INEQUALITIES OF HERMITE-HADAMARD TYPE FOR
\(\varphi\)-CONVEX FUNCTIONS

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Abstract. Some inequalities of Hermite-Hadamard type for \(\varphi\)-convex functions defined on real intervals are given.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let \(I\) be an interval in \(\mathbb{R}\).

Definition 1 ([37]). We say that \(f : I \to \mathbb{R}\) is a Godunova-Levin function or that \(f\) belongs to the class \(Q(I)\) if \(f\) is non-negative and for all \(x, y \in I\) and \(t \in (0, 1)\) we have
\[
(1.1) \quad f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y).
\]

Some further properties of this class of functions can be found in [28], [29], [31], [43], [46] and [47]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([31]). We say that a function \(f : I \to \mathbb{R}\) belongs to the class \(P(I)\) if it is nonnegative and for all \(x, y \in I\) and \(t \in [0, 1]\) we have
\[
(1.2) \quad f(tx + (1-t)y) \leq f(x) + f(y).
\]

Obviously \(Q(I)\) contains \(P(I)\) and for applications it is important to note that also \(P(I)\) contain all nonnegative monotone, convex and quasi convex functions, i.e. nonnegative functions satisfying
\[
(1.3) \quad f(tx + (1-t)y) \leq \max\{f(x), f(y)\}
\]
for all \(x, y \in I\) and \(t \in [0, 1]\).

For some results on \(P\)-functions see [31] and [44] while for quasi convex functions, the reader can consult [30].

Definition 3 ([7]). Let \(s\) be a real number, \(s \in (0, 1]\). A function \(f : [0, \infty) \to [0, \infty)\) is said to be \(s\)-convex (in the second sense) or Breckner \(s\)-convex if
\[
(1.4) \quad f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
for all \(x, y \in [0, \infty)\) and \(t \in [0, 1]\).

1991 Mathematics Subject Classification. 26D15; 25D10.

Key words and phrases. Convex functions, Integral inequalities, \(h\)-Convex functions.
For some properties of this class of functions see [1], [2], [7], [8], [26], [27], [38], [40] and [49].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of \( h \)-convex functions as follows.

**Definition 4** ([52]). Let \( h : J \rightarrow [0, 1) \) with \( h \) not identical to 0. We say that \( f : I \rightarrow [0, 1) \) is an \( h \)-convex function if for all \( x, y \in I \) we have
\[
(1.4) \quad f \left( tx + (1 - t) y \right) \leq h(t) f(x) + h(1 - t) f(y)
\]
for all \( t \in (0, 1) \).

For some results concerning this class of functions see [52], [6], [41], [50], [48] and [51].

We can introduce now another class of functions.

**Definition 5.** We say that the function \( f : I \rightarrow [0, 1) \) is of \( s \)-Godunova-Levin type, with \( s \in [0, 1] \), if
\[
(1.5) \quad f \left( tx + (1 - t) y \right) \leq \frac{1}{t^s} f(x) + \frac{1}{(1 - t)^s} f(y),
\]
for all \( t \in (0, 1) \) and \( x, y \in I \).

We observe that for \( s = 0 \) we obtain the class of \( P \)-functions while for \( s = 1 \) we obtain the class of Godunova-Levin. If we denote by \( Q_s(I) \) the class of \( s \)-Godunova-Levin functions defined on \( I \), then we obviously have
\[
P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)
\]
for \( 0 \leq s_1 \leq s_2 \leq 1 \).

The following inequality holds for any convex function \( f \) defined on \( \mathbb{R} \)
\[
(1.6) \quad (b - a)f \left( \frac{a + b}{2} \right) \leq \int_a^b f(x)dx < (b - a)\frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}.
\]

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [42]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite’s note in *Mathesis* [42]. Since (1.6) was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[25], [32]-[35] and [45].

The following inequality of Hermite-Hadamard type holds [48]

**Theorem 1.** Assume that the function \( f : I \rightarrow [0, \infty) \) is an \( h \)-convex function with \( h \in L[0, 1] \). Let \( y, x \in I \) with \( y \neq x \) and assume that the mapping \( [0, 1] \ni t \mapsto f \left( (1 - t) x + ty \right) \) is Lebesgue integrable on \([0, 1] \). Then
\[
(1.7) \quad \frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f(u) du \leq \left[ f(x) + f(y) \right] \int_0^1 h(t) dt.
\]
functions. Also, if we put
\[ f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{2}. \]

If we write (1.7) for the case of \( P \)-type functions \( f : I \to [0,\infty) \), i.e., \( h(t) = 1 \), \( t \in [0,1] \), then we get the inequality
\[ \frac{1}{2} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq f(x) + f(y), \]
that has been obtained for functions of real variable in [31].

If \( f \) is Breckner \( s \)-convex on \( I \), for \( s \in (0,1) \), then by taking \( h(t) = t^s \) in (1.7) we get
\[ 2^{s-1} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{s+1}, \]
that was obtained for functions of a real variable in [26].

If \( f : I \to (0,\infty) \) is of \( s \)-Godunova-Levin type, with \( s \in [0,1] \), then
\[ \frac{1}{2s+t} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{1-s}. \]

We notice that for \( s = 1 \) the first inequality in (1.11) still holds, i.e.
\[ \frac{1}{4} f \left( \frac{x+y}{2} \right) \leq \int_0^1 f \left( [(1-t)x + ty] \right) dt. \]

The case for functions of real variables was obtained for the first time in [31].

2. \( \varphi \)-Convex Functions

We introduce the following class of \( h \)-convex functions.

**Definition 6.** Let \( \varphi : (0,1) \to (0,\infty) \) a measurable function. We say that the function \( f : I \to [0,\infty) \) is a \( \varphi \)-convex function on the interval \( I \) if for all \( x,y \in I \) we have
\[ f(tx + (1-t)y) \leq t \varphi(t) f(x) + (1-t) \varphi(1-t) f(y) \]
for all \( t \in (0,1) \).

If we denote \( \ell(t) = t \), the identity function, then it is obvious that \( f \) is \( h \)-convex with \( h = \ell \varphi \). Also, all the examples from the introduction can be seen as \( \varphi \)-convex functions with appropriate choices of \( \varphi \).

If we take \( \varphi(t) = \frac{1}{1-t} \) with \( s \in [0,1] \) then we get the class of \( s \)-Godunova-Levin functions. Also, if we put \( \varphi(t) = t^{s-1} \) with \( s \in (0,1) \), then we get the concept of Breckner \( s \)-convexity. We notice that for all these examples we have
\[ \varphi_+(0) = \lim_{t \to 0+} \varphi(t) = \infty. \]

The case of convex functions, i.e. when \( \varphi(t) = 1 \) is the only example from above for which \( \varphi_+(0) \) is finite, namely \( \varphi_+(0) = 1 \).

Consider the family of functions, for \( p > 1 \) and \( k > 0 \)
\[ \delta(p,k) : [0,1] \to \mathbb{R}_+, \delta(p,k)(t) = k (1-t)^p + 1. \]
We observe that \( \delta_+ (p, k) (0) = \delta (p, k) (0) = k + 1, \delta (p, k) \) is strictly decreasing on \([0, 1]\) and \( \delta (p, k) (1) = 1. \)

**Definition 7.** We say that the function \( f : I \to [0, \infty) \) is a \( \delta (p, k) \)-convex function on the interval \( I \) if for all \( x, y \in I \) we have
\[
(2.3) \quad f (tx + (1 - t) y) \leq t [k (1 - t)^p + 1] f (x) + (1 - t) (kt^p + 1) f (y)
\]
for all \( t \in (0, 1). \)

It is obvious that any nonnegative convex function is a \( \delta(p, k) \)-convex function for any \( p > 1 \) and \( k > 0. \)

For \( m > 0 \) we consider the family of functions
\[
\eta (m) : [0, 1] \to \mathbb{R}_+, \eta (m) (t) = \exp \left[ m (1 - t) \right].
\]
We observe that \( \eta_+ (m) (0) = \eta (m) (0) = \exp (m), \eta (m) \) is strictly decreasing on \([0, 1]\) and \( \eta (m) (t) \geq \eta (m) (1) = 1. \)

**Definition 8.** We say that the function \( f : I \to [0, \infty) \) is a \( \eta (m) \)-convex function on the interval \( I \) if for all \( x, y \in I \) we have
\[
(2.4) \quad f (tx + (1 - t) y) \leq t \exp [m (1 - t)] f (x) + (1 - t) \exp (mt) f (y)
\]
for all \( t \in (0, 1). \)

It is obvious that any nonnegative convex function is a \( \eta (m) \)-convex function for any \( m > 0. \)

There are many other examples one can consider. In fact any continuous function \( \varphi : [0, 1] \to [1, \infty) \) can generate a class of \( \varphi \)-convex function that contains the class of nonnegative convex functions.

Utilising Theorem 1 we can state the following result.

**Theorem 2.** Assume that the function \( f : I \to [0, \infty) \) is a \( \varphi \)-convex function with \( \ell \varphi \in L [0, 1]. \) Let \( y, x \in I \) with \( y \neq x \) and assume that the mapping \([0, 1] \ni t \mapsto f [(1 - t) x + ty] \) is Lebesgue integrable on \([0, 1]\). Then
\[
(2.5) \quad \frac{1}{\varphi \left( \frac{1}{2} \right)} f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f (u) \, du \leq [f (x) + f (y)] \int_0^1 \ell \varphi (t) \, dt.
\]

The proof follows from (1.7) by taking \( h (t) = t \ell \varphi (t), \ t \in (0, 1). \)

**Remark 1.** We notice that, since \( \int_0^1 \ell \varphi (t) \, dt \) can be seen as the expectation of a random variable \( X \) with the density function \( \varphi, \) the inequality (2.5) provides a connection to Probability Theory and motivates the introduction of \( \varphi \)-convex function as a natural concept, having available many examples of density functions \( \varphi \) that arise in applications.

We have the following particular cases:

**Corollary 1.** Assume that the function \( f : I \to [0, \infty) \) is a \( \alpha \delta (p, k) \)-convex function on the interval \( I \) with \( p > 1 \) and \( k > 0. \) Let \( y, x \in I \) with \( y \neq x \) and assume that the mapping \([0, 1] \ni t \mapsto f [(1 - t) x + ty] \) is Lebesgue integrable on \([0, 1]\). Then
\[
(2.6) \quad \frac{2p}{k + 2p} f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f (u) \, du
\]
\[
\leq [f (x) + f (y)] \left[ \frac{1}{2} + \frac{k}{(p + 1) (p + 2)} \right].
\]
Inequalities of Hermite-Hadamard Type

3. Some Results for Differentiable Functions

If we assume that the function \( f : I \rightarrow [0, \infty) \) is differentiable on the interior of \( I \) denoted by \( \bar{I} \) then we have the following "gradient inequality" that will play an essential role in the following.

Theorem 3. Let \( \varphi : (0, 1) \rightarrow (0, \infty) \) a measurable function and such that the right limit \( \varphi^+ (0) \) exists and is finite, the left limit \( \varphi^- (1) = 1 \) and the left derivative in 1 denoted \( \varphi^- (1) \) exists and is finite. If the function \( f : I \rightarrow [0, \infty) \) is differentiable on \( \bar{I} \) and \( \varphi \)-convex, then

\[
\varphi^+ (0) f (x) - \varphi^- (1) f (y) \geq f' (y) (x - y)
\]

for any \( x, y \in \bar{I} \) with \( x \neq y \).

Proof. Since \( f \) is \( \varphi \)-convex on \( I \), then

\[
t \varphi (t) f (x) + (1 - t) \varphi (1 - t) f (y) \geq f (tx + (1 - t) y)
\]

for any \( t \in (0, 1) \) and for any \( x, y \in \bar{I} \), which is equivalent to

\[
t \varphi (t) f (x) + [(1 - t) \varphi (1 - t) - 1] f (y) \geq f (tx + (1 - t) y) - f (y)
\]

and by dividing by \( t > 0 \) we get

\[
\varphi (t) f (x) + \left[ \frac{(1 - t) \varphi (1 - t) - 1}{t} \right] f (y) \geq \frac{f (tx + (1 - t) y) - f (y)}{t}
\]

for any \( t \in (0, 1) \).
Now, since \( f \) is differentiable on \( y \in \hat{I} \), then we have

\[
\lim_{t \to 0^+} \frac{f(tx + (1 - t)y) - f(y)}{t} = \lim_{t \to 0^+} \frac{f(y + t(x - y)) - f(y)}{t} = (x - y) f'(y)
\]

for any \( x \in \hat{I} \) with \( x \neq y \).

Also since \( \varphi\_+ (1) = 1 \) and \( \varphi\_' (1) \) exists and is finite, we have

\[
\lim_{t \to 0^+} \frac{(1 - t) \varphi(1 - t) - 1}{t} = \lim_{s \to 1^+} \frac{s \varphi(s) - 1}{1 - s} = - \lim_{s \to 1^+} \frac{s \varphi(s) - \varphi(1)}{s - 1} = - \varphi\_' (1) - 1.
\]

Taking the limit over \( t \to 0^+ \) in (3.2) and utilizing (3.3) and (3.4) we get the desired result (3.1).

**Remark 2.** If we assume that

\[
\varphi\_+ (0) - \varphi\_- (1) \geq \varphi\_' (1),
\]

then the inequality (3.1) also holds for \( x = y \).

There are numerous examples of such functions, for instance, if \( \varphi(t) = k (1 - t)^p + 1, t \in [0, 1] \) \( (p > 0, k > 0) \) then \( \varphi\_+ (0) = k + 1, \varphi\_- (1) = 1 \) and \( \varphi\_' (1) = 0 \), which satisfy the condition (3.5).

If we take \( \varphi(t) = \exp [m (1 - t)] \) \( (m > 0) \), then \( \varphi\_+ (0) = \exp m, \varphi\_- (1) = 1 \) and \( \varphi\_' (1) = -m \). We have

\[
\varphi\_+ (0) - \varphi\_- (1) - \varphi\_' (1) = e^m - 1 + m > 0
\]

for \( m > 0 \).

The following result holds:

**Theorem 4.** Let \( \varphi : (0, 1) \to (0, \infty) \) a measurable function and such that the right limit \( \varphi\_+ (0) \) exists and is finite, the left limit \( \varphi\_- (1) = 1 \) and the left derivative in 1 denoted \( \varphi\_- (1) \) exists and is finite. Assume also that \( \varphi\_- (1) > -1 \). If the function \( f : I \to [0, \infty) \) is differentiable on \( \hat{I} \) and \( \varphi \)-convex, then

\[
\frac{\varphi\_+ (0)}{\varphi\_- (1) + 1} \cdot \frac{f(x) + f(y)}{2} \geq \frac{1}{y - x} \int_x^y f(u) \, du \geq \frac{\varphi\_- (1) + 1}{\varphi\_+ (0)} f\left(\frac{x + y}{2}\right)
\]

for any \( x, y \in I \).

**Proof.** Assume that \( y > x \) with \( x, y \in I \). From (3.1) we get

\[
\varphi\_+ (0) f(u) - [\varphi\_- (1) + 1] f\left(\frac{x + y}{2}\right) \geq f\left(\frac{x + y}{2}\right) \left( x - \frac{x + y}{2} \right)
\]

for any \( u \in [x, y] \) with \( u \neq \frac{x + y}{2} \).
Integrating this inequality over $u$ on $[x, y]$ we get
\[
\varphi_+ (0) \int_x^y f(u) \, du - \left[ \varphi' (1) + 1 \right] (y - x) f \left( \frac{x + y}{2} \right) \\
\geq f' \left( \frac{x + y}{2} \right) \int_x^y \left( u - \frac{x + y}{2} \right) \, du = 0
\]
which implies (3.6).

The case $y < x$ goes likewise and the proof of the second inequality in (3.6) is completed.

Assume that $y > x$ with $x, y \in I$.
From (3.1) we get
\[
\begin{align*}
\varphi_+ (0) f(x) - \left[ \varphi' (1) + 1 \right] f ((1-t)x + ty) \\
\geq f' ((1-t)x + ty) (x - (1-t)x - ty) \\
= tf' ((1-t)x + ty) (x - y)
\end{align*}
\]
for any $t \in (0, 1)$ and
\[
\begin{align*}
\varphi_+ (0) f(y) - \left[ \varphi' (1) + 1 \right] f ((1-t)x + ty) \\
\geq f' ((1-t)x + ty) (y - (1-t)x - ty) \\
= (1-t)f' ((1-t)x + ty) (y - x)
\end{align*}
\]
for any $t \in (0, 1)$.

Now, if we multiply (3.7) by $1 - t$, (3.8) by $t$ and add the obtained inequalities, then we get
\[
\varphi_+ (0) [(1-t)f(x) + tf(y)] \geq [\varphi' (1) + 1] f ((1-t)x + ty)
\]
for any $t \in (0, 1)$, that is of interest in itself as well.

Now, if we integrate this inequality on $[0, 1]$ we get
\[
\begin{align*}
\varphi_+ (0) \left[ f(x) \int_0^1 (1-t) \, dt + f(y) \int_0^1 t \, dt \right] \\
\geq [\varphi' (1) + 1] \int_0^1 f ((1-t)x + ty) \, dt.
\end{align*}
\]
Since
\[
\int_0^1 (1-t) \, dt = \int_0^1 t \, dt = \frac{1}{2}
\]
and
\[
\int_0^1 f ((1-t)x + ty) \, dt = \frac{1}{y-x} \int_x^y f(u) \, du,
\]
then by (3.11) we get the desired inequality (3.7).

Remark 3. Since the function $f$ takes nonnegative values, then the second inequality in (3.6) and the inequality (3.10) are trivially satisfied if $\varphi' (1) + 1 \leq 0$, so we must assume that $\varphi' (1) + 1 > 0$.

This condition is satisfied for the function $\varphi(t) = k (1-t)^p + 1$, $t \in [0, 1]$ ($p > 1, k > 0$). If $\varphi(t) = \exp [m (1-t)]$ ($m > 0$) then the condition $\varphi' (1)+1 = 1-m > 0$ is satisfied only for $m \in (0, 1)$.
Now, if we write the inequality (3.6) for \( \varphi(t) = k(1 - t)^p + 1 \), we get
\[
(k + 1) \frac{f(x) + f(y)}{2} \geq \frac{1}{y - x} \int_x^y f(u) \, du \geq \frac{1}{k + 1} f\left(\frac{x + y}{2}\right)
\]
From (2.6) we also have
\[
(f(x) + f(y)) \left[ \frac{1}{2} + \frac{k}{(p + 1)(p + 2)} \right] \geq \frac{1}{y - x} \int_x^y f(u) \, du \geq \frac{2^p}{k + 2^p} f\left(\frac{x + y}{2}\right).
\]
Since
\[
\frac{2^p}{k + 2^p} - \frac{1}{k + 1} = \frac{2^p k + 2^p - k}{(k + 2^p)(k + 1)} = \frac{(2^p - 1) k}{(k + 2^p)(k + 1)} \geq 0
\]
and
\[
\frac{k + 1}{2} - \frac{1}{k + 1} - \frac{k}{(p + 1)(p + 2)} = \frac{k}{2} - \frac{k}{(p + 1)(p + 2)} \geq 0
\]
it follows that the inequality (3.12) is better than (3.11).

Now, consider the family of functions
\[
\vartheta(k, p, q) := k t^p (1 - t)^q + 1
\]
where \( k > 0, p > 0 \) and \( q > 1 \).

**Definition 9.** We say that the function \( f : I \to [0, \infty) \) is a \( \vartheta(k, p, q) \)-convex function on the interval \( I \) if for all \( x, y \in I \) we have
\[
f(tx + (1 - t)y) \leq t [k t^p (1 - t)^q + 1] f(x) + (1 - t) [k (1 - t)^p t^q + 1] f(y)
\]
for all \( t \in (0, 1) \).

We observe that this class contains the class of nonnegative convex functions for any \( k > 0, p > 0 \) and \( q > 1 \).

**Corollary 3.** If the function \( f : I \to [0, \infty) \) is differentiable on \( I \) and \( \vartheta(k, p, q) \)-convex with \( k > 0, p > 0 \) and \( q > 1 \) then
\[
\frac{f(x) + f(y)}{2} \geq \frac{1}{y - x} \int_x^y f(u) \, du \geq f\left(\frac{x + y}{2}\right)
\]
for any \( x, y \in I \).

If we write the inequality (2.5) for \( \varphi = \vartheta(k, p, q) \), then we get
\[
\frac{1}{k \left(\frac{1}{2}\right)^{p+q} + 1} f\left(\frac{x + y}{2}\right) \leq \frac{1}{y - x} \int_x^y f(u) \, du \\
\leq [f(x) + f(y)] \left[ k \beta(p + 2, q + 1) + \frac{1}{2} \right]
\]
where
\[
\beta(u, v) := \int_a^1 t^{u-1} (1 - t)^{v-1}, u, v > 0
\]
is Euler’s Beta function.
Since
\[
\frac{1}{k \left(\frac{1}{2}\right)^{p+q} + 1} < 1 \text{ and } k \beta(p + 2, q + 1) + \frac{1}{2} \geq \frac{1}{2},
\]
it follows that the inequality (3.14) is better than (3.15).
Now, more generally, assume that

$$\varphi (g, q) : [0, 1] \to [1, \infty), \varphi (g, q) (t) = g (t) (1 - t)^q + 1$$

where $g : [0, 1] \to [0, \infty)$ is continuous and $q > 1$.

We then have

$$\varphi_+ (g, q) (0) = g (0) + 1, \quad \varphi_- (g, q) (1) = 1, \quad \varphi_- (g, q) (1) = 0$$

and

$$\varphi \left( \frac{1}{2} \right) = g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q + 1, \quad \int_0^1 t \varphi (t) \, dt = \int_0^1 t (1 - t)^q g (t) \, dt + \frac{1}{2}.$$

If we apply Theorem 2 to the function $\varphi (g, q)$ we have

$$(3.16) \quad [f (x) + f (y)] \left[ \int_0^1 t (1 - t)^q g (t) \, dt + \frac{1}{2} \right] \geq \frac{1}{y - x} \int_x^y f (u) \, du$$

$$\geq \frac{1}{g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q + 1} f \left( \frac{x + y}{2} \right).$$

If we apply Theorem 4 to the same function $\varphi (g, q)$ we also have

$$(3.17) \quad (g (0) + 1) \frac{f (x) + f (y)}{2} \geq \frac{1}{y - x} \int_x^y f (u) \, du$$

$$\geq \frac{1}{g (0) + 1} f \left( \frac{x + y}{2} \right).$$

Consider the difference

$$\Delta_1 := \frac{1}{g (0) + 1} - \frac{1}{g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q + 1}$$

$$= \frac{g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q - g (0)}{[g (0) + 1] [g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q + 1]}$$

and the difference

$$\Delta_2 := \int_0^1 t (1 - t)^q g (t) \, dt + \frac{1}{2} - \frac{g (0) + 1}{2}$$

$$= \int_0^1 t (1 - t)^q g (t) \, dt - \frac{1}{2} g (0).$$

We observe that if $\Delta_1, \Delta_2 \geq (\leq) 0$ then the double inequality (3.17) is better (worse) than (3.17).

If we take $g (0) = 0$, then (3.17) is better than (3.16) for any $q > 1$.

If we take $g (t) = kt + 1$, $k > 0$ then

$$\Delta_1 = \frac{g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q - g (0)}{[g (0) + 1] [g \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^q + 1]} = \frac{k \left( \frac{1}{2} \right)^{q+1}}{k \left( \frac{1}{2} \right)^{q+1} + 1} > 0$$

showing that the second inequality in (3.17) is better than the same inequality in (3.16) for any $k > 0$ and $q > 1$. 
We also have
\[ \Delta_2 = \int_0^1 t (1 - t)^q g(t) \, dt - \frac{1}{2} q(0) = \int_0^1 t (1 - t)^q (kt + 1) \, dt - \frac{1}{2} \]
\[ = k \int_0^1 t^2 (1 - t)^q \, dt + \int_0^1 t (1 - t)^q \, dt - \frac{1}{2} \]
\[ = k \beta(3, q + 1) + \beta(2, q + 1) - \frac{1}{2}. \]

If we take
\[ k > \frac{\frac{1}{2} - \beta(2, q + 1)}{\beta(3, q + 1)} = \frac{\frac{1}{2} - \frac{1}{(q+1)(q+2)}}{\beta(3, q + 1)} \]
\[ = \frac{(q + 1)(q + 2) - 2}{2(q + 1)(q + 2) \beta(3, q + 1)} (> 0) \]
then \( \Delta_2 > 0 \) showing that the first inequality in (3.17) is better than the first inequality in (3.16).

If we take
\[ 0 < k < \frac{(q + 1)(q + 2) - 2}{2(q + 1)(q + 2) \beta(3, q + 1)} \]
then \( \Delta_2 < 0 \) showing that the first inequality in (3.17) is worse than the first inequality in (3.16).

**Conclusion 1.** The inequalities (2.5) and (3.6) are not comparable, meaning that some time one is better than the other, depending on the \( \varphi \)-convex function involved.

4. **Some Related Results**

If we apply Theorem 2 on the subintervals \( [x, \frac{x+y}{2}] \) and \( [\frac{x+y}{2}, y] \) (provided \( x < y \)) and add the corresponding inequalities we get:

**Proposition 1.** Assume that the function \( f : I \to [0, \infty) \) is a \( \varphi \)-convex function with \( \ell \varphi \in L[0, 1] \). Let \( y, x \in I \) with \( y \neq x \) and assume that the mappings \( [0, 1] \ni t \mapsto f \left( (1 - t) x + \frac{t + x}{2} y \right) \) are Lebesgue integrable on \( [0, 1] \). Then
\[
\int_x^y f(u) \, du \leq \frac{1}{y - x} \int_0^1 t \varphi(t) \, dt.
\]

Also, by Theorem 4 we have

**Proposition 2.** Let \( \varphi : (0, 1) \to (0, \infty) \) a measurable function and such that the right limit \( \varphi_+(0) \) exists and is finite, the left limit \( \varphi_-(1) = 1 \) and the left derivative in 1 denoted \( \varphi_+'(1) \) exists and is finite. Assume also that \( \varphi_-'(1) > -1 \). If the function \( f : I \to [0, \infty) \) is differentiable on \( I \) and \( \varphi \)-convex, then
\[
\frac{\varphi_-'(1) + 1}{\varphi_+(0)} \left[ f \left( \frac{3x + y}{4} \right) + f \left( \frac{x + 3y}{4} \right) \right]
\]
\[ \leq \frac{1}{y - x} \int_x^y f(u) \, du \leq \left[ f \left( \frac{x + y}{2} \right) + \frac{f(x) + f(y)}{2} \right] \varphi_+(0) \frac{\varphi_-'(1) + 1}{\varphi_+(0)} \]
for any \( x, y \in I \).
Now we can prove the following result as well:

**Theorem 5.** Let \( \varphi : (0, 1) \rightarrow (0, \infty) \) a measurable function and such that the right limit \( \varphi_+ (0) \) exists and is finite, the left limit \( \varphi_- (1) = 1 \) and the left derivative in 1 denoted \( \varphi_- (1) \) exists and is finite. Assume also that \( \varphi_- (1) > -2 \). If the function \( f : I \rightarrow [0, \infty) \) is differentiable on \( I \) and \( \varphi \)-convex, then

\[
(4.3) \quad \frac{x + y}{2} \int_x^y f(u) \, du \leq \frac{\varphi_+ (0)}{\varphi_- (1) + 2} f \left( \frac{x + y}{2} \right) + \frac{1}{\varphi_- (1) + 2} f(x) + f(y)
\]

for any \( x, y \in I \).

**Proof.** Assume that \( x < y \). From the inequality (3.1) we have

\[
(4.4) \quad \varphi_+ (0) f \left( \frac{x + y}{2} \right) - \left[ \varphi_- (1) + 1 \right] f(u) \geq f'(u) \left( \frac{x + y}{2} - u \right)
\]

for any \( u \in [x, y] \) with \( u \neq \frac{x + y}{2} \).

Integrating over \( u \in [x, y] \) and dividing by \( y - x \) we have

\[
(4.5) \quad \varphi_+ (0) f \left( \frac{x + y}{2} \right) - \left[ \varphi_- (1) + 1 \right] \frac{1}{y - x} \int_x^y f(u) \, du \geq \frac{1}{y - x} \int_x^y f'(u) \left( \frac{x + y}{2} - u \right) \, du.
\]

Integrating by parts, we have

\[
\int_x^y f'(u) \left( \frac{x + y}{2} - u \right) \, du = \left. \left( \frac{x + y}{2} - u \right) f(u) \right|_x^y + \int_x^y f(u) \, du = \int_x^y f(u) \, du - \frac{f(y) + f(x)}{2} (y - x)
\]

and by (4.5) we get

\[
\varphi_+ (0) f \left( \frac{x + y}{2} \right) - \left[ \varphi_- (1) + 1 \right] \frac{1}{y - x} \int_x^y f(u) \, du \geq \frac{1}{y - x} \int_x^y f(u) \, du - \frac{f(y) + f(x)}{2},
\]

which is equivalent to

\[
\varphi_+ (0) f \left( \frac{x + y}{2} \right) + \frac{f(y) + f(x)}{2} \geq \frac{1}{y - x} \int_x^y f(u) \, du + \left[ \varphi_- (1) + 1 \right] \frac{1}{y - x} \int_x^y f(u) \, du = \left[ \varphi_- (1) + 2 \right] \frac{1}{y - x} \int_x^y f(u) \, du.
\]

Since \( \varphi_- (1) + 2 > 0 \), then on dividing by \( \varphi_- (1) + 2 \) we get the desired result (4.3). \( \square \)
Remark 4. We observe that
\[ \frac{\varphi_+(0)}{\varphi'_-(1) + 2} < \frac{\varphi_+(0)}{\varphi'_-(1) + 1} \]
and if we assume that \( \varphi \) is taken to satisfy the condition
\[ \varphi_+(0) > \frac{\varphi'_-(1) + 1}{\varphi'_-(1) + 2} \in (0, 1), \]
then
\[ \frac{1}{\varphi'_-(1) + 2} < \frac{\varphi_+(0)}{\varphi'_-(1) + 1} \]
and the inequality (4.3) is better than the second inequality in (4.2).

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