \( f(t) \) is a log-convex function on \( \mathbb{R}_+ \).

Proof. By computing, we have
\[ (\log f(t))'' = -\frac{x^t (\log x)^2}{(x^t - 1)^2} + \frac{1}{t^2}. \]

We need only prove \((\log f(t))'' \geq 0\). It equivalent to
\[ t^2 x^t (\log x)^2 \leq (x^t - 1)^2. \]
In both sides the inequality (6), extracting the square root and dividing by \( x^t \), then the inequality (6) equivalent to
\[ g(t) := x^{\frac{t}{2}} - x^{-\frac{t}{2}} - t \log x \geq 0. \]
When \( x > 1 \), \( g'(x) = \frac{1}{2} \log x \left( x^{\frac{t}{2}} - x^{-\frac{t}{2}} - 2 \right) \geq 0 \), hence \( g(t) \) is increasing on \( \mathbb{R}_+ \), and then \( g(t) \geq g(0) = 0 \), that is \((\log f(t))'' \geq 0\).

The proof of Lemma 5 is completed. \( \square \)

3. Proof of Main Results

Proof of Theorem 1: For any \( 1 \leq i_1 < \cdots < i_k \leq n \), by Lemma 3 and Lemma 4, it follows that \( \ln \sum_{j=1}^{k} f(x_{i_j}) \) is convex on \( I^k \). Obviously, \( \ln \sum_{j=1}^{k} f(x_{i_j}) \) is also convex on \( I^n \), and then \( F_k^*(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \log \sum_{j=1}^{k} f(x_{i_j}) \) is convex on \( I^n \). Furthermore, it is clear that \( \log F_k^*(x) \) is symmetric on \( I^n \), by Lemma 1, it follows that \( \log F_k^*(x) \) is Schur-convex on \( I^n \), and then from Lemma 2 we conclude that \( F_k^*(x) \) is also Schur-convex on \( I^n \).

The proof of Theorem 1 is completed.

Proof of Theorem 2: For \( x \in I \subset \mathbb{R}_+ \) and \( x_1 \neq x_2 \), we have
\[
\Delta = (\log x_1 - \log x_2) \left( x_1 \frac{\partial F_k^*}{\partial x_1} - x_2 \frac{\partial F_k^*}{\partial x_2} \right)
= (\log x_1 - \log x_2) \left( x_1 \frac{\partial F_k^*}{\partial x_1} - x_1 \frac{\partial F_k^*}{\partial x_2} + x_1 \frac{\partial F_k^*}{\partial x_2} - x_2 \frac{\partial F_k^*}{\partial x_2} \right)
= x_1 \log x_1 - \log x_2 \left( x_1 - x_2 \right) \left( \frac{\partial F_k^*}{\partial x_1} - \frac{\partial F_k^*}{\partial x_2} \right) + \frac{\partial F_k^*}{\partial x_2} \left( x_1 - x_2 \right) (\log x_1 - \log x_2).
\]

Since \( F_k^*(x) \) is Schur-convex on \( I^n \), by Theorem A, we have
\[ (x_1 - x_2) \left( \frac{\partial F_k^*}{\partial x_1} - \frac{\partial F_k^*}{\partial x_2} \right) \geq 0. \]
Notice that \( f \) and \( \log t \) is increasing, we have \( \frac{\partial F_k^*}{\partial x_1} \geq 0 \), \( \frac{\log x_1 - \log x_2}{x_1 - x_2} \geq 0 \) and \( (x_1 - x_2) (\log x_1 - \log x_2) \geq 0 \), so that \( \Delta \geq 0 \), by Theorem B, it follows that \( F_k^*(x) \) is Schur-geometric convex on \( I^n \).

Proof of Theorem 3: The proof of Theorem 3 similar to Theorem 2, the detailed proof is left to the reader.

Remark 1. If using the decision theorems, i.e. Theorem A, Theorem B and Theorem C respectively direct to prove Theorem 1, Theorem 2 and Theorem 3, I am afraid not above proofs are simple, interested readers may wish to try.
The symmetric function

\[ Q_k(x) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \frac{1 + x_{i_1}}{1 - x_{i_1}}, \quad k = 1, \ldots, n. \]  

(7)

is Schur-convex function, Schur-geometrically and harmonically convex function on \((0,1)^n\). And for \(x \in (0,1)^n\), we have

\[ \prod_{1 \leq i_1 < \ldots < i_k \leq n} \frac{1 + x_{i_1}}{1 - x_{i_1}} \geq \left( \frac{k(n + s)}{n - s} \right)^{C_k}, \quad k = 1, \ldots, n. \]  

(8)

where \(s = \sum_{i=1}^{n} x_i\) and \(C_k = \frac{n!}{k!(n-k)!}\).

Proof. Let \(f(x) = \frac{1+x}{1-x}, x \in (0,1)\). By computing, we have \(f'(x) = \frac{2}{(1-x)^2} > 0\) and \(\log(f(x))'' = \frac{4x}{(1+x)^2(1-x)^2} \geq 0\), that is \(f\) is an increasing log-convex function. By Theorem 1, Theorem 2 and Theorem 3, it follows that \(Q_k(x)\) is respectively Schur-convex function, Schur-geometrically and harmonically convex function on \((0,1)^n\).

Since \(y = (\frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}) < x = (x_1, x_2, \ldots, x_n)\), from Schur-convexity of \(G_k(x)\), it follows that \(Q_k(y) \leq Q_k(x)\), i.e. inequality (7) holds.

The proof of Theorem 4 is completed. \(\square\)

Specially, taking \(k = 1, s = 1\), from the inequality (8) we can get the known Klamkin inequality:

\[ \prod_{i=1}^{n} \frac{1 + x_i}{1 - x_i} \geq \left( \frac{n + 1}{n - 1} \right)^n. \]  

(9)

By analogous proof with Theorem 4, we can obtain the following theorem.

Theorem 5. The symmetric function

\[ R_k(x) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1}, \quad k = 1, \ldots, n. \]  

(10)

is Schur-convex function, Schur-geometrically and harmonically convex function on \([\frac{1}{2}, 1)^n\). And for \(x \in [\frac{1}{2}, 1)^n\), we have

\[ \prod_{1 \leq i_1 < \ldots < i_k \leq n} \frac{x_{i_1}}{1 - x_{i_1}} \geq \left( \frac{ks}{n - s} \right)^{C_k}, \quad k = 1, \ldots, n. \]  

(11)

where \(s = \sum_{i=1}^{n} x_i\) and \(C_k = \frac{n!}{k!(n-k)!}\).

Theorem 6. The symmetric function

\[ D_k(x) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1}, \quad k = 1, \ldots, n. \]  

(12)

is Schur-convex on \(\mathbb{R}^n_+\) and Schur-geometric and harmonic convex on \([e^{-1}, \infty)^n\).

And for \(x \in \mathbb{R}^n_+\), we have

\[ \prod_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \geq \left( k[A(x)]^A(x) \right)^{C_k}, \quad k = 1, \ldots, n. \]  

(13)

where \(A(x) = \frac{1}{n} \sum_{i=1}^{n} x_i\) and \(C_k = \frac{n!}{k!(n-k)!}\).
Proof. It is not difficult to verify that $x^r$ is log-convex function on $(0, \infty)$ and increasing on $[e^{-1}, \infty)$. By Theorem 1, Theorem 2 and Theorem 3, it follows that $D_k(x)$ is Schur-convex on $\mathbb{R}_+^n$ and Schur-geometric and harmonic convex on $[e^{-1}, \infty)^n$.

Since $y = (A(x), A(x), ..., A(x)) < x = (x_1, x_2, ..., x_n)$, from Schur-convexity of $D_k(x)$, it follows that $D_k(y) \leq D_k(x)$, i.e. inequality (11) holds.

The proof of Theorem 6 is completed.  

From Lemma 5 and Theorem 1, we can obtain the following Theorem 2.

**Theorem 7.** Let $x > 1$.

\[ P_k(t) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} \frac{x^{i_j} - 1}{t_{i_j}}, \quad k = 1, \ldots, n. \]  

is Schur-convex on $\mathbb{R}_+^n$. And for $t \in \mathbb{R}_+^n$, we have

\[ \prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} \frac{x^{i_j} - 1}{t_{i_j}} \geq \left( \frac{k(x^{A(t)} - 1)}{A(t)} \right)^{C_k^n}, \quad k = 1, \ldots, n. \]  

where $A(t) = \frac{1}{n} \sum_{i=1}^{n} t_i$ and $C_k^n = \frac{n!}{k!(n-k)!}$.

Specially, taking $n = 2, k = 1$ and $t = (m + r, m - r)$, from the inequality (15) we can get the known inequality:

\[ (x^{m-r} - 1)(x^{m+r} - 1) \geq \left( 1 - \frac{r^2}{m^2} \right) (x^{m} - 1)^2, \]  

where $r \in \mathbb{N}, m \geq 2, r < m$.

**Theorem 8.** Let $0 < \mu(E) < \infty, 1 \leq p < \infty$ and let

\[ N_p(f) = \left( \frac{1}{\mu(E)} \int_E |f|^p d\mu \right)^{\frac{1}{p}}. \]  

Then

\[ B_k(p) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{j=1}^{k} (N_{p_i}(f))^{p_i}, \quad k = 1, \ldots, n. \]  

is Schur-convex function, Schur-geometrically and harmonically convex function on $[1, \infty)^n$.

Proof. Since $(N_{p_i}(f))^p$ is an increasing log-convex function (see[7], p.36), from Theorem 1, Theorem 2 and Theorem 3, it follows that Theorem 7 holds.  

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