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REFINEMENTS OF THE OSTROWSKI INEQUALITY IN TERMS OF THE CUMULATIVE VARIATION AND APPLICATIONS

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ABSTRACT. Refinements of the Ostrowski inequality for functions of bounded variation in terms of the cumulative variation function are given. Applications for selfadjoint operators on complex Hilbert spaces are also provided.

1. Introduction

In order to extend the classical *Ostrowski's inequality* for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, the author obtained in 1999 (see [16] or the RGMIA preprint version of [18]) the following result

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{1.1}$$

for any $x \in [a,b]$ and f a function of bounded variation on $[a,b]$. Here $\bigvee_a^b (f)$ denotes the *total variation* of f on $[a,b]$ and the constant $\frac{1}{2}$ is best possible in (1.1). The best inequality one can obtain from (1.1) is the *midpoint inequality*, namely

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f), \tag{1.2}$$

for which the constant $\frac{1}{2}$ is also sharp.

For recent related results, see [1]-[4], [6]-[10], [12]-[14], [24]-[28] and [30]-[42].

The main aim of the present paper is to provide some refinements of the inequalities (1.1) and (1.2) in terms of the cumulative variation function. Applications for selfadjoint operators on complex Hilbert spaces are also given.

2. Refinements of the Ostrowski Inequality

For a function of bounded variation $v : [a,b] \to \mathbb{C}$ we define the *Cumulative Variation Function* (CVF) $V : [a,b] \to [0,\infty)$ by

$$V(t) := \bigvee_a^t (v)$$

the total variation of v on the interval $[a,t]$ with $t \in [a,b]$.

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It is know that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous in a point $c \in [a,b]$ if and only if the generating function v is continuing in that point. If v is Lipschitzian with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \leq L|t-s| \text{ for any } t, s \in [a,b]$$

then V is also Lipschitzian with the same constant.

The following lemma is of interest in itself as well, see also [21].

Lemma 1. *Let $f, u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\, du(t)\right| \leq \int_a^b |f(t)|\, d\left(\bigvee_a^t (u)\right) \leq \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u). \tag{2.1}$$

The following result may be stated.

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned} &\left|\int_a^b f(t)\, dt - f(x)(b-a)\right| \\ &\leq \int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt \\ &\leq (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f) \\ &\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a), \end{cases} \end{aligned} \tag{2.2}$$

for any $x \in [a,b]$.

Proof. We start with the equality

$$f(x)(b-a) - \int_a^b f(t)\, dt = \int_a^x (t-a)\, df(t) + \int_x^b (t-b)\, df(t) \tag{2.3}$$

that holds for any $x \in [a,b]$ and $f : [a,b] \to \mathbb{C}$ a function of bounded variation on $[a,b]$ (see [16] or [18]).

Taking the modulus in (2.3) and using the property (2.1) we have

$$\begin{aligned} &\left|f(x)(b-a) - \int_a^b f(t)\, dt\right| \\ &\leq \left|\int_a^x (t-a)\, df(t)\right| + \left|\int_x^b (t-b)\, df(t)\right| \\ &\leq \int_a^x (t-a)\, d\left(\bigvee_a^t (f)\right) + \int_x^b (b-t)\, d\left(\bigvee_a^t (f)\right) \end{aligned} \tag{2.4}$$

for any $x \in [a,b]$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
(2.5)\qquad \int_a^x (t-a)\, d\left(\bigvee_a^t (f)\right) &= (t-a)\bigvee_a^t (f)\Bigg|_a^x - \int_a^x \left(\bigvee_a^t (f)\right) dt \\
&= (x-a)\bigvee_a^x (f) - \int_a^x \left(\bigvee_a^t (f)\right) dt \\
&= \int_a^x \left(\bigvee_a^x (f) - \bigvee_a^t (f)\right) dt \\
&= \int_a^x \left(\bigvee_t^x (f)\right) dt
\end{aligned}$$

and

$$\begin{aligned}
(2.6)\qquad \int_x^b (b-t)\, d\left(\bigvee_a^t (f)\right) &= (b-t)\bigvee_a^t (f)\Bigg|_x^b + \int_x^b \left(\bigvee_a^t (f)\right) dt \\
&= \int_x^b \left(\bigvee_a^t (f)\right) dt - (b-x)\bigvee_a^x (f) \\
&= \int_x^b \left(\bigvee_a^t (f) - \bigvee_a^x (f)\right) dt \\
&= \int_x^b \left(\bigvee_x^t (f)\right) dt
\end{aligned}$$

for any $x \in [a,b]$.

Utilising (2.4)-(2.6) we deduce the first inequality in (2.2).

Since $\bigvee_t^x (f) \le \bigvee_a^x (f)$ for $t \in [a,x]$ and $\bigvee_x^t (f) \le \bigvee_x^b (f)$ for $t \in [x,b]$, then

$$\begin{aligned}
&\int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt \\
\le\ & (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f)
\end{aligned}$$

for any $x \in [a,b]$, which proves the second inequality in (2.2).

The last part is obvious by the max properties and the fact that for $c,d \in \mathbb{R}$ we have

$$\max\{c,d\} = \frac{c+d+|c-d|}{2}.$$

The details are omitted. □

The following midpoint inequality holds:

Corollary 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$
\begin{aligned}
(2.7)\qquad & \left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \\
& \le \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}}(f)\right)dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f)\right)dt \le \frac{1}{2}(b-a)\bigvee_a^b (f).
\end{aligned}
$$

The first inequality in (2.7) is sharp and the constant $\frac{1}{2}$ in the second, is best possible.

Proof. We must prove only the sharpness of the inequalities in (2.7).

If we consider the function $f:[a,b]\to\mathbb{R}$ given by

$$
f(t) := \begin{cases} 0, & t\in[a,\frac{a+b}{2}) \\ 1, & t=\frac{a+b}{2} \\ 0 & t\in(\frac{a+b}{2},b] \end{cases}
$$

then this function is of bounded variation, we observe that $\bigvee_{\frac{a+b}{2}}^t(f)=1$ for any $t\in(\frac{a+b}{2},b]$ and $\bigvee_t^{\frac{a+b}{2}}(f)=1$ for any $t\in[a,\frac{a+b}{2})$. Also, we have $\bigvee_a^b(f)=2$.

If we replace these values in (2.7) we obtain in all terms the same quantity $b-a$. □

Corollary 2. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If $p\in(a,b)$ is a median point in variation, namely $\bigvee_a^p(f)=\bigvee_p^b(f)$, then we have the inequality*

$$
\begin{aligned}
(2.8)\qquad & \left|\int_a^b f(t)\,dt - f(p)(b-a)\right| \\
& \le \int_a^p \left(\bigvee_t^p (f)\right)dt + \int_p^b \left(\bigvee_p^t (f)\right)dt \le \frac{1}{2}(b-a)\bigvee_a^b(f).
\end{aligned}
$$

The first inequality in (2.2) is useful when some properties for the CVF are available, like for instance below:

Corollary 3. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If for $x\in(a,b)$ there exist $L_x>0$ and $\alpha>-1$ such that*

$$
(2.9)\qquad \left|\bigvee_x^t(f)\right| \le L_x|t-x|^\alpha \text{ for any } t\in[a,b]\setminus\{x\},
$$

then

$$
\begin{aligned}
(2.10)\qquad & \left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
& \le \frac{1}{\alpha+1}L_x\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right].
\end{aligned}
$$

In particular, for $\alpha = 1$ *in (2.9) we get from (2.10) that*

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] L_x (b-a)^2. \tag{2.11}$$

Remark 1. *If the CVF* $\bigvee_a^t(f)$ *is* K*-Lipschitzian, i.e.,*

$$\left|\bigvee_s^t(f)\right| \leq K|t-s| \text{ for any } t,s \in [a,b],$$

then

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \leq \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] L (b-a)^2. \tag{2.12}$$

for any $x \in [a,b]$*.*

Corollary 4. *If there exists a constant* $L_{\frac{a+b}{2}} > 0$ *and* $\alpha > -1$ *such that*

$$\left|\bigvee_{\frac{a+b}{2}}^t(f)\right| \leq L_{\frac{a+b}{2}}\left|t-\frac{a+b}{2}\right|^{\alpha} \text{ for any } t \in [a,b] \setminus \left\{\frac{a+b}{2}\right\}, \tag{2.13}$$

then we have the midpoint inequality

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \leq \frac{1}{2^{\alpha}(\alpha+1)} L_{\frac{a+b}{2}}(b-a)^{\alpha+1}. \tag{2.14}$$

In particular, if we take $\alpha = 1$ *in (2.13), then we get from (2.14)*

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \leq \frac{1}{4} L_{\frac{a+b}{2}}(b-a)^2. \tag{2.15}$$

The constant $\frac{1}{4}$ *is best possible in (2.15).*

Proof. First, we notice that if $h : [a,b] \to \mathbb{C}$ is of bounded variation, then $|h| : [a,b] \to [0,\infty)$ is of bounded variation and

$$\bigvee_a^b(|h|) \leq \bigvee_a^b(h). \tag{2.16}$$

Indeed, by the continuity property of the modulus, we have that

$$\sum_{j=0}^{n-1} \left||h(t_{j+1})| - |h(t_j)|\right| \leq \sum_{j=0}^{n-1} |h(t_{j+1}) - h(t_j)|$$

for any division $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$, which, by taking the supremum over all divisions of $[a,b]$, produces the desired inequality (2.16).

If we consider the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(s) := \left|s - \frac{a+b}{2}\right|$ then, by denoting with $\mathbf{e}$ the identity function on $[a,b]$, i.e. $\mathbf{e}(t) = t, t \in [a,b]$, we have

$$\begin{aligned}\left|\bigvee_{\frac{a+b}{2}}^{t}(f_0)\right| &= \left|\bigvee_{\frac{a+b}{2}}^{t}\left(\left|\mathbf{e}-\frac{a+b}{2}\right|\right)\right| \\ &\leq \left|\bigvee_{\frac{a+b}{2}}^{t}\left(\mathbf{e}-\frac{a+b}{2}\right)\right| = \left|\bigvee_{\frac{a+b}{2}}^{t}(\mathbf{e})\right| = \left|t - \frac{a+b}{2}\right|\end{aligned}$$

for any $t \in [a,b]$.

Therefore the function f_0 satisfies the condition (2.13) for $\alpha = 1$ and with the constant $L_{\frac{a+b}{2}} = 1$. Since

$$\int_a^b f_0(t)\,dt = \int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{1}{4}(b-a)^2$$

then we obtain in both sides of the inequality (2.15) the same quantity $\frac{1}{4}(b-a)^2$. □

Remark 2. *The inequalities (2.12) and (2.15) are known in the case of Lipschitzian functions with the constant $L > 0$. We obtained them here under weaker conditions for the function f. This show that the refinement in terms of the CVF for the Ostrowski inequality (2.2) is also useful to extend known results to larger classes of functions.*

Theorem 2. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\ &\leq \int_a^x \left(\bigvee_t^x (|f(x) - f|)\right) dt + \int_x^b \left(\bigvee_x^t (|f(x) - f|)\right) dt \\ &\leq (x-a)\bigvee_a^x (|f(x) - f|) + (b-x)\bigvee_x^b (|f(x) - f|) \\ &\leq (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f) \\ &\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a), \end{cases}\end{aligned} \tag{2.17}$$

for any $x \in [a,b]$.

Proof. Observe that

$$\begin{aligned}\left|f(x)(b-a) - \int_a^b f(t)\,dt\right| &= \left|\int_a^b [f(x) - f(t)]\,dt\right| \\ &\leq \int_a^b |f(x) - f(t)|\,dt\end{aligned} \tag{2.18}$$

for any $x \in [a,b]$.

For a fixed $x \in [a,b]$, define the function $g_x : [a,b] \to [0,\infty)$ by $g_x(t) := |f(x) - f(t)|$. We observe that g_x is of bounded variation on $[a,b]$ and

$$\left| g_x(x)(b-a) - \int_a^b g_x(t)\,dt \right| = \int_a^b |f(x) - f(t)|\,dt. \tag{2.19}$$

Writing the inequality (2.2) for the function g_x we have

$$\begin{aligned}
&\left| g_x(x)(b-a) - \int_a^b g_x(t)\,dt \right| \\
&\leq \int_a^x \left(\bigvee_t^x (|f(x) - f|) \right) dt + \int_x^b \left(\bigvee_x^t (|f(x) - f|) \right) dt \\
&\leq (x-a) \bigvee_a^x (|f(x) - f|) + (b-x) \bigvee_x^b (|f(x) - f|).
\end{aligned} \tag{2.20}$$

Utilising (2.18)-(2.20) we deduce the first two inequalities in (2.17).

By the inequality (2.16) we have

$$\bigvee_a^x (|f(x) - f|) \leq \bigvee_a^x (f(x) - f) = \bigvee_a^x (f)$$

and

$$\bigvee_x^b (|f(x) - f|) \leq \bigvee_x^b (f(x) - f) = \bigvee_x^b (f),$$

which proves the third inequality in (2.17). □

Corollary 5. *If $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then*

$$\begin{aligned}
&\left| \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \right| \\
&\leq \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} \left(\left| f\left(\frac{a+b}{2}\right) - f \right| \right) \right) dt \\
&+ \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t \left(\left| f\left(\frac{a+b}{2}\right) - f \right| \right) \right) dt \\
&\leq \frac{1}{2}(b-a) \bigvee_a^b \left(\left| f\left(\frac{a+b}{2}\right) - f \right| \right) \leq \frac{1}{2}(b-a) \bigvee_a^b (f).
\end{aligned} \tag{2.21}$$

All inequalities in (2.21) are sharp.

Proof. If we consider the function $f : [a,b] \to \mathbb{R}$, with

$$f(t) := \begin{cases} 0, & t \in [a, \frac{a+b}{2}) \\ 1, & t = \frac{a+b}{2} \\ 0 & t \in (\frac{a+b}{2}, b], \end{cases}$$

then this function is of bounded variation, we observe that

$$\left|f\left(\frac{a+b}{2}\right)-f(t)\right| = \begin{cases} 1, & t\in[a,\frac{a+b}{2}) \\ 0, & t=\frac{a+b}{2} \\ 1 & t\in(\frac{a+b}{2},b], \end{cases}$$

$$\bigvee_{t}^{\frac{a+b}{2}}\left(\left|f\left(\frac{a+b}{2}\right)-f\right|\right) = 1 \text{ for } t\in[a,\frac{a+b}{2}),$$

$$\bigvee_{\frac{a+b}{2}}^{t}\left(\left|f\left(\frac{a+b}{2}\right)-f\right|\right) = 1 \text{ for } t\in(\frac{a+b}{2},b]$$

and

$$\bigvee_{a}^{b}\left(\left|f\left(\frac{a+b}{2}\right)-f\right|\right) = 2.$$

Replacing these values in (2.21) we get in all terms of this inequality the same quantity $b-a$. □

Corollary 6. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. If $q\in(a,b)$ is a point for which*

$$\bigvee_{a}^{q}(|f(q)-f|) = \bigvee_{q}^{b}(|f(q)-f|),$$

then

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - f(q)(b-a)\right| \\ &\leq \int_a^q\left(\bigvee_{t}^{q}(|f(q)-f|)\right)dt + \int_q^b\left(\bigvee_{q}^{t}(|f(q)-f|)\right)dt \\ &\leq \frac{1}{2}\bigvee_{a}^{b}(|f(q)-f|) \leq \frac{1}{2}(b-a)\bigvee_{a}^{b}(f). \end{aligned} \tag{2.22}$$

Remark 3. *Since*

$$\begin{aligned} &(x-a)\bigvee_{a}^{x}(|f(x)-f|) + (b-x)\bigvee_{x}^{b}(|f(x)-f|) \\ &\leq \begin{cases} \max\{x-a,b-x\}\bigvee_a^b|f(x)-f| \\ \\ \max\left\{\bigvee_a^x(|f(x)-f|),\bigvee_x^b(|f(x)-f|)\right\}(b-a) \end{cases} \\ &= \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b|f(x)-f| \\ \\ \left[\frac{1}{2}\bigvee_a^b|f(x)-f| + \frac{1}{2}\left|\bigvee_a^x(|f(x)-f|)-\bigvee_x^b(|f(x)-f|)\right|\right], \end{cases} \end{aligned}$$

then from (2.17) we also have the string of inequalities

$$
\begin{aligned}
(2.23)\qquad & \left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
& \leq \int_a^x \left(\bigvee_t^x (|f(x)-f|)\right) dt + \int_x^b \left(\bigvee_x^t (|f(x)-f|)\right) dt \\
& \leq (x-a)\bigvee_a^x (|f(x)-f|) + (b-x)\bigvee_x^b (|f(x)-f|) \\
& \leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right] \bigvee_a^b |f(x)-f| \\ \\ \left[\frac{1}{2}\bigvee_a^b |f(x)-f| + \frac{1}{2}\left|\bigvee_a^x (|f(x)-f|) - \bigvee_x^b (|f(x)-f|)\right|\right] \\ \times (b-a), \end{cases}
\end{aligned}
$$

for any $x \in [a,b]$.

Remark 4. *If the function* $f : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing on* $[a,b]$, *then*

$$
\begin{aligned}
(2.24)\qquad & \left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
& \leq \int_a^b sgn(x-t)[f(x)-f(t)]\,dt \\
& \leq (x-a)[f(x)-f(a)] + (b-x)[f(b)-f(x)] \\
& \leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right][f(b)-f(a)], \\ \\ \left[\frac{1}{2}[f(b)-f(a)] + \frac{1}{2}\left|f(x)-\frac{f(a)+f(b)}{2}\right|\right](b-a), \end{cases}
\end{aligned}
$$

for any $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$, *we get the trapezoid inequality*

$$
\begin{aligned}
(2.25)\qquad & \left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \\
& \leq \int_a^b sgn\left(\frac{a+b}{2}-t\right)\left[f\left(\frac{a+b}{2}\right)-f(t)\right] dt \leq \frac{1}{2}[f(b)-f(a)](b-a).
\end{aligned}
$$

Moreover, if $p \in (a,b)$ *is such that*

$$
f(p) = \frac{f(a)+f(b)}{2},
$$

then we have the trapezoid inequality

$$
\begin{aligned}
(2.26)\qquad & \left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt\right| \\
& \leq \int_a^b sgn(p-t)\left[\frac{f(a)+f(b)}{2}-f(t)\right] dt \leq \frac{1}{2}[f(b)-f(a)](b-a).
\end{aligned}
$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, \text{ for } -\infty<s\leq\lambda,\\ \\ 0, \text{ for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [23, p. 256]:

Theorem 3 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) *$E_\lambda\leq E_{\lambda'}$ for $\lambda\leq\lambda'$;*
b) *$E_{m-0}=0, E_M=I$ and $E_{\lambda+0}=E_\lambda$ for all $\lambda\in\mathbb{R}$;*
c) *We have the representation*

$$A=\int_{m-0}^{M}\lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left\|\varphi(A)-\sum_{k=1}^{n}\varphi(\lambda_k')\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\leq\varepsilon$$

whenever

$$\begin{cases}\lambda_0<m=\lambda_1<...<\lambda_{n-1}<\lambda_n=M,\\ \\ \lambda_k-\lambda_{k-1}\leq\delta \textit{ for } 1\leq k\leq n,\\ \\ \lambda_k'\in[\lambda_{k-1},\lambda_k] \textit{ for } 1\leq k\leq n\end{cases}$$

this means that

$$\varphi(A)=\int_{m-0}^{M}\varphi(\lambda)\,dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 7. *With the assumptions of Theorem 3 for A, E_λ and φ we have the representations*

$$\varphi(A)x=\int_{m-0}^{M}\varphi(\lambda)\,dE_\lambda x \quad \textit{for all } x\in H$$

and

$$\langle\varphi(A)x,y\rangle=\int_{m-0}^{M}\varphi(\lambda)\,d\langle E_\lambda x,y\rangle \quad \textit{for all } x,y\in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [22].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \leq \langle (E_\beta - E_\alpha)x, x\rangle \langle (E_\beta - E_\alpha)y, y\rangle, \tag{3.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle E_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

Remark 5. *For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (3.4) the inequality*

$$\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \langle (I - E_{m-\varepsilon})x, x\rangle^{1/2} \langle (I - E_{m-\varepsilon})y, y\rangle^{1/2} \tag{3.5}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\|\,\|y\|, \tag{3.6}$$

where $\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+}\left[\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

The following result holds:

Theorem 4. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda\,|\lambda \in Sp(A)\}$*

$=:\max Sp(A)$. *If* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *is the spectral family of the bounded selfadjoint operator* A, *then for any* $v\in[m,M]$ *we have*

$$\begin{aligned}
&|\langle [M(I-E_v)+mE_v-A]x,y\rangle| \\
&\leq \int_{m-0}^{v}\langle (E_v-E_t)x,x\rangle^{1/2}\langle (E_v-E_t)y,y\rangle^{1/2}dt \\
&+\int_{v}^{M}\langle (E_t-E_v)x,x\rangle^{1/2}\langle (E_t-E_v)y,y\rangle^{1/2}dt \\
&\leq \langle E_vx,x\rangle^{1/2}\langle E_vy,y\rangle^{1/2}(v-m) \\
&+\langle (I-E_v)x,x\rangle^{1/2}\langle (I-E_v)y,y\rangle^{1/2}(M-v) \\
&\leq \left[\frac{1}{2}(M-m)+\left|v-\frac{m+M}{2}\right|\right] \\
&\times\left[\langle E_vx,x\rangle^{1/2}\langle E_vy,y\rangle^{1/2}+\langle (I-E_v)x,x\rangle^{1/2}\langle (I-E_v)y,y\rangle^{1/2}\right] \\
&\leq \left[\frac{1}{2}(M-m)+\left|v-\frac{m+M}{2}\right|\right]\|x\|\|y\|
\end{aligned}\tag{3.7}$$

for any $x,y\in H$.

In particular,

$$\begin{aligned}
&\left|\left\langle\left[M\left(I-E_{\frac{m+M}{2}}\right)+mE_{\frac{m+M}{2}}-A\right]x,y\right\rangle\right| \\
&\leq \int_{m-0}^{\frac{m+M}{2}}\left\langle\left(E_{\frac{m+M}{2}}-E_t\right)x,x\right\rangle^{1/2}\left\langle\left(E_{\frac{m+M}{2}}-E_t\right)y,y\right\rangle^{1/2}dt \\
&+\int_{v}^{M}\left\langle\left(E_t-E_{\frac{m+M}{2}}\right)x,x\right\rangle^{1/2}\left\langle\left(E_t-E_{\frac{m+M}{2}}\right)y,y\right\rangle^{1/2}dt \\
&\leq \frac{1}{2}(M-m)\left[\left\langle E_{\frac{m+M}{2}}x,x\right\rangle^{1/2}\left\langle E_{\frac{m+M}{2}}y,y\right\rangle^{1/2}\right. \\
&\left.+\left\langle\left(I-E_{\frac{m+M}{2}}\right)x,x\right\rangle^{1/2}\left\langle\left(I-E_{\frac{m+M}{2}}\right)y,y\right\rangle^{1/2}\right] \\
&\leq \frac{1}{2}(M-m)\|x\|\|y\|
\end{aligned}\tag{3.8}$$

for any $x,y\in H$.

Proof. For $\varepsilon>0$, by applying the inequality (2.2) on the interval $[m-\varepsilon,M]$ we can write that

$$\begin{aligned}
&\left|\langle E_vx,y\rangle(M-m+\varepsilon)-\int_{m-\varepsilon}^{M}\langle E_tx,y\rangle dt\right| \\
&\leq \int_{m-\varepsilon}^{v}\left(\bigvee_{t}^{v}\left(\langle E_{(\cdot)}x,y\rangle\right)\right)dt+\int_{v}^{M}\left(\bigvee_{v}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)\right)dt
\end{aligned}\tag{3.9}$$

for any $x,y\in H$.

Utilising Lemma 2 we also have

$$
\begin{aligned}
&\int_{m-\varepsilon}^{v}\left(\bigvee_{t}^{v}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right)dt+\int_{v}^{M}\left(\bigvee_{v}^{t}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right)dt \\
&\leq \int_{m-\varepsilon}^{v}\left\langle\left(E_v-E_t\right)x,x\right\rangle^{1/2}\left\langle\left(E_v-E_t\right)y,y\right\rangle^{1/2}dt \\
&+\int_{v}^{M}\left\langle\left(E_t-E_v\right)x,x\right\rangle^{1/2}\left\langle\left(E_t-E_v\right)y,y\right\rangle^{1/2}dt \\
&\leq \left\langle\left(E_v-E_{m-\varepsilon}\right)x,x\right\rangle^{1/2}\left\langle\left(E_v-E_{m-\varepsilon}\right)y,y\right\rangle^{1/2}\left(v-m+\varepsilon\right) \\
&+\left\langle\left(E_M-E_v\right)x,x\right\rangle^{1/2}\left\langle\left(E_M-E_v\right)y,y\right\rangle^{1/2}\left(M-v\right)
\end{aligned}
\tag{3.10}
$$

for any $x, y \in H$.

Letting $\varepsilon \to 0+$ in (3.9) and (3.10) produces the inequalities

$$
\begin{aligned}
&\left|\left\langle E_v x,y\right\rangle\left(M-m\right)-\int_{m-0}^{M}\left\langle E_t x,y\right\rangle dt\right| \\
&\leq \int_{m-0}^{v}\left\langle\left(E_v-E_t\right)x,x\right\rangle^{1/2}\left\langle\left(E_v-E_t\right)y,y\right\rangle^{1/2}dt \\
&+\int_{v}^{M}\left\langle\left(E_t-E_v\right)x,x\right\rangle^{1/2}\left\langle\left(E_t-E_v\right)y,y\right\rangle^{1/2}dt \\
&\leq \left\langle E_v x,x\right\rangle^{1/2}\left\langle E_v y,y\right\rangle^{1/2}\left(v-m\right) \\
&+\left\langle\left(I-E_v\right)x,x\right\rangle^{1/2}\left\langle\left(I-E_v\right)y,y\right\rangle^{1/2}\left(M-v\right)
\end{aligned}
\tag{3.11}
$$

for any $x, y \in H$.

Integrating by parts in the Riemann-Stieltjes integral and utilising the representation (3.3), we have

$$
\begin{aligned}
&\left\langle E_v x,y\right\rangle\left(M-m\right)-\int_{m-0}^{M}\left\langle E_t x,y\right\rangle dt \\
&=\left\langle E_v x,y\right\rangle\left(M-m\right)-\left[\left\langle E_t x,y\right\rangle t\Big|_{m-0}^{M}-\int_{m-0}^{M}t d\left\langle E_t x,y\right\rangle\right] \\
&=\left\langle E_v x,y\right\rangle\left(M-m\right)-\left\langle x,y\right\rangle M+\left\langle Ax,y\right\rangle \\
&=\left\langle\left[A-mE_v-\left(I-M\right)E_v\right]x,y\right\rangle
\end{aligned}
\tag{3.12}
$$

for any $x, y \in H$.

By (3.11) and (3.12) we obtain the first two inequalities in (3.7).

The rest is obvious and we omit the details. □

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