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SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTIONS (I)

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Abstract. In this paper, some two parameters perturbed Ostrowski type inequalities for absolutely continuous functions are established.

1. Introduction

We start with the following result that generalizes Ostrowski's inequality for real valued differentiable functions whose derivative are bounded.

Theorem 1 (Dragomir, 2003 [20])**.** *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. Suppose that there exist the functions m_i, $M_i:[a,b]\to\mathbb{R}$ $\left(i=\overline{1,2}\right)$ with the properties:*

$$m_1(x)\le f'(t)\le M_1(x)\quad \text{for a.e. } t\in[a,x] \tag{1.1}$$

and

$$m_2(x)\le f'(t)\le M_2(x)\quad \text{for a.e. } t\in(x,b]. \tag{1.2}$$

Then we have the inequalities:

$$\begin{aligned}&\frac{1}{2(b-a)}\left[m_1(x)(x-a)^2-M_2(x)(b-x)^2\right]\\&\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\le\frac{1}{2(b-a)}\left[M_1(x)(x-a)^2-m_2(x)(b-x)^2\right].\end{aligned}\tag{1.3}$$

The constant $\frac{1}{2}$ is sharp on both sides.

In the case that the derivative is globally bounded on $[a,b]$ by two constants, then we have:

Corollary 1. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is, there exists the constants $M>m$ such that*

$$-\infty<m\le f'(t)\le M<\infty \text{ for a.e. } t\in[a,b], \tag{1.4}$$

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then we have the inequality

$$\begin{aligned}(1.5)\qquad & \frac{1}{2(b-a)}\left[m(x-a)^2 - M(b-x)^2\right] \\ & \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & \leq \frac{1}{2(b-a)}\left[M(x-a)^2 - m(b-x)^2\right]\end{aligned}$$

for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best in both inequalities.

We may rewrite Corollary 1 in the following equivalent manner:

Corollary 2. *With the assumptions on Corollary 1, we have:*

$$\begin{aligned}(1.6)\qquad & \left| f(x) - \left(x - \frac{a+b}{2}\right)\left(\frac{M+m}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ & \leq \frac{1}{2}(M-m)(b-a)\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\end{aligned}$$

for all $x \in [a,b]$.

Remark 1. *If we assume that $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)| < \infty$, then obviously we may choose in (1.5) $m = \|f'\|_\infty$ and $M = \|f'\|_\infty$, obtaining Ostrowski's inequality for absolutely continuous functions whose derivatives are essentially bounded:*

$$\begin{aligned}\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| & \leq \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] \\ & = \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty,\end{aligned}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ here is best.

Remark 2. *Ostrowski's inequality for absolutely continuous mappings in terms of $\|f'\|_\infty$ basically states that*

$$\begin{aligned}(1.7)\qquad -\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right] & \leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ & \leq \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2 + (b-x)^2\right]\end{aligned}$$

for all $x \in [a,b]$.

Now, if we assume that (1.1) and (1.2) hold, then $-\|f'\|_\infty \le m_1(x),\ m_2(x)$ *and* $M_1(x),\ M_2(x) \le \|f'\|_\infty$*, which implies:*

$$\begin{aligned}
(1.8)\quad & -\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right] \\
&\le \frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&\le \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right] \\
&\le \frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right].
\end{aligned}$$

Thus, the inequality (1.3) may also be regarded as a refinement of the classical Ostrowski result.

An important particular case is $x = \frac{a+b}{2}$ providing the following corollary.

Corollary 3. *Assume that the derivative* $f' : [a,b] \to \mathbb{R}$ *satisfy the conditions:*

$$(1.9)\qquad -\infty < m_1 \le f'(t) \le M_1 < \infty \text{ for a.e. } t \in \left[a, \frac{a+b}{2}\right]$$

and

$$(1.10)\qquad -\infty < m_2 \le f'(t) \le M_2 < \infty \text{ for a.e. } t \in \left(\frac{a+b}{2}, b\right].$$

Then we have the inequalities

$$\begin{aligned}
(1.11)\qquad \frac{1}{8}(m_1 - M_2)(b-a) &\le f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&\le \frac{1}{8}(M_1 - m_2)(b-a).
\end{aligned}$$

The constant $\frac{1}{8}$ *is the best in both inequalities.*

Finally, if we know some global bounds for the derivative f' on $[a,b]$, then we may state the following corollary.

Corollary 4. *Under the assumptions of Corollary 1, we have the midpoint inequality:*

$$(1.12)\qquad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}(M-m)(b-a).$$

The constant $\frac{1}{8}$ *is best.*

For other Ostrowski type inequalities see [1]-[19] and [21]-[42].

Motivated by the above results, we establish in this paper some perturbed Ostrowski type inequalities for complex valued differentiable functions whose derivatives are either bounded or of bounded variation. Applications for midpoint inequalities are provided as well.

2. Some Identities

We start with the following identity that will play an important role in the following:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned} f(x)&+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt, \end{aligned} \tag{2.1}$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Utilising the integration by parts formula in the Lebesgue integral, we have

$$\begin{aligned} &\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt \\ &=(t-a)\left[f(t)-\lambda_1(x)t\right]\Big|_a^x-\int_a^x\left[f(t)-\lambda_1(x)t\right]dt \\ &=(x-a)\left[f(x)-\lambda_1(x)x\right]-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right) \\ &=(x-a)f(x)-\lambda_1(x)x(x-a)-\int_a^x f(t)\,dt+\frac{1}{2}\lambda_1(x)\left(x^2-a^2\right) \\ &=(x-a)f(x)-\int_a^x f(t)\,dt-\frac{1}{2}(x-a)^2\lambda_1(x) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt \\ &=(t-b)\left[f(t)-\lambda_2(x)t\right]\Big|_x^b-\int_x^b\left[f(t)-\lambda_2(x)t\right]dt \\ &=(b-x)\left[f(x)-\lambda_2(x)x\right]-\int_x^b f(t)\,dt+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right) \\ &=(b-x)f(x)-\int_x^b f(t)\,dt-(b-x)\lambda_2(x)x+\frac{1}{2}\lambda_2(x)\left(b^2-x^2\right) \\ &=(b-x)f(x)-\int_x^b f(t)\,dt+\frac{1}{2}(b-x)^2\lambda_2(x). \end{aligned} \tag{2.3}$$

If we add the identities (2.2) and (2.3) and divide by $b-a$ we deduce the desired identity (2.1). □

Corollary 5. *With the assumption in Lemma 1, we have for any $\lambda(x)\in\mathbb{C}$ that*

$$\begin{aligned} f(x)&+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda(x)\right]dt. \end{aligned} \tag{2.4}$$

Remark 3. *If we take $\lambda(x) = 0$ in (2.4), then we get Montgomery's identity for absolutely continuous functions, i.e.*

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt, \tag{2.5}$$

for $x \in [a,b]$.

We have the following midpoint representation:

Corollary 6. *With the assumption in Lemma 1, we have for any $\lambda_1, \lambda_2 \in \mathbb{C}$ that*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \lambda_1\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \lambda_2\right]dt. \tag{2.6}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \lambda\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \lambda\right]dt. \tag{2.7}$$

Remark 4. *The identity (2.1) has many particular cases of interest.*

If $x \in (a,b)$ is a point of differentiability for the absolutely continuous function $f : [a,b] \to \mathbb{C}$, then we have the equality:

$$f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'(x)\right]dt. \tag{2.8}$$

In particular we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right]dt \\ &\quad + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right]dt \end{aligned} \tag{2.9}$$

provided $f'\left(\frac{a+b}{2}\right)$ exists and is finite.

For $x \in (a,b)$, if we take in (2.1)

$$\lambda_1(x) = \frac{f(x) - f(a)}{x-a} \text{ and } \lambda_2(x) = \frac{f(b) - f(x)}{b-x},$$

then we get, after some elementary calculations,

$$\frac{1}{2}\left[f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.10}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\frac{f(x)-f(a)}{x-a}\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\frac{f(b)-f(x)}{b-x}\right]dt.$$

In particular, we have

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.11}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{b-a}{2}}\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\frac{f(b)-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]dt.$$

If we assume that the lateral derivatives $f'_+(a)$ and $f'_-(b)$ exist and are finite, then we have from (2.1) for $\lambda_1(x)=f'_+(a)$ and $\lambda_2(x)=f'_-(b)$

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.12}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'_+(a)\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'_-(b)\right]dt,$$

for all $x\in[a,b]$.

In particular, we have

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.13}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-f'_+(a)\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-f'_-(b)\right]dt.$$

If we take in (2.1) $\lambda_2(x)=\lambda_2(x)=f'\left(\frac{a+b}{2}\right)$, provided this derivative exists and is finite, then we get

$$f(x)+\left(\frac{a+b}{2}-x\right)f'\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.14}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt,$$

for all $x\in[a,b]$.

If we assume that the derivatives $f'_{+}(a)$, $f'_{-}(b)$ and $f'(x)$ exist and are finite, then by taking

$$\lambda_1(x)=\frac{f'_{+}(a)+f'(x)}{2} \text{ and } \lambda_2(x)=\frac{f'(x)+f'_{-}(b)}{2}$$

in (2.1) we get

$$\begin{aligned} f(x)&+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &+\frac{1}{4(b-a)}\left[(b-x)^2 f'_{-}(b)-(x-a)^2 f'_{+}(a)\right] \\ &=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\frac{f'_{+}(a)+f'(x)}{2}\right]dt \\ &+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\frac{f'(x)+f'_{-}(b)}{2}\right]dt. \end{aligned} \tag{2.15}$$

In particular, we have

$$\begin{aligned} f&\left(\frac{a+b}{2}\right)+\frac{1}{16}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\frac{f'_{+}(a)+f'\left(\frac{a+b}{2}\right)}{2}\right]dt \\ &+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\frac{f'\left(\frac{a+b}{2}\right)+f'_{-}(b)}{2}\right]dt. \end{aligned} \tag{2.16}$$

3. Inequalities for Bounded Derivatives

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\begin{aligned} &\bar{U}_{[a,b]}(\gamma,\Gamma) \\ &:=\left\{f:[a,b]\to\mathbb{C}\,|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}\,|\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 7. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma)=\{f:[a,b]\to\mathbb{C}\mid\ &(\operatorname{Re}\Gamma-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\gamma)\\&+(\operatorname{Im}\Gamma-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\gamma)\geq 0 \text{ for a.e. } t\in[a,b]\}.\end{aligned}\tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma)\geq\operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma)\geq\operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}\mid\ &\operatorname{Re}(\Gamma)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\gamma)\\&\text{and }\operatorname{Im}(\Gamma)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\gamma)\ \text{ for a.e. } t\in[a,b]\}.\end{aligned}\tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma).\tag{3.4}$$

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in(a,b)$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f'\in\bar{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap\bar{U}_{[x,b]}(\gamma_2,\Gamma_2)$, then we have*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&\left.+\frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2}-(x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right|\\&\leq\frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)\\&\leq\frac{1}{4}(b-a)\\&\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\{|\Gamma_1-\gamma_1|,|\Gamma_2-\gamma_2|\}.\\[2ex]\left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[|\Gamma_1-\gamma_1|^q+|\Gamma_2-\gamma_2|^q\right]^{1/q}\\ p>1,\frac{1}{p}+\frac{1}{q}=1,\\[2ex]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right].\end{cases}\end{aligned}\tag{3.5}$$

Proof. Since $f' \in \bar{U}_{[a,x]}(\gamma_1, \Gamma_1) \cap \bar{U}_{[x,b]}(\gamma_2, \Gamma_2)$, then by taking the modulus in (2.1) for $\lambda_1(x) = \frac{\Gamma_1+\gamma_1}{2}$ and $\lambda_2(x) = \frac{\Gamma_2+\gamma_2}{2}$ we get

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right.\\
&\left. + \frac{1}{2(b-a)}\left[(b-x)^2\frac{\Gamma_2+\gamma_2}{2} - (x-a)^2\frac{\Gamma_1+\gamma_1}{2}\right]\right| \\
&\leq \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{\Gamma_1+\gamma_1}{2}\right]dt\right| \\
&+ \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{\Gamma_2+\gamma_2}{2}\right]dt\right| \\
&\leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{\Gamma_1+\gamma_1}{2}\right|dt \\
&+ \frac{1}{b-a}\int_x^b (t-b)\left|f'(t) - \frac{\Gamma_2+\gamma_2}{2}\right|dt \\
&\leq \frac{1}{b-a}\frac{|\Gamma_1-\gamma_1|}{2}\int_a^x (t-a)\,dt + \frac{1}{b-a}\frac{|\Gamma_2-\gamma_2|}{2}\int_x^b (b-t)\,dt \\
&= \frac{1}{4}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^2 + |\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^2\right](b-a)
\end{aligned}$$

and the first inequality in (3.5) is proved.

The last part follows by Hölder's inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{1/\alpha}\left(n^\beta + q^\beta\right)^{1/\beta},$$

where $m, n, p, q \geq 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. □

Corollary 8. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x \in (a,b)$. Suppose that $\gamma, \Gamma \in \mathbb{C}$ with $\gamma \neq \Gamma$, and $f' \in \bar{U}_{[a,b]}(\gamma, \Gamma)$, then we have*

$$\begin{aligned}
&\left| f(x) + \left(\frac{a+b}{2} - x\right)\frac{\Gamma+\gamma}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma|\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a).
\end{aligned} \tag{3.6}$$

In particular, we have

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}|\Gamma-\gamma|(b-a). \tag{3.7}$$

Remark 5. *If the derivative $f' : [a,b] \to \mathbb{R}$ is bounded above and below, that is, there exists the constants $M > m$ such that*

$$-\infty < m \leq f'(t) \leq M < \infty \text{ for a.e. } t \in [a,b],$$

then we recapture from (3.6) the inequality (1.6).

Remark 6. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$ with $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f'\in\bar{U}_{\left[a,\frac{a+b}{2}\right]}(\gamma_1,\Gamma_1)\cap\bar{U}_{\left[\frac{a+b}{2},b\right]}(\gamma_2,\Gamma_2)$, then we have from (3.5) that*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{8}(b-a)\left(\frac{\Gamma_2+\gamma_2}{2}-\frac{\Gamma_1+\gamma_1}{2}\right)\right| \le\frac{1}{16}\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right](b-a). \tag{3.8}$$

4. Inequalities for Derivatives of Bounded Variation

Assume that the function $f:I\to\mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b]\subset\mathring{I}$. Then, as in (2.15), we have the equality

$$\begin{aligned}&f(x)+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&+\frac{1}{4(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]\\&=\frac{1}{b-a}\int_a^x(t-a)\left[f'(t)-\frac{f'(a)+f'(x)}{2}\right]dt\\&+\frac{1}{b-a}\int_x^b(t-b)\left[f'(t)-\frac{f'(x)+f'(b)}{2}\right]dt,\end{aligned}\tag{4.1}$$

for any $x\in[a,b]$.

Theorem 3. *Let $f:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b]\subset\mathring{I}$. If the derivative $f':\mathring{I}\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\right.\\&\left.+\frac{1}{4(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]\right|\\&\le\frac{1}{4}\left[\left(\frac{x-a}{b-a}\right)^2\bigvee_a^x(f')+\left(\frac{b-x}{b-a}\right)^2\bigvee_x^b(f')\right](b-a)\\&\le\frac{1}{4}(b-a)\\&\times\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b(f')+\frac{1}{2}\left|\bigvee_a^x(f')-\bigvee_x^b(f')\right|\right],\\[2ex]\left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x(f')\right]^q+\left[\bigvee_x^b(f')\right]^q\right]^{1/q}\\ p>1,\frac{1}{p}+\frac{1}{q}=1,\\[2ex]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f'),\end{cases}\end{aligned}\tag{4.2}$$

for any $x \in [a,b]$ *.*

Proof. Taking the modulus in (4.1) we have

$$\begin{aligned}
(4.3) \qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\
& \left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\
& \leq \frac{1}{b-a}\left|\int_a^x (t-a)\left[f'(t) - \frac{f'(a)+f'(x)}{2}\right] dt\right| \\
& + \frac{1}{b-a}\left|\int_x^b (t-b)\left[f'(t) - \frac{f'(x)+f'(b)}{2}\right] dt\right| \\
& \leq \frac{1}{b-a}\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt \\
& + \frac{1}{b-a}\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt.
\end{aligned}$$

Since $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}
\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| &= \frac{|f'(t) - f'(a) + f'(t) - f'(x)|}{2} \\
&\leq \frac{1}{2}\left[|f'(t) - f'(a)| + |f'(x) - f'(t)|\right] \\
&\leq \frac{1}{2}\bigvee_a^x (f')
\end{aligned}$$

for any $t \in [a,x]$ and, similarly,

$$\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| \leq \frac{1}{2}\bigvee_x^b (f')$$

for any $t \in [x,b]$.

Then

$$\begin{aligned}
\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(x)}{2}\right| dt &\leq \frac{1}{2}\bigvee_a^x (f')\int_a^x (t-a)\,dt \\
&= \frac{1}{4}(x-a)^2\bigvee_a^x (f')
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (b-t)\left|f'(t) - \frac{f'(x)+f'(b)}{2}\right| dt &\leq \frac{1}{2}\bigvee_x^b (f')\int_x^b (b-t)\,dt \\
&= \frac{1}{4}(b-x)^2\bigvee_x^b (f')
\end{aligned}$$

and by (4.3) we get the desired inequality (4.2).

The last part follows by Hölder's inequality

$$mn + pq \le (m^\alpha + p^\alpha)^{1/\alpha} \left(n^\beta + q^\beta\right)^{1/\beta},$$

where $m, n, p, q \ge 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. $\square$

Corollary 9. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{16}(b-a)\left[f'(b) - f'(a)\right] \right| \le \frac{1}{16}(b-a)\bigvee_a^b (f'). \tag{4.4}$$

Remark 7. *If $p \in (a,b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p (f') = \bigvee_p^b (f')$, then under the assumptions of Theorem 3, we have*

$$\begin{aligned} &\left| f(p) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - p\right) f'(p) \right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-p)^2 f'(b) - (p-a)^2 f'(a)\right] \right| \\ &\le \frac{1}{8}(b-a)\left[\frac{1}{4} + \left(\frac{p - \frac{a+b}{2}}{b-a}\right)^2\right]\bigvee_a^b (f'). \end{aligned} \tag{4.5}$$

5. Inequalities for Lipschitzian Derivatives

We say that $v : [a,b] \to \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|v(t) - v(s)| \le L|t - s| \text{ for any } t, s \in [a,b].$$

Theorem 4. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \end{aligned} \tag{5.1}$$

$$\leq \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^{3} K_{1}(x)+\left(\frac{b-x}{b-a}\right)^{3} K_{2}(x)\right](b-a)^{2}$$

$$\leq \frac{1}{8}(b-a)^{2}$$

$$\times\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^{3}+\left(\frac{b-x}{b-a}\right)^{3}\right]\max\left\{K_{1}(x), K_{2}(x)\right\}, \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[K_{1}^{q}(x)+K_{2}^{q}(x)\right]^{1/q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^{3}\left[K_{1}(x)+K_{2}(x)\right].\end{cases}$$

Proof. Since $f':\mathring{I}\to\mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a,x]$ and constant $K_2(x)$ on $[x,b]$, then

$$\begin{aligned}\left|f'(t)-\frac{f'(a)+f'(x)}{2}\right| &= \frac{\left|f'(t)-f'(a)+f'(t)-f'(x)\right|}{2} \\ &\leq \frac{1}{2}\left[\left|f'(t)-f'(a)\right|+\left|f'(x)-f'(t)\right|\right] \\ &\leq \frac{1}{2}K_{1}(x)\left[|t-a|+|x-t|\right] \\ &= \frac{1}{2}K_{1}(x)(x-a)\end{aligned}$$

for any $t\in[a,x]$ and, similarly,

$$\begin{aligned}\left|f'(t)-\frac{f'(x)+f'(b)}{2}\right| &\leq \frac{1}{2}K_{2}(x)\left[|t-x|+|b-t|\right] \\ &= \frac{1}{2}K_{2}(x)(b-x)\end{aligned}$$

for any $t\in[x,b]$.

Then

$$\begin{aligned}\int_{a}^{x}(t-a)\left|f'(t)-\frac{f'(a)+f'(x)}{2}\right|dt &\leq \frac{1}{2}K_{1}(x)(x-a)\int_{a}^{x}(t-a)\,dt \\ &= \frac{1}{8}(x-a)^{3}K_{1}(x)\end{aligned}$$

and

$$\begin{aligned}\int_{x}^{b}(b-t)\left|f'(t)-\frac{f'(x)+f'(b)}{2}\right|dt &\leq \frac{1}{2}K_{2}(x)(b-x)\int_{x}^{b}(b-t)\,dt \\ &= \frac{1}{8}(b-x)^{3}K_{2}(x).\end{aligned}$$

Making use of the inequality (4.3) we deduce the first bound in (5.1).

The second part is obvious. □

Corollary 10. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a, b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is Lipschitzian with the constant K on $[a, b]$ then*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\ &\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\ &\le \frac{1}{8}\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right] K (b-a)^2 \end{aligned} \tag{5.2}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) + \frac{1}{16}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{1}{32} K (b-a)^2 . \end{aligned} \tag{5.3}$$

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