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A FUNCTIONAL GENERALIZATION OF OSTROWSKI INEQUALITY VIA MONTGOMERY IDENTITY

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Abstract. We show in this paper amongst other that, if $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$ then

$$\Phi\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq (\geq) \frac{1}{b-a}\left[\int_a^x \Phi\left[(t-a) f'(t)\right] dt + \int_x^b \Phi\left[(t-b) f'(t)\right] dt\right]$$

for any $x \in [a, b]$.

Natural applications for power and exponential functions are provided as well. Bounds for the Lebesgue p-norms of the deviation of a function from its integral mean are also given.

1. Introduction

Comparison between functions and integral means are incorporated in Ostrowski type inequalities as follows.

The first result in this direction is due to Ostrowski [20].

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a) M \tag{1.1}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following results for absolutely continuous functions hold (see [16] – [18]).

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha+1}+\left(\frac{b-x}{b-a}\right)^{\alpha+1}\right]^{\frac{1}{\alpha}} & \text{if } f'\in L_\beta[a,b], \\ \times(b-a)^{\frac{1}{\alpha}}\|f'\|_\beta & \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & \alpha>1; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_{[a,b],r}:=\left(\int_a^b|g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from the Fink result in [19] on choosing $n=1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [14] and the references therein for earlier contributions):

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x)-f(y)|\leq H|x-y|^r,\ \textit{for all}\ \ x,y\in[a,b], \tag{1.3}$$

where $r\in(0,1]$ and $H>0$ are fixed. Then, for all $x\in[a,b]$, we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{1.4}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [6])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)L, \tag{1.5}$$

where $x\in[a,b]$. Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

Theorem 4. *Assume that $f : [a,b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{1.6}$$

for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [5] (see also the monograph [15]).

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a,b]$, we have the inequality:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ &\leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\, dt \right\} \\ &\leq \frac{1}{b-a} \{ (x-a) [f(x) - f(a)] + (b-x) [f(b) - f(x)] \} \\ &\leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned} \tag{1.7}$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [11]:

Theorem 6. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in (a,b)$ one has the inequality*

$$\begin{aligned} &\frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \\ &\leq \int_a^b f(t)\, dt - (b-a) f(x) \\ &\leq \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right]. \end{aligned} \tag{1.8}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x = a$ or $x = b$.

For other Ostrowski's type inequalities for the Lebesgue integral, see [1]-[6] and [12]. Inequalities for the Riemann-Stieltjes integral may be found in [7], [9] while the generalization for isotonic functionals was provided in [10]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [13].

2. A Generalization of Ostrowski's Inequality

The following result holds:

Theorem 7. *Let* $f:[a,b]\to\mathbb{R}$ *be absolutely continuous on* $[a,b]$. *If* $\Phi:\mathbb{R}\to\mathbb{R}$ *is convex (concave) on* $\mathbb{R}$ *then we have the inequalities*

$$\Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq (\geq) \frac{1}{b-a}\left[\int_a^x \Phi[(t-a)f'(t)]\,dt+\int_x^b \Phi[(t-b)f'(t)]\,dt\right] \tag{2.1}$$

for any $x\in[a,b]$.

Proof. Utilising the Montgomery identity

$$\begin{aligned} f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{b-a}\left[\int_a^x (t-a)f'(t)\,dt+\int_x^b (t-b)f'(t)\,dt\right] \\ &= \frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right) \\ &\quad+\frac{b-x}{b-a}\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right), \end{aligned} \tag{2.2}$$

which holds for any $x\in(a,b)$ and the convexity of $\Phi:\mathbb{R}\to\mathbb{R}$, we have

$$\begin{aligned} \Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right) &\leq \frac{x-a}{b-a}\Phi\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right) \\ &\quad+\frac{b-x}{b-a}\Phi\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right) \end{aligned} \tag{2.3}$$

for any $x\in(a,b)$, which is an inequality of interest in itself as well.

If we use Jensen's integral inequality

$$\Phi\left(\frac{1}{d-c}\int_c^d g(t)\,dt\right)\leq\frac{1}{d-c}\int_c^d \Phi[g(t)]\,dt$$

we have

$$\Phi\left(\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right)\leq\frac{1}{x-a}\int_a^x \Phi[(t-a)f'(t)]\,dt \tag{2.4}$$

and

$$\Phi\left(\frac{1}{b-x}\int_x^b (t-b)f'(t)\,dt\right)\leq\frac{1}{b-x}\int_x^b \Phi[(t-b)f'(t)]\,dt \tag{2.5}$$

for any $x\in(a,b)$.

Making use of (2.3)-(2.5) we get the desired result (2.1) for the convex functions.

If $x=b$, then

$$f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b (t-a)\,f'(t)\,dt$$

and by Jensen's inequality we get

$$\Phi\left(f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)\leq\frac{1}{b-a}\int_a^b \Phi\left[(t-a)\,f'(t)\right]dt,$$

which proves the inequality (2.1) for $x=b$.

The same argument can be applied for $x=a$.

The case of concave functions goes likewise and the theorem is proved. □

Corollary 1. *With the assumptions of Theorem 7 we have*

$$\Phi(0)\leq(\geq)\frac{1}{b-a}\int_a^b \Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \tag{2.6}$$
$$\leq(\geq)\frac{1}{(b-a)^2}\left[\int_a^b (b-x)\,\Phi\left[(x-a)\,f'(x)\right]dx\right.$$
$$\left.+\int_a^b (x-a)\,\Phi\left[(x-b)\,f'(x)\right]dx\right].$$

Proof. By Jensen's integral inequality we have

$$\frac{1}{b-a}\int_a^b \Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx$$
$$\geq(\leq)\,\Phi\left[\frac{1}{b-a}\int_a^b\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx\right]$$
$$=\Phi(0),$$

which proves the first inequality in (2.6).

Integrating the inequality (2.1) over x we have

$$\frac{1}{b-a}\int_a^b \Phi\left(f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \tag{2.7}$$
$$\leq(\geq)\frac{1}{(b-a)^2}\int_a^b\left[\int_a^x \Phi\left[(t-a)\,f'(t)\right]dt+\int_x^b \Phi\left[(t-b)\,f'(t)\right]dt\right]dx.$$

Integrating by parts we have

$$\int_a^b\left(\int_a^x \Phi\left[(t-a)\,f'(t)\right]dt\right)dx$$
$$=x\int_a^x \Phi\left[(t-a)\,f'(t)\right]dt\Big|_a^b-\int_a^b x\,d\left(\int_a^x \Phi\left[(t-a)\,f'(t)\right]dt\right)$$
$$=b\int_a^b \Phi\left[(t-a)\,f'(t)\right]dt-\int_a^b x\Phi\left[(x-a)\,f'(x)\right]dx$$
$$=\int_a^b (b-x)\,\Phi\left[(x-a)\,f'(x)\right]dx$$

and

$$
\begin{aligned}
& \int_a^b \left(\int_x^b \Phi\left[(t-b) f'(t)\right] dt \right) dx \\
= \; & x \left(\int_x^b \Phi\left[(t-b) f'(t)\right] dt \right) \Bigg|_a^b - \int_a^b x d\left(\int_x^b \Phi\left[(t-b) f'(t)\right] dt \right) \\
= \; & -a \left(\int_a^b \Phi\left[(t-b) f'(t)\right] dt \right) + \int_a^b x \Phi\left[(x-b) f'(x)\right] dx \\
= \; & \int_a^b (x-a) \Phi\left[(x-b) f'(x)\right] dx.
\end{aligned}
$$

Utilising the inequality (2.7) we deduce the desired inequality (2.6). □

Remark 1. *If we write the inequality (2.1) for the convex function* $\Phi(x) = |x|^p$, $p \geq 1$ *then we get the inequality*

$$
\begin{aligned}
& \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right|^p \\
& \leq \frac{1}{b-a} \left[\int_a^x (t-a)^p |f'(t)|^p dt + \int_x^b (b-t)^p |f'(t)|^p dt \right]
\end{aligned} \tag{2.8}
$$

for $x \in [a,b]$.

Utilising Hölder's inequality we have

$$
\begin{aligned}
B(x) & := \int_a^x (t-a)^p |f'(t)|^p dt + \int_x^b (b-t)^p |f'(t)|^p dt \\
& \leq \begin{cases} \frac{(x-a)^{p+1}}{p+1} \|f'\|_{[a,x],\infty}^p & \text{if } f' \in L_\infty [a,x]; \\ \frac{(x-a)^{p+1/\alpha}}{(p\alpha+1)^{1/\alpha}} \|f'\|_{[a,x],p\beta}^p & \text{if } f' \in L_{p\beta}[a,x], \\ & \alpha > 1, 1/\alpha + 1/\beta = 1; \\ (x-a)^p \|f'\|_{[a,x],p}^p \end{cases} \\
& + \begin{cases} \frac{(b-x)^{p+1}}{p+1} \|f'\|_{[x,b],\infty}^p & \text{if } f' \in L_\infty [x,b]; \\ \frac{(b-x)^{p+1/\alpha}}{(p\alpha+1)^{1/\alpha}} \|f'\|_{[x,b],p\beta}^p & \text{if } f' \in L_{p\beta}[x,b], \\ & \alpha > 1, 1/\alpha + 1/\beta = 1; \\ (b-x)^p \|f'\|_{[x,b],p}^p \end{cases}
\end{aligned} \tag{2.9}
$$

for $x \in [a,b]$.

Utilising the inequalities (2.8) and (2.9) we have for $x \in [a,b]$ that

$$
\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \\
&\leq \frac{1}{(b-a)(p+1)}\left[(x-a)^{p+1}\|f'\|^p_{[a,x],\infty} + (b-x)^{p+1}\|f'\|^p_{[x,b],\infty} \right] \\
&\leq \frac{1}{(p+1)}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1} \right](b-a)^p \|f'\|^p_{[a,b],\infty}
\end{aligned} \tag{2.10}
$$

provided $f' \in L_\infty [a,b]$,

$$
\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \\
&\leq \frac{1}{(b-a)(p\alpha+1)^{1/\alpha}}\left[(x-a)^{p+1/\alpha}\|f'\|^p_{[a,x],p\beta} + (b-x)^{p+1/\alpha}\|f'\|^p_{[x,b],p\beta} \right] \\
&\leq \frac{1}{(p\alpha+1)^{1/\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{p+1/\alpha} + \left(\frac{b-x}{b-a}\right)^{p+1/\alpha} \right](b-a)^{p-1/\beta} \|f'\|^p_{[a,b],p\beta}
\end{aligned} \tag{2.11}
$$

provided $f' \in L_{p\beta}[a,b]$, $\alpha > 1, 1/\alpha + 1/\beta = 1$ and

$$
\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \\
&\leq \frac{1}{b-a}\left[(x-a)^{p}\|f'\|^p_{[a,x],p} + (b-x)^{p}\|f'\|^p_{[x,b],p} \right] \\
&\leq \max\left\{ \left(\frac{x-a}{b-a}\right)^{p}, \left(\frac{b-x}{b-a}\right)^{p} \right\}(b-a)^{p-1} \|f'\|^p_{[a,b],p} \\
&= \left\{ \frac{1}{2} + \left| \frac{x-\frac{a+b}{2}}{b-a} \right| \right\}^p (b-a)^{p-1}\|f'\|^p_{[a,b],p}
\end{aligned} \tag{2.12}
$$

provided $f' \in L_p[a,b]$.

Remark 2. *If we take $p = 1$ in the above inequalities (2.11)-(2.12), then we obtain some inequalities similar to the Ostrowski type inequalities from Theorem 2, namely*

$$
\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \frac{1}{2(b-a)}\left[(x-a)^{2}\|f'\|_{[a,x],\infty} + (b-x)^{2}\|f'\|_{[x,b],\infty} \right] \\
&\leq \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^{2} + \left(\frac{b-x}{b-a}\right)^{2} \right](b-a) \|f'\|_{[a,b],\infty} \\
&= \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right](b-a)\|f'\|_{[a,b],\infty}
\end{aligned} \tag{2.13}
$$

for $x \in [a,b]$, *provided* $f' \in L_\infty [a,b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{2.14}$$
$$\leq \frac{1}{(b-a)(\alpha+1)^{1/\alpha}} \left[(x-a)^{1+1/\alpha} \|f'\|_{[a,x],\beta} + (b-x)^{1+1/\alpha} \|f'\|_{[x,b],\beta} \right]$$
$$\leq \frac{1}{(\alpha+1)^{1/\alpha}} \left[\left(\frac{x-a}{b-a}\right)^{1+1/\alpha} + \left(\frac{b-x}{b-a}\right)^{1+1/\alpha} \right] (b-a)^{1/\alpha} \|f'\|_{[a,b],\beta}$$

for $x \in [a,b]$, *provided* $f' \in L_\beta [a,b]$, $\alpha > 1, 1/\alpha + 1/\beta = 1$ *and*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{2.15}$$
$$\leq \frac{1}{b-a} \left[(x-a) \|f'\|_{[a,x],1} + (b-x) \|f'\|_{[x,b],1} \right]$$
$$= \left\{ \frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right\} \|f'\|_{[a,b],1}$$

for $x \in [a,b]$.

3. Applications for p-Norms

We have the following inequalities for Lebesgue norms of the deviation of a function from its integral mean:

Theorem 8. *Let* $f : [a,b] \to \mathbb{R}$ *be absolutely continuous on* $[a,b]$.

(i) *If* $f' \in L_\infty [a,b]$, *then*

$$\left\| f - \frac{1}{b-a} \int_a^b f(t)\,dt \right\|_{[a,b],p} \leq \left[\frac{2}{(p+1)(p+2)} \right]^{1/p} (b-a)^{1+\frac{1}{p}} \|f'\|_{[a,b],\infty}. \tag{3.1}$$

(ii) *If* $f' \in L_{p\beta} [a,b]$, *with* $\alpha > 1, 1/\alpha + 1/\beta = 1$ *then*

$$\left\| f - \frac{1}{b-a} \int_a^b f(t)\,dt \right\|_{[a,b],p} \tag{3.2}$$
$$\leq \left[\frac{2}{(p\alpha+1)^{1/\alpha} (p + 1/\alpha + 1)} \right]^{1/p} \|f'\|_{[a,b],p\beta} (b-a)^{1+\frac{1}{\alpha p}}.$$

(iii) *We have*

$$\left\| f - \frac{1}{b-a} \int_a^b f(t)\,dt \right\|_{[a,b],p} \leq \frac{1}{2} \left(\frac{2^{p+1}-1}{p+1} \right)^{1/p} (b-a) \|f'\|_{[a,b],p}. \tag{3.3}$$

Proof. Integrating on $[a,b]$ the inequality (2.10) we have

$$\begin{aligned} &\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dx \\ &\leq \frac{1}{(b-a)(p+1)} \|f'\|_{[a,b],\infty}^p \int_a^b \left[(x-a)^{p+1} + (b-x)^{p+1} \right] dx \\ &= \frac{1}{(b-a)(p+1)} \|f'\|_{[a,b],\infty}^p \left[\frac{2(b-a)^{p+2}}{p+2} \right] \\ &= \frac{2}{(p+1)(p+2)} \|f'\|_{[a,b],\infty}^p (b-a)^{p+1} \end{aligned} \tag{3.4}$$

which is equivalent with (3.1).

Integrating the inequality (2.11)

$$\begin{aligned} &\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dx \\ &\leq \frac{1}{(b-a)(p\alpha+1)^{1/\alpha}} \|f'\|_{[a,b],p\beta}^p \int_a^b \left[(x-a)^{p+1/\alpha} + (b-x)^{p+1/\alpha} \right] dx \\ &= \frac{1}{(b-a)(p\alpha+1)^{1/\alpha}} \|f'\|_{[a,b],p\beta}^p \left[\frac{2(b-a)^{p+1/\alpha+1}}{p+1/\alpha+1} \right] \\ &= \frac{2}{(p\alpha+1)^{1/\alpha}(p+1/\alpha+1)} \|f'\|_{[a,b],p\beta}^p (b-a)^{p+1/\alpha} \end{aligned} \tag{3.5}$$

which is equivalent with (3.2).

Integrating the inequality (2.12) we have

$$\begin{aligned} &\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dx \\ &\leq \frac{1}{b-a} \|f'\|_{[a,b],p}^p \int_a^b \max\left\{ (x-a)^p, (b-x)^p \right\} dx. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} &\int_a^b \max\left\{ (x-a)^p, (b-x)^p \right\} dx \\ &= \int_a^{\frac{a+b}{2}} (b-x)^p\,dx + \int_{\frac{a+b}{2}}^b (x-a)^p\,dx \\ &= -\frac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1} + \frac{(b-a)^{p+1}}{p+1} + \frac{(b-a)^{p+1}}{p+1} - \frac{\left(\frac{b-a}{2}\right)^{p+1}}{p+1} \\ &= \frac{1}{p+1}\left(\frac{2^{p+1}-1}{2^p} \right)(b-a)^{p+1} \end{aligned}$$

then from (3.6) we get

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dx \leq \frac{1}{p+1}\left(\frac{2^{p+1}-1}{2^p} \right)(b-a)^p \|f'\|_{[a,b],p}^p ,$$

which is equivalent with (3.3). □

4. Applications for the Exponential

If we write the inequality (2.1) for the convex function $\Phi(x) = \exp(x)$ then we get the inequality

$$\exp\left[f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] \le \frac{1}{b-a}\left[\int_a^x \exp\left[(t-a) f'(t)\right] dt + \int_x^b \exp\left[(t-b) f'(t)\right] dt\right] \tag{4.1}$$

for $x \in [a,b]$.

If we write the inequality (2.1) for the convex function $\Phi(x) = \cosh(x) := \frac{e^x + e^{-x}}{2}$ then we get the inequality

$$\cosh\left[f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right] \le \frac{1}{b-a}\left[\int_a^x \cosh\left[(t-a) f'(t)\right] dt + \int_x^b \cosh\left[(t-b) f'(t)\right] dt\right] \tag{4.2}$$

for $x \in [a,b]$.

Utilising the inequality (4.1) we have the following multiplicative version of Ostrowski's inequality:

Theorem 9. *Let $f : [a,b] \to (0,\infty)$ be absolutely continuous on $[a,b]$. Then we have the inequalities*

$$\frac{f(x)}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \le \frac{1}{b-a}\left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right] dt + \int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right] dt\right] \tag{4.3}$$

for any $x \in [a,b]$ and

$$\frac{\int_a^b f(x)\,dx}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \le \frac{1}{b-a}\left[\int_a^b (b-x)\exp\left[(x-a)\frac{f'(x)}{f(x)}\right] dx + \int_a^b (x-a)\exp\left[(x-b)\frac{f'(x)}{f(x)}\right] dx\right]. \tag{4.4}$$

Proof. If we replace f by $\ln f$ in (4.1) we get

$$\begin{aligned}(4.5)\qquad & \exp\left[\ln f(x)-\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right] \\ & \leq \frac{1}{b-a}\left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt+\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right]\end{aligned}$$

for any $x\in[a,b]$.

Since

$$\begin{aligned}& \exp\left[\ln f(x)-\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right] \\ & =\exp\left[\ln f(x)-\ln\left\{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)\right\}\right] \\ & =\exp\left[\ln\left(\frac{f(x)}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}\right)\right] \\ & =\frac{f(x)}{\exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right)}\end{aligned}$$

for any $x\in[a,b]$, then we get from (4.5) the desired inequality (4.3).

If we integrate the inequality (4.3) we get

$$\begin{aligned}(4.6)\qquad & \frac{\int_a^b f(x)\,dx}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \\ & \leq \frac{1}{b-a}\int_a^b\left[\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt+\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right]dx.\end{aligned}$$

Integrating by parts we have

$$\begin{aligned}& \int_a^b\left(\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\right)dx \\ & =x\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\bigg|_a^b-\int_a^b xd\left(\int_a^x \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt\right) \\ & =b\int_a^b \exp\left[(t-a)\frac{f'(t)}{f(t)}\right]dt-\int_a^b x\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx \\ & =\int_a^b (b-x)\exp\left[(x-a)\frac{f'(x)}{f(x)}\right]dx\end{aligned}$$

and

$$\int_a^b \left(\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right)dx$$
$$= x\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\bigg|_a^b - \int_a^b x d\left(\int_x^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt\right)$$
$$= -a\int_a^b \exp\left[(t-b)\frac{f'(t)}{f(t)}\right]dt + \int_a^b x\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx$$
$$= \int_a^b (x-a)\exp\left[(x-b)\frac{f'(x)}{f(x)}\right]dx,$$

then by (4.6) we deduce the desired inequality (4.4). □

5. Applications for Midpoint-Inequalities

We have from the inequality (2.1) written for $-f$ the following result:

Proposition 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then from (2.1) we have the inequalities*

$$\Phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right) \le (\ge)\frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \Phi\left[(b-t)f'(t)\right]dt + \int_a^{\frac{a+b}{2}} \Phi\left[(a-t)f'(t)\right]dt\right]. \tag{5.1}$$

If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then by Hermite-Hadamard inequality we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \ge f\left(\frac{a+b}{2}\right).$$

We can state the following result in which the function Φ is assumed be convex only on $[0,\infty)$ or $(0,\infty)$.

Proposition 2. *If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$. If $\Phi:[0,\infty),(0,\infty)\to\mathbb{R}$ is convex (concave) on $[0,\infty)$ or $(0,\infty)$, then (5.1) holds true.*

If $f:[a,b]\to\mathbb{R}$ is strictly convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$, then by taking $\Phi(x)=\ln x$, which is strictly concave on $(0,\infty)$, we get the logarithmic inequality

$$\ln\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right) \ge \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \ln\left[(b-t)f'(t)\right]dt + \int_a^{\frac{a+b}{2}} \ln\left[(a-t)f'(t)\right]dt\right]. \tag{5.2}$$

If $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, monotonic nondecreasing on $\left[a,\frac{a+b}{2}\right]$ and monotonic nonincreasing $\left[a,\frac{a+b}{2}\right]$, then by taking $\Phi(x)=x^q$, with $q\in(0,1)$ we also have

$$\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right)^q \tag{5.3}$$
$$\geq \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b [(b-t)\,f'(t)]^q\,dt + \int_a^{\frac{a+b}{2}} [(a-t)\,f'(t)]^q\,dt\right].$$

If $\Phi : [0,\infty), (0,\infty) \to \mathbb{R}$ is convex (concave) on $[0,\infty)$ or $(0,\infty)$, and if we take $f(t) := \left|t - \frac{a+b}{2}\right|^p$, $p \geq 1$, then we get from (5.1)

$$\Phi\left(\frac{(b-a)^p}{2^p(p+1)}\right) \leq (\geq) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^b \Phi\left[p(b-t)\left(t-\frac{a+b}{2}\right)^{p-1}\right]dt\right. \tag{5.4}$$
$$\left.+\int_a^{\frac{a+b}{2}} \Phi\left[(t-a)\left(\frac{a+b}{2}-t\right)^{p-1}\right]dt\right].$$

Let us recall the following means:

(a) The *arithmetic mean*

$$A = A(a,b) := \frac{a+b}{2}, a, b \geq 0;$$

(b) The *geometric mean*

$$G = G(a,b) := \sqrt{ab}; a, b \geq 0;$$

(c) The *harmonic mean*

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; a, b > 0;$$

(d) The *logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \quad a, b > 0;$$

(e) The *identric mean*

$$I = I(a,b) := \begin{cases} a \text{ if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \text{ if } a \neq b \end{cases};$$

(f) The *p-logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} a \text{ if } a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} \text{ if } a \neq b \end{cases};$$

where $p \in \mathbb{R}\setminus\{0,-1\}$.

The following inequality is well-known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also well-known that L_p is monotonically increasing over p, assuming that $L_0 = I$ and $L_{-1} = L$.

Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$.

Now, if we take $f(t) = \frac{1}{t}$ in (5.1), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\Phi\left(\frac{A(a,b) - L(a,b)}{A(a,b)\,L(a,b)}\right) \le (\ge) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} \Phi\left(\frac{t-b}{t^2}\right) dt + \int_{a}^{\frac{a+b}{2}} \Phi\left(\frac{t-a}{t^2}\right) dt\right]. \tag{5.5}$$

If we take $f(t) = -\ln t$ in (5.1), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\Phi\left(\ln\left(\frac{A(a,b)}{I(a,b)}\right)\right) \le (\ge) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} \Phi\left(\frac{t-b}{t}\right) dt + \int_{a}^{\frac{a+b}{2}} \Phi\left(\frac{t-a}{t}\right) dt\right]. \tag{5.6}$$

If we take $f(t) = t^p$, $p \in \mathbb{R}\backslash\{0,-1\}$ in (5.1), where $t \in [a,b] \subset (0,\infty)$, then we have

$$\Phi\left(L_p^p(a,b) - A^p(a,b)\right) \le (\ge) \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} \Phi\left[p(b-t)\,t^{p-1}\right] dt + \int_{a}^{\frac{a+b}{2}} \Phi\left[p(a-t)\,t^{p-1}\right] dt\right]. \tag{5.7}$$

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