NORM INEQUALITIES OF ČEBYŠEV TYPE FOR POWER SERIES IN BANACH ALGEBRAS

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ABSTRACT. Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$, a Banach algebra, with $xy = yx$.

In this paper we establish some upper bounds for the norm of the *Čebyšev type difference*

$$f(\lambda) f(\lambda xy) - f(\lambda x) f(\lambda y)$$

provide that the complex number λ and the vectors $x, y \in \mathcal{B}$ are such that the series in the above expression are convergent. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

1. INTRODUCTION

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$(1.1) \qquad C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt.$$

In 1935, Grüss [18] showed that

$$(1.2) \qquad |C(f, g)| \le \frac{1}{4}(M - m)(N - n),$$

provided that there exists the real numbers m, M, n, N such that

$$(1.3) \qquad m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a, b].$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [4], states that

$$(1.4) \qquad |C(f, g)| \le \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b - a)^2,$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t \in [a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f, g : [a, b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$ while $\|f'\|_\infty = ess \sup_{t \in [a,b]} |f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [23]:

$$(1.5) \qquad |C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty,$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [21] in which he proved that

$$(1.6) \qquad |C(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a),$$

provided that f,g are absolutely continuous and $f',g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [2] have proved the following results:

$$(1.7) \qquad |C(f,g)| \leq \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}},$$

where $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$ or $p=1$ and $q=\infty$, and

$$(1.8) \qquad |C(f,g)| \leq \inf_{\gamma \in \mathbb{R}}\|g-\gamma\|_1 \cdot \frac{1}{b-a}\, ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|,$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ $(p>1, \frac{1}{p}+\frac{1}{q}=1; p=1, q=\infty$ or $p=\infty, q=1)$.

Notice that for $q=\infty, p=1$ in (1.7) we obtain

$$(1.9) \qquad |C(f,g)| \leq \inf_{\gamma \in \mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$

$$\leq \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$

and if g satisfies (1.3), then

$$(1.10) \qquad |C(f,g)| \leq \inf_{\gamma \in \mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$

$$\leq \left\|g-\frac{n+N}{2}\right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$

$$\leq \frac{1}{2}(N-n)\cdot\frac{1}{b-a}\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [5]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [3].

For other recent results on the Grüss inequality, see [1], [6]-[11], [17], [19], [22], [24], [25] and the references therein.

In order to consider a *Čebyšev type functional* for functions of vectors in Banach algebras, we need some preliminary definitions and results as follows.

2. Some Facts in Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \rightarrow [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \, \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*.

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}\mathcal{B}$. If $a, b \in \mathrm{Inv}\mathcal{B}$ then $ab \in \mathrm{Inv}\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}\mathcal{B}$;
(ii) $\{a \in \mathcal{B} : \|1 - b\| < 1\} \subset \mathrm{Inv}\mathcal{B}$;
(iii) $\mathrm{Inv}\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}\mathcal{B} \ni a \longmapsto a^{-1} \in \mathrm{Inv}\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : \ \lambda - a \in \mathrm{Inv}\mathcal{B}\} ;$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \rightarrow \mathrm{Inv}\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma) .$$

We also have that $\sigma(a) \subset \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|a\|\}$. The *spectral radius* of a is defined as $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \leq \nu(a)\nu(b) \ \text{and} \ \nu(a + b) \leq \nu(a) + \nu(b) .$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \rightarrow \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have

$$a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi;$$

(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \ (|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \, \|a^j\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important

functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a + b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1 - a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1 - a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x + y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\, dt = x^{-1}\left[\exp(bx) - \exp(ax)\right].$$

Moreover, if x and y are commuting and $y - x$ is invertible, then

$$\int_0^1 \exp((1 - s)x + sy)\, ds = \int_0^1 \exp(s(y - x))\exp(x)\, ds$$
$$= \left(\int_0^1 \exp(s(y - x))\, ds\right)\exp(x)$$
$$= (y - x)^{-1}\left[\exp(y - x) - I\right]\exp(x)$$
$$= (y - x)^{-1}\left[\exp(y) - \exp(x)\right].$$

Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$ with $xy = yx$. In this paper we establish some upper bounds for the norm of the *Čebyšev type difference*

$$(2.1) \qquad\qquad f(\lambda)f(\lambda xy) - f(\lambda x)f(\lambda y)$$

provide that the complex number λ and the vectors $x, y \in \mathcal{B}$ are such that the series in (2.1) are convergent. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

Inequalities for functions of operators in Hilbert spaces may be found in the recent monographs [12], [13], [16] and the references therein.

3. The Results

We denote by $\mathbb{C}$ the set of all complex numbers. Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \le R \le \infty$, but we consider only the case $0 < R \le \infty$.

Denote by:

$$D(0, R) = \begin{cases} \{z \in \mathbb{C} : |z| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0, R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0, R) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0, R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } \|x\| < R \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map:

$$x \mapsto \tilde{f}(x) : B(0, R) \to \mathcal{B}, \tilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, $\tilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \le \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions have that:

Theorem 1. *Let $\lambda \in \mathbb{C}$ such that $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ and let $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le 1$ and $xy = yx$. Then:*

(i) *We have*

$$(3.1) \qquad \left\| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \right\|$$
$$\le \sqrt{2} \psi \min\{\|x - 1\|, \|y - 1\|\} f_A\left(|\lambda|^2\right)$$

where:

$$(3.2) \qquad \psi^2 := s_0 s_2 - s_1^2.$$

(ii) *We also have*

$$(3.3) \qquad \left\| \tilde{f}(\lambda \cdot 1) \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \tilde{f}(\lambda y) \right\|$$
$$\le \sqrt{2} \min\{\|x - 1\|, \|y - 1\|\} f_A(|\lambda|)$$
$$\times \left\{ f_A(|\lambda|) \left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - [|\lambda| f_A'(|\lambda|)]^2 \right\}^{1/2}.$$

Proof. For $m \geq 1$ and since $xy = yx$ we have

(3.4)
$$\sum_{n=0}^{m} \sum_{j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j \left(x^n - x^j \right) y^n$$

$$= \sum_{n=0}^{m} \sum_{j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j x^n y^n - \sum_{n=0}^{m} \sum_{j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j x^j y^n$$

$$= \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n x^n y^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n$$

$$= \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n$$

for any $\lambda \in \mathbb{C}$.

Taking the norm in (3.4) we have

(3.5)
$$\left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \right\|$$

$$\leq \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| \left(x^n - x^j \right) y^n \right\|$$

$$\leq \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| x^n - x^j \right\| \|y^n\|$$

$$\leq \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| x^n - x^j \right\| \|y\|^n$$

$$\leq \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| x^n - x^j \right\|$$

$$= 2 \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| x^n - x^j \right\|,$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Observe that

(3.6)
$$L := \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| x^n - x^j \right\|$$

$$= \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| \sum_{\ell=j}^{n-1} \left(x^{\ell+1} - x^\ell \right) \right\|$$

$$= \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\| \sum_{\ell=j}^{n-1} x^\ell (x - 1) \right\|$$

$$\leq \|x - 1\| \sum_{0 \leq j < n \leq m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \sum_{\ell=j}^{n-1} \|x\|^\ell$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

We have

$$\sum_{\ell=j}^{n-1} \|x\|^{\ell} \le (n-j) \max_{\ell \in \{j,...,n-1\}} \|x\|^{\ell} \le (n-j) \max_{\ell \in \{1,...,m-1\}} \|x\|^{\ell}$$

and then

$$(3.7) \qquad L \le \|x-1\| \max_{\ell \in \{1,...,m-1\}} \|x\|^{\ell} \sum_{0 \le j < n \le m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n-j).$$

From the first inequality in (3.7) and since $\|x\| < 1$ we have

$$(3.8) \qquad \left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \right\|$$

$$\le 2 \|x-1\| \sum_{0 \le j < n \le m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j (n-j)$$

$$= \|x-1\| \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j|.$$

(i) Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums

$$\sum_{n=0}^{m} \sum_{j=0}^{m} p_{n,j} a_{n,j} b_{n,j} \le \left(\sum_{n=0}^{m} \sum_{j=0}^{m} p_{n,j} a_{n,j}^2 \right)^{1/2} \left(\sum_{n=0}^{m} \sum_{j=0}^{m} p_{n,j} b_{n,j}^2 \right)^{1/2}$$

where $p_{n,j}, a_{n,j}, b_{n,j} \ge 0$ for $n,j \in \{0,...,m\}$, we have

$$(3.9) \qquad \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j|$$

$$\le \left(\sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^{2n} |\lambda|^{2j} \right)^{1/2} \left(\sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |n-j|^2 \right)^{1/2}$$

$$= \sqrt{2} \left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^{2n} \right) \left[\sum_{n=0}^{m} |\alpha_n| \sum_{n=0}^{m} n^2 |\alpha_n| - \left(\sum_{n=0}^{m} n |\alpha_n| \right)^2 \right]^{1/2}$$

for any $\lambda \in \mathbb{C}$ and $m \ge 1$.

From (3.8) and (3.9) we get the inequality

$$(3.10) \qquad \left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \right\|$$

$$\le \sqrt{2} \|x-1\| \left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^{2n} \right)$$

$$\times \left[\sum_{n=0}^{m} |\alpha_n| \sum_{n=0}^{m} n^2 |\alpha_n| - \left(\sum_{n=0}^{m} n |\alpha_n| \right)^2 \right]^{1/2}.$$

Since the series

$$\sum_{j=0}^{\infty} \alpha_j \lambda^j, \sum_{n=0}^{\infty} \alpha_n \lambda^n (xy)^n, \sum_{j=0}^{\infty} \alpha_j \lambda^j x^j, \sum_{n=0}^{\infty} \alpha_n \lambda^n y^n$$

are convergent in $\mathcal{B}$, $\sum_{n=0}^{\infty} |\alpha_n| |\lambda|^{2n}$ is convergent and the limit

$$\lim_{m \to \infty} \left[\sum_{n=0}^{m} |\alpha_n| \sum_{n=0}^{m} n^2 |\alpha_n| - \left(\sum_{n=0}^{m} n |\alpha_n| \right)^2 \right]^{1/2}$$

exists, then by letting $m \to \infty$ in (3.10) we deduce the desired result in (3.1) for x. Due to the commutativity of x with y, a similar result can be stated for y, and taking the minimum, we deduce the desired result.

(ii) Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums

$$\sum_{n=0}^{m} \sum_{j=0}^{m} p_{n,j} a_{n,j} \leq \left(\sum_{n=0}^{m} \sum_{j=0}^{m} p_{n,j} \right)^{1/2} \left(\sum_{n=0}^{m} \sum_{j=0}^{m} p_{n,j} a_{n,j}^2 \right)^{1/2}$$

where $p_{n,j}, a_{n,j} \geq 0$ for $n, j \in \{0, ..., m\}$, we also have

$$(3.11) \quad \sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n - j|$$

$$\leq \left(\sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \right)^{1/2} \left(\sum_{n=0}^{m} \sum_{j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n - j|^2 \right)^{1/2}$$

$$= \sqrt{2} \left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^n \right)$$

$$\times \left[\sum_{n=0}^{m} |\alpha_n| |\lambda|^n \sum_{n=0}^{m} n^2 |\alpha_n| |\lambda|^n - \left(\sum_{n=0}^{m} n |\alpha_n| |\lambda|^n \right)^2 \right]^{1/2}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

From (3.8) and (3.11) we have

$$(3.12) \quad \left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \right\|$$

$$\leq \sqrt{2} \|x - 1\| \left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^n \right)$$

$$\times \left[\sum_{n=0}^{m} |\alpha_n| |\lambda|^n \sum_{n=0}^{m} n^2 |\alpha_n| |\lambda|^n - \left(\sum_{n=0}^{m} n |\alpha_n| |\lambda|^n \right)^2 \right]^{1/2}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

If we denote $f(u) := \sum_{n=0}^{\infty} \alpha_n u^n$, then for $|u| < R$ we have

$$\sum_{n=0}^{\infty} n \alpha_n u^n = u f'(u)$$

and

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^n = u \left(u g' \left(u \right) \right)'.$$

However

$$u \left(u g' \left(u \right) \right)' = u g' \left(u \right) + u^2 g'' \left(u \right)$$

and then

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^n = u g' \left(u \right) + u^2 g'' \left(u \right).$$

Therefore

$$\sum_{n=0}^{\infty} n^2 \left| \alpha_n \right| \left| \lambda \right|^n = \left| \lambda \right| f_A' \left(\left| \lambda \right| \right) + \left| \lambda \right|^2 f_A'' \left(\left| \lambda \right| \right)$$

and

$$\sum_{n=0}^{m} n \left| \alpha_n \right| \left| \lambda \right|^n = \left| \lambda \right| f' \left(\left| \lambda \right| \right)$$

for $\left| \lambda \right| < R$.

Since all the series whose partial sums are involved in (3.12) are convergent, then by letting $m \to \infty$ in (3.12) we deduce the desired inequality (3.3) for x. Due to the commutativity of x with y, a similar result can be stated for y, and taking the minimum, we deduce the desired result. $\square$

Remark 1. *If $R = \infty$, Theorem 1 holds true. Moreover, in this case the restrictions $\left\| x \right\|, \left\| y \right\| \leq 1$ need no longer be imposed.*

Remark 2. *We observe that if the power series $f \left(\lambda \right) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ has the radius of convergence $R > 1$, then*

$$\sum_{n=0}^{\infty} \left| \alpha_n \right| = f_A \left(1 \right), \quad \sum_{n=0}^{\infty} n^2 \left| \alpha_n \right| = f_A' \left(1 \right) + f_A'' \left(1 \right)$$

and

$$\sum_{n=0}^{\infty} n \left| \alpha_n \right| = f_A' \left(1 \right).$$

In this case ψ is finite and

$$\psi = \lim_{m \to \infty} \left[\sum_{n=0}^{m} \left| \alpha_n \right| \sum_{n=0}^{m} n^2 \left| \alpha_n \right| - \left(\sum_{n=0}^{m} n \left| \alpha_n \right| \right)^2 \right]^{1/2}$$

$$= \left\{ f_A \left(1 \right) \left[f_A' \left(1 \right) + f_A'' \left(1 \right) \right] - \left[f_A' \left(1 \right) \right]^2 \right\}^{1/2}.$$

Therefore, if $\lambda \in \mathbb{C}$ with $\left| \lambda \right|, \left| \lambda \right|^2, \left| \lambda \right| \left\| x \right\|, \left| \lambda \right| \left\| y \right\| < R$, then from (3.1) we have

(3.13)
$$\left\| \tilde{f} \left(\lambda \cdot 1 \right) \tilde{f} \left(\lambda x y \right) - \tilde{f} \left(\lambda x \right) \tilde{f} \left(\lambda y \right) \right\|$$

$$\leq \sqrt{2} \left\{ f_A \left(1 \right) \left[f_A' \left(1 \right) + f_A'' \left(1 \right) \right] - \left[f_A' \left(1 \right) \right]^2 \right\}^{1/2}$$

$$\times \min \left\{ \left\| x - 1 \right\|, \left\| y - 1 \right\| \right\} f_A \left(\left| \lambda \right|^2 \right).$$

Corollary 1. *Under the assumptions of Theorem 1 we have the inequalities*

(3.14) $$\left\| \tilde{f}\left(\lambda \cdot 1\right) \tilde{f}\left(\lambda x^2\right) - \tilde{f}^2\left(\lambda x\right)\right\| \le \sqrt{2}\psi \left\| x - 1\right\| f_A\left(\left|\lambda\right|^2\right)$$

provided $\lambda \in \mathbb{C}$ *with* $\left|\lambda\right|, \left|\lambda\right|^2, \left|\lambda\right| \left\| x\right\| < R,$ *and*

(3.15) $$\left\| \tilde{f}\left(\lambda \cdot 1\right) \tilde{f}\left(\lambda x^2\right) - \tilde{f}^2\left(\lambda x\right)\right\|$$
$$\le \sqrt{2}\left\| x - 1\right\| f_A\left(\left|\lambda\right|\right)$$
$$\times \left\{ f_A\left(\left|\lambda\right|\right) \left[\left|\lambda\right| f_A'\left(\left|\lambda\right|\right) + \left|\lambda\right|^2 f_A''\left(\left|\lambda\right|\right)\right] - \left[\left|\lambda\right| f_A'\left(\left|\lambda\right|\right)\right]^2 \right\}^{1/2}$$

provided $\lambda \in \mathbb{C}$ *with* $\left|\lambda\right|, \left|\lambda\right| \left\| x\right\| < R.$

Theorem 2. *Let* $f\left(\lambda\right) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *be a function defined by power series with complex coefficients and convergent on the open disk* $D\left(0, R\right) \subset \mathbb{C}, R > 0$ *and* $x, y \in \mathcal{B}$ *with* $xy = yx$ *and* $\left\| x\right\|, \left\| y\right\| < 1.$
If $\lambda \in \mathbb{C}$ *with* $\left|\lambda\right|, \left|\lambda\right| \left\| x\right\|, \left|\lambda\right| \left\| y\right\| < R,$ *then*

(3.16) $$\left\| \tilde{f}\left(\lambda \cdot 1\right) \tilde{f}\left(\lambda xy\right) - \tilde{f}\left(\lambda x\right) \tilde{f}\left(\lambda y\right)\right\|$$
$$\le \min\left\{ \frac{\left\| x - 1\right\|}{1 - \left\| x\right\|}, \frac{\left\| y - 1\right\|}{1 - \left\| y\right\|}\right\} \left[f_A^2\left(\left|\lambda\right|\right) - f_{A^2}\left(\left|\lambda\right|^2\right)\right]$$

where

(3.17) $$f_{A^2}\left(\lambda\right) := \sum_{n=0}^{\infty} \left|\alpha_n\right|^2 \lambda^n$$

has the radius of convergence $R^2.$

Proof. As pointed out in (3.6), we have

(3.18) $$L \le \left\| x - 1\right\| \sum_{0 \le j < n \le m} \left|\alpha_n\right| \left|\alpha_j\right| \left|\lambda\right|^n \left|\lambda\right|^j \sum_{\ell=j}^{n-1} \left\| x\right\|^\ell$$
$$\le \left\| x - 1\right\| \sum_{\ell=0}^{m-1} \left\| x\right\|^\ell \sum_{0 \le j < n \le m} \left|\alpha_n\right| \left|\alpha_j\right| \left|\lambda\right|^n \left|\lambda\right|^j$$

for any $\lambda \in \mathbb{C}$ and $m \ge 1.$
 Denote
$$K_m := \sum_{0 \le j < n \le m} \left|\alpha_n\right| \left|\alpha_j\right| \left|\lambda\right|^n \left|\lambda\right|^j.$$

We obviously have

$$K_m = \frac{1}{2}\left(\sum_{n,j=0}^{m} \left|\alpha_n\right| \left|\alpha_j\right| \left|\lambda\right|^n \left|\lambda\right|^j - \sum_{n=0}^{m} \left|\alpha_n\right|^2 \left|\lambda\right|^{2n}\right)$$
$$= \frac{1}{2}\left[\left(\sum_{n=0}^{m} \left|\alpha_n\right| \left|\lambda\right|^n\right)^2 - \sum_{n=0}^{m} \left|\alpha_n\right|^2 \left|\lambda\right|^{2n}\right].$$

From (3.8) and (3.18) we get the inequality

$$(3.19) \quad \left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \right\|$$

$$\leq \|x - 1\| \sum_{\ell=0}^{m-1} \|x\|^\ell$$

$$\times \left[\left(\sum_{n=0}^{m} |\alpha_n| \, |\lambda|^n \right)^2 - \sum_{n=0}^{m} |\alpha_n|^2 \, |\lambda|^{2n} \right],$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Since all the series whose partial sums are involved in (3.19) are convergent, then by letting $m \to \infty$ in (3.19) we deduce the desired inequality (3.16) for x. Due to the commutativity of x with y, a similar result can be stated for y, and taking the minimum, we deduce the desired result. □

Remark 3. *Since the power series $f_{A^2}(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|^2 \lambda^n$ is not easy to compute, we can provide some bounds for the quantity*

$$D_f(|\lambda|) := f_A^2(|\lambda|) - f_{A^2}\left(|\lambda|^2\right)$$

where $|\lambda| < R$, as follows.

If $|\lambda| < 1$ and $a_{\ell_\infty} := \sup_{n \in \mathbb{N}} \{|a_n|\} < \infty$, then

$$K_m \leq a_{\ell_\infty}^2 \sum_{0 \leq j < n \leq m} |\lambda|^n \, |\lambda|^j$$

$$= \frac{1}{2} a_{\ell_\infty}^2 \left[\left(\sum_{n=0}^{m} |\lambda|^n \right)^2 - \sum_{n=0}^{m} |\lambda|^{2n} \right]$$

and by taking $m \to \infty$ in this inequality we get

$$(3.20) \quad D_f(|\lambda|) \leq \frac{1}{2} a_{\ell_\infty}^2 \left[\left(\frac{1}{1 - |\lambda|} \right)^2 - \frac{1}{1 - |\lambda|^2} \right]$$

for $|\lambda| < 1$.

If $|\lambda| < 1$ and

$$a_{\ell_1} := \lim_{m \to \infty} \left[\left(\sum_{n=0}^{m} |\alpha_n| \right)^2 - \sum_{n=0}^{m} |\alpha_n|^2 \right] < \infty$$

then

$$K_m \leq \max_{n \in \{0, \ldots, m\}} |\lambda|^{2n} \sum_{0 \leq j < n \leq m} |\alpha_n| \, |\alpha_j|$$

$$\leq \frac{1}{2} \left[\left(\sum_{n=0}^{m} |\alpha_n| \right)^2 - \sum_{n=0}^{m} |\alpha_n|^2 \right]$$

and by taking $m \to \infty$ in this inequality we get

$$(3.21) \quad D_f(|\lambda|) \leq \frac{1}{2} a_{\ell_1}$$

for $|\lambda| < 1$.

If the series $\sum_{n=0}^{\infty} |\alpha_n|$ and $\sum_{n=0}^{\infty} |\alpha_n|^2$ are convergent, then

$$(3.22) \qquad D_f(|\lambda|) \le \frac{1}{2}\left[\left(\sum_{n=0}^{\infty} |\alpha_n|\right)^2 - \sum_{n=0}^{\infty} |\alpha_n|^2\right]$$

for $|\lambda| < 1$.

If $|\lambda| < 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and

$$a_{\ell_q} := \lim_{m\to\infty}\left[\left(\sum_{n=0}^{m} |\alpha_n|^q\right)^2 - \sum_{n=0}^{m} |\alpha_n|^{2q}\right] < \infty$$

then by Hölder's inequality we have

$$K_m \le \left(\sum_{0\le j<n\le m} |\alpha_n|^q |\alpha_j|^q\right)^{1/q}\left(\sum_{0\le j<n\le m} |\lambda|^{pn} |\lambda|^{pj}\right)^{1/p}$$

$$\le \left\{\frac{1}{2}\left[\left(\sum_{n=0}^{m} |\alpha_n|^q\right)^2 - \sum_{n=0}^{m} |\alpha_n|^{2q}\right]\right\}^{1/q}$$

$$\times \left\{\frac{1}{2}\left[\left(\sum_{n=0}^{m} |\lambda|^{pn}\right)^2 - \sum_{n=0}^{m} |\lambda|^{2pn}\right]\right\}^{1/p}$$

and by taking $m \to \infty$ in this inequality we get

$$(3.23) \qquad D_f(|\lambda|) \le \frac{1}{2}a_{\ell_q}^{1/q}\left[\left(\frac{1}{1-|\lambda|^p}\right)^2 - \frac{1}{1-|\lambda|^{2p}}\right]^{1/p}$$

for $|\lambda| < 1$.

If the series $\sum_{n=0}^{\infty} |\alpha_n|^q$ and $\sum_{n=0}^{\infty} |\alpha_n|^{2q}$ are convergent, then

$$(3.24) \qquad D_f(|\lambda|) \le \frac{1}{2}\left[\left(\sum_{n=0}^{\infty} |\alpha_n|^q\right)^2 - \sum_{n=0}^{\infty} |\alpha_n|^{2q}\right]^{1/p}$$

$$\times \left[\left(\frac{1}{1-|\lambda|^p}\right)^2 - \frac{1}{1-|\lambda|^{2p}}\right]^{1/p}$$

for $|\lambda| < 1$.

The following result also holds:

Theorem 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < 1$.*

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|^p, |\lambda| \, \|x\|, |\lambda| \, \|y\| < R$, then

$$(3.25) \qquad \left\| \tilde{f}(\lambda \cdot 1) \, \tilde{f}(\lambda xy) - \tilde{f}(\lambda x) \, \tilde{f}(\lambda y) \right\|$$

$$\leq \frac{1}{2} \min \left\{ \frac{\|x - 1\|}{(1 - \|x\|^p)^{1/p}}, \frac{\|y - 1\|}{(1 - \|y\|^p)^{1/p}} \right\}$$

$$\times \varphi^{1/q} \left[f_A^2 \left(|\lambda|^p \right) - f_{A^2} \left(|\lambda|^{2p} \right) \right]^{1/p},$$

where

$$(3.26) \qquad \varphi := \lim_{m \to \infty} \sum_{n,j=0}^{m} |\alpha_n| \, |\alpha_j| \, |n - j|$$

is assumed to exists and be finite.

Proof. Using Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and (3.6), we have

$$(3.27) \qquad L \leq \|x - 1\| \sum_{0 \leq j < n \leq m} |\alpha_n| \, |\alpha_j| \, |\lambda|^n \, |\lambda|^j \, (n - j)^{1/q} \left(\sum_{\ell=j}^{n-1} \|x\|^{\ell p} \right)^{1/p}$$

$$\leq \|x - 1\| \left(\sum_{\ell=0}^{m-1} \|x\|^{\ell p} \right)^{1/p} \sum_{0 \leq j < n \leq m} |\alpha_n| \, |\alpha_j| \, |\lambda|^n \, |\lambda|^j \, (n - j)^{1/q}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Applying Hölder's inequality once more we have

$$(3.28) \qquad \sum_{0 \leq j < n \leq m} |\alpha_n| \, |\alpha_j| \, |\lambda|^n \, |\lambda|^j \, (n - j)^{1/q}$$

$$\leq \left(\sum_{0 \leq j < n \leq m} |\alpha_n| \, |\alpha_j| \, |\lambda|^n \, (n - j) \right)^{1/q} \left(\sum_{0 \leq j < n \leq m} |\alpha_n| \, |\alpha_j| \, |\lambda|^{pn} \, |\lambda|^{pj} \right)^{1/p}$$

$$= \left(\frac{1}{2} \sum_{n,j=0}^{m} |\alpha_n| \, |\alpha_j| \, |n - j| \right)^{1/q}$$

$$\times \left(\frac{1}{2} \left[\left(\sum_{n=0}^{m} |\alpha_n| \, |\lambda|^{np} \right)^2 - \sum_{n=0}^{m} |\alpha_n|^2 \, |\lambda|^{2np} \right] \right)^{1/p}$$

$$= \frac{1}{2} \left(\sum_{n,j=0}^{m} |\alpha_n| \, |\alpha_j| \, |n - j| \right)^{1/q}$$

$$\times \left[\left(\sum_{n=0}^{m} |\alpha_n| \, |\lambda|^{np} \right)^2 - \sum_{n=0}^{m} |\alpha_n|^2 \, |\lambda|^{2np} \right]^{1/p}$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

From (3.8) and (3.28) we get the inequality

$$(3.29) \qquad \left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \right\|$$

$$\leq \frac{1}{2} \|x - 1\| \left(\sum_{\ell=0}^{m-1} \|x\|^{\ell p} \right)^{1/p} \left(\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |n - j| \right)^{1/q}$$

$$\times \left[\left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^{np} \right)^2 - \sum_{n=0}^{m} |\alpha_n|^2 |\lambda|^{2np} \right]^{1/p},$$

for any $\lambda \in \mathbb{C}$ and $m \geq 1$.

Since all the series whose partial sums are involved in (3.29) are convergent, then by letting $m \to \infty$ in (3.29)) we deduce the desired inequality (3.25) for x. Due to the commutativity of x with y, a similar result can be stated for y, and taking the minimum, we deduce the desired result. □

Remark 4. *Observe that*

$$\left[f_A^2 \left(|\lambda|^p \right) - f_{A^2} \left(|\lambda|^{2p} \right) \right]^{1/p} = D_f^{1/p} \left(|\lambda|^p \right)$$

and then further bounds for the inequality (3.25) may be provided by the use of Remark 3. However the details are not mentioned here.

We can obtain a simpler upper bound for φ as follows.

Using the Cauchy-Bunyakovsky-Schwarz inequality for double sums

$$\sum_{n=0}^{m} \sum_{j=0}^{m} p_{i,j} a_{i,j} \leq \left(\sum_{n=0}^{m} \sum_{j=0}^{m} p_{i,j} \right)^{1/2} \left(\sum_{n=0}^{m} \sum_{j=0}^{m} p_{i,j} a_{i,j}^2 \right)^{1/2}$$

where $p_{i,j}, a_{i,j} \geq 0$ for $i, j \in \{0, ..., m\}$, we have

$$(3.30) \qquad \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |n - j|$$

$$\leq \left(\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| \right)^{1/2} \left(\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |n - j|^2 \right)^{1/2}$$

$$= \sqrt{2} \sum_{n=0}^{m} |\alpha_n| \left[\sum_{n=0}^{m} |\alpha_n| \sum_{n=0}^{m} n^2 |\alpha_n| - \left(\sum_{n=0}^{m} n |\alpha_n| \right)^2 \right]^{1/2}$$

for $m \geq 1$.

If the series $\sum_{n=0}^{\infty} |\alpha_n|$ is finite and ψ defined by (3.2) is finite, then from (3.30) we have

$$(3.31) \qquad \varphi \leq \sqrt{2} \sum_{n=0}^{\infty} |\alpha_n| \psi.$$

We observe that, if the power series $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ has the radius of convergence $R > 1$, then ψ is finite and

$$\psi = \left\{ f_A(1)\left[f'_A(1) + f''_A(1)\right] - \left[f'_A(1)\right]^2 \right\}^{1/2}.$$

We have from (3.31) the inequality

$$(3.32) \qquad \varphi \leq \sqrt{2} f_A(1) \left\{ f_A(1)\left[f'_A(1) + f''_A(1)\right] - \left[f'_A(1)\right]^2 \right\}^{1/2}$$

4. Some Examples

As some natural examples that are useful for applications, we can point out that, if

$$(4.1) \qquad f(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C};$$

$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$

$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$(4.2) \qquad f_A(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1);$$

$$g_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$

$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$

$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$(4.3) \qquad \exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C},$$

$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$$_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

If we apply the inequality (3.13) to the exponential function, then we have

$$(4.4) \quad \|\exp[\lambda(1+xy)] - \exp[\lambda(x+y)]\| \leq \sqrt{2}e \min\{\|x-1\|, \|y-1\|\} \exp\left(|\lambda|^2\right)$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$.

If we take $y = -x$ in (4.4), then we get

$$(4.5) \qquad \left\|\exp\left[\lambda\left(1-x^2\right)\right] - 1\right\| \leq \sqrt{2}e \min\{\|x-1\|, \|x+1\|\} \exp\left(|\lambda|^2\right)$$

for any $x \in \mathcal{B}$ with $\|x\| < 1$ and $\lambda \in \mathbb{C}$

If we apply the inequality (3.3) for the exponential functions we also have

$$(4.6) \qquad \|\exp[\lambda(1+xy)] - \exp[\lambda(x+y)]\|$$
$$\leq \sqrt{2} \min\{\|x-1\|, \|y-1\|\} |\lambda|^{1/2} \exp(2|\lambda|),$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$.

If we take $y = -x$ in (4.6), then we get

$$(4.7) \qquad \left\|\exp\left[\lambda\left(1-x^2\right)\right] - 1\right\| \leq \sqrt{2} \min\{\|x-1\|, \|x+1\|\} |\lambda|^{1/2} \exp(2|\lambda|)$$

Now, consider the function $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$, $\lambda \in D(0,1)$. If we apply the inequality (3.3) for this function, then we get the result:

$$(4.8) \qquad \left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\|$$
$$\leq \sqrt{2} \min\{\|x-1\|, \|y-1\|\} |\lambda|^{1/2} (1-|\lambda|)^{-3}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.

We have in particular the inequalities

$$(4.9) \qquad \left\|(1-\lambda)^{-1}(1-\lambda x^2)^{-1} - (1-\lambda x)^{-2}\right\| \leq \sqrt{2}\|x-1\| |\lambda|^{1/2} (1-|\lambda|)^{-3}$$

and

$$(4.10) \qquad \left\| (1 - \lambda)^{-1} \left(1 + \lambda x^2 \right)^{-1} - \left(1 - \lambda^2 x^2 \right)^{-1} \right\|$$

$$\leq \sqrt{2} \min \left\{ \|x - 1\|, \|x + 1\| \right\} |\lambda|^{1/2} \left(1 - |\lambda| \right)^{-3}$$

for any $x \in \mathcal{B}$ with $\|x\| < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$.

Now, if we take $\lambda = \frac{1}{\gamma}$ with $|\gamma| > 1$ then we get from (4.8) the inequality

$$\left\| \gamma^2 (\gamma - 1)^{-1} (\gamma - xy)^{-1} - \gamma^2 (\gamma - x)^{-1} (\gamma - y)^{-1} \right\|$$

$$\leq \sqrt{2} \min \left\{ \|x - 1\|, \|y - 1\| \right\} |\gamma|^{-1/2} \left(|\gamma| - 1 \right)^{-3} |\gamma|^3$$

which is equivalent with

$$\left\| (\gamma - 1)^{-1} (\gamma - xy)^{-1} - (\gamma - x)^{-1} (\gamma - y)^{-1} \right\|$$

$$\leq \sqrt{2} \min \left\{ \|x - 1\|, \|y - 1\| \right\} |\gamma|^{1/2} \left(|\gamma| - 1 \right)^{-3}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\gamma \in \mathbb{C}$ with $|\gamma| > 1$.

If we use the *resolvent function* notation we then have the following inequality:

$$(4.11) \qquad \left\| (\gamma - 1)^{-1} R_{xy} (\gamma) - R_x (\gamma) R_y (\gamma) \right\|$$

$$\leq \sqrt{2} \min \left\{ \|x - 1\|, \|y - 1\| \right\} |\gamma|^{1/2} \left(|\gamma| - 1 \right)^{-3}$$

for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\gamma \in \mathbb{C}$ with $|\gamma| > 1$.

In particular, we have

$$(4.12) \qquad \left\| (\gamma - 1)^{-1} R_{x^2} (\gamma) - R_x^2 (\gamma) \right\| \leq \sqrt{2} \|x - 1\| |\gamma|^{1/2} \left(|\gamma| - 1 \right)^{-3}$$

for any $x \in \mathcal{B}$ with $\|x\| < 1$ and $\gamma \in \mathbb{C}$ with $|\gamma| > 1$.

Remark 5. *Similar inequalities may be stated for the other power series mentioned at the beginning of this paragraph. However, the details are not presented here.*

References

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