Some Inequalities for $f$-Divergences Via Slater’s Inequality for Convex Functions

S. S. Dragomir$^{1,2}$

Abstract. Some inequalities for $f$-divergence measures by the use of Slater’s inequality for convex functions of a real variable are established.

1. Introduction

Given a convex function $f : \mathbb{R}^+ \to \mathbb{R}^+$, the $f$-divergence functional

$$D_f (p, q) := \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right)$$

was introduced in Csiszár [3], [4] as a generalized measure of information, a “distance function” on the set of probability distributions $\mathbb{P}^n$. The restriction here to discrete distribution is only for convenience, similar results hold for general distributions. As in Csiszár [4], we interpret undefined expressions by

$$f(0) = \lim_{t \to 0^+} f(t), \quad 0f\left( \frac{0}{0} \right) = 0$$

$$0f\left( \frac{a}{0} \right) = \lim_{\varepsilon \to 0^+} f\left( \frac{a}{\varepsilon} \right) = a \lim_{t \to \infty} \frac{f(t)}{t}, \quad a > 0.$$  

The following results were essentially given by Csiszár and Körner [5].

**Theorem 1.** If $f : \mathbb{R}^+ \to \mathbb{R}$ is convex, then $D_f (p, q)$ is jointly convex in $p$ and $q$.

The following lower bound for the $f$-divergence functional also holds.

**Theorem 2.** Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be convex. Then for every $p, q \in \mathbb{R}_+^n$, we have the inequality:

$$D_f (p, q) \geq \sum_{i=1}^{n} q_i f \left( \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} q_i} \right).$$

1991 Mathematics Subject Classification. Primary 94Xxx; Secondary 26D15.
If \( f \) is strictly convex, equality holds in (1.2) iff
\[
\frac{p_1}{q_1} = \frac{p_2}{q_2} = \ldots = \frac{p_n}{q_n}.
\]

Corollary 1. Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) be convex and normalized, i.e.,
\[
f(1) = 0.
\]
Then for any \( p, q \in \mathbb{R}_n^+ \) with
\[
\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i
\]
we have the inequality
\[
D_f(p, q) \geq 0.
\]
If \( f \) is strictly convex, the equality holds in (1.6) iff \( p_i = q_i \) for all \( i \in \{1, \ldots, n\} \).

In particular, if \( p, q \) are probability vectors, then (1.5) is assured. Corollary 1 then shows, for strictly convex and normalized \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \),
\[
D_f(p, q) \geq 0 \quad \text{for all} \quad p, q \in \mathbb{P}^n.
\]
The equality holds in (1.7) iff \( p = q \).

These are “distance properties”. However, \( D_f \) is not a metric: It violates the triangle inequality, and is asymmetric, i.e., for general \( p, q \in \mathbb{R}_n^+ \), \( D_f(p, q) \neq D_f(q, p) \).

In the examples below we obtain, for suitable choices of the kernel \( f \), some of the best known distance functions \( D_f \) used in mathematical statistics [15], information theory [2]-[24] and signal processing [13], [19].

Example 1. (Kullback-Leibler) For
\[
f(t) := t \log t, \quad t > 0
\]
the \( f \)-divergence is
\[
D_f(p, q) = \sum_{i=1}^{n} p_i \log \left( \frac{p_i}{q_i} \right),
\]
the Kullback-Leibler distance [17]-[18].

Example 2. (Hellinger) Let
\[
f(t) = \frac{1}{2} \left( 1 - \sqrt{t} \right)^2, \quad t > 0.
\]
Then \( D_f \) gives the Hellinger distance [1]
\[
D_f(p, q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2,
\]
which is symmetric.

Example 3. (Renyi) For \( \alpha > 1 \), let
\[
f(t) = t^\alpha, \quad t > 0.
\]
Then
\[
D_f(p, q) = \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha},
\]
which is the $\alpha$-order entropy \[23\].

**Example 4.** (\(\chi^2\)-distance) Let
\[ f(t) = (t - 1)^2, \quad t > 0. \]
Then
\[ D_f(p, q) = \sum_{i=1}^{n} \left( \frac{p_i - q_i}{q_i} \right)^2 \]
is the \(\chi^2\)-distance between \(p\) and \(q\).

Finally, we have:

**Example 5.** (Variational distance). Let \(f(t) = |t - 1|, \quad t > 0\). The corresponding divergence, called the variational distance, is symmetric,
\[ D_f(p, q) = \sum_{i=1}^{n} |p_i - q_i|. \]

For other examples of divergence measures, see the paper \[16\] by J. N. Kapur, where further references are given.

2. Slater Type Inequalities

Suppose that \(I\) is an interval of real numbers with interior \(\hat{I}\) and \(f : I \to \mathbb{R}\) is a convex function on \(I\). Then \(f\) is continuous on \(\hat{I}\) and has finite left and right derivatives at each point of \(\hat{I}\). Moreover, if \(x, y \in \hat{I}\) and \(x < y\), then
\[ D^- f(x) \leq D^+ f(x) \leq D^- f(y) \leq D^+ (y), \]
which shows that both \(D^- f\) and \(D^+ f\) are nondecreasing functions on \(\hat{I}\). It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function \(f : I \to \mathbb{R}\), the subdifferential of \(f\) denoted by \(\partial f\) is the set of all functions \(\varphi : I \to [-\infty, \infty]\) such that \(\varphi(\hat{I}) \subseteq \mathbb{R}\) and
\[ f(x) \geq f(a) + (x-a)\varphi(a) \quad \text{for any} \quad x, a \in I. \]

It is also well known that if \(f\) is convex on \(I\), then \(\partial f\) is nonempty, \(D^+ f, D^- f \in \partial f\) and if \(\varphi \in \partial f\), then
\[ D^- f(x) \leq \varphi(x) \leq D^+ f(x) \]
for every \(x \in \hat{I}\). In particular, \(\varphi\) is a nondecreasing function. If \(f\) is differentiable convex on \(\hat{I}\), then \(\partial f = \{f'\}\).

The following result is well known in literature as Slater’s inequality. For the original proof due to Slater, see \[25\]. For related results, see Chapter I of the book \[21\] or Chapter 2 of the book \[22\].

We shall here follow the presentation in \[6, pp. 129-130\] where a slightly more general result for Slater’s inequality is provided:

**Lemma 1.** Let \(f : I \to \mathbb{R}\) be a nondecreasing (nonincreasing) convex function on \(I\), \(x_i \in I, \quad p_i \geq 0\) with \(P_n = \sum_{i=1}^{n} p_i > 0\) and for a given \(\varphi \in \partial f\) assume that \(\sum_{i=1}^{n} p_i \varphi(x_i) \neq 0\). Then one has the inequality
\[ \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq f \left( \frac{\sum_{i=1}^{n} p_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \right). \]
Proof. Let us give the proof for the case of nondecreasing functions only. In this case \( \varphi(x) \geq 0 \) for any \( x \in I \) and

\[
\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I
\]

being a convex combination of \( x_i \in I \) with the nonnegative weights

\[
\frac{p_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)}, \quad i \in \{1, \ldots, n\}.
\]

Now, if we use the inequality (2.2) we deduce

\[
f(x) - f(x_i) \geq (x - x_i) \varphi(x_i) \quad \text{for any} \quad x, x_i \in I, \quad i \in \{1, \ldots, n\}.
\]

Multiplying (2.4) by \( p_i/P_n \geq 0 \) and summing over \( i \in \{1, \ldots, n\} \), we deduce

\[
f(x) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \geq x \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \varphi(x_i)
\]

for any \( x \in I \). If in (2.5) we choose

\[
x = \frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)},
\]

then we deduce the desired inequality (2.3).

If we would like to drop the assumption of monotonicity for the function \( f \), then we can state and prove in a similar way the following result (see also [6]):

**Lemma 2.** Let \( f : I \to \mathbb{R} \) be a convex function, \( x_i \in I, \quad p_i \geq 0 \) with \( P_n > 0 \) and \( \sum_{i=1}^{n} p_i \varphi(x_i) \neq 0 \) for a given \( \varphi \in \partial f \). If

\[
\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I,
\]

then the inequality (2.3) holds true.

**Proof.** Since

\[
\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I,
\]

hence we can use the inequality (2.4) and proceed as in the above Lemma 1. The details are omitted.

The following inequality is well known in literature as Karamata’s inequality, see [21, pp. 298] or [22, p. 212]:

**Lemma 3.** Assume that \( 0 < a \leq a_i \leq A < \infty, \quad 0 < b \leq b_i \leq B < \infty \) for each \( i \in \{1, \ldots, n\} \). Then for \( p_i > 0, \quad \sum_{i=1}^{n} p_i = 1 \), one has the inequalities

\[
K^{-2} \leq \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i} \leq K^2
\]

with \( K = \frac{\sqrt{\alpha B + \sqrt{AB}}}{\sqrt{\alpha B + \sqrt{AB}}} > 1 \).

Using Karamata’s result, we may point out the following reverse of Jensen’s inequality that may be useful in applications.
Let \( f : [0, \infty) \to \mathbb{R} \) be a monotonic nondecreasing convex function. Assume that \( 0 < r \leq x_i \leq R < \infty \) for each \( i \in \{1, \ldots, n\} \), \( (p_i)_{i=1, \ldots, n} \) is a probability distribution and for a given \( \varphi \in \partial f \) consider
\[
K(r, R) = \frac{\sqrt{r\varphi(r)} + \sqrt{R\varphi(R)}}{\sqrt{r\varphi(R)} + \sqrt{R\varphi(r)}}.
\]
Then we have the inequality
\[
\sum_{i=1}^{n} p_i f(x_i) \leq f\left( K^2(r, R) \sum_{i=1}^{n} p_i x_i \right).
\]

**Proof.** From Lemma 3 we know that
\[
\sum_{i=1}^{n} p_i f(x_i) \leq f\left( \sum_{i=1}^{n} p_i x_i \varphi(x_i) \right).
\]
If we apply Karamata’s inequality for \( a_i = x_i \), \( b_i = \varphi(x_i) \), we get successively
\[
f\left( \sum_{i=1}^{n} p_i x_i \varphi(x_i) \right) = f\left( \frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \right) \leq
\[
\leq f\left( K^2(r, R) \sum_{i=1}^{n} p_i x_i \right),
\]
since, obviously, \( \varphi(x_i) \in [\varphi(r), \varphi(R)] \) being monotonic nondecreasing on \([r, R]\). The inequality (2.7) is thus proved. \( \square \)

### 3. Some Inequalities for \( f \)-Divergences

The following result may be stated:

**Theorem 3.** Let \( f : [0, \infty] \to \mathbb{R} \) be a differentiable, convex and normalized function, i.e. \( f(1) = 0 \) and \( 0 \leq r \leq 1 \leq R \leq \infty \). If there exists a real number \( m \) so that
\[
-\infty < m \leq f'(x) \quad \text{for any} \quad x \in (r, R),
\]
then for any probability distribution \( p, q \in \mathcal{P} \) with
\[
r \leq \frac{p_i}{q_i} \leq R \quad \text{for any} \quad i \in \{1, \ldots, n\}
\]
(if \( r = 0 \) and \( R = \infty \), the assumption (3.2) is always satisfied), one has the inequality
\[
0 \leq D_f(p, q) \leq f\left( \frac{D_{\Phi_*}(p, q) - m}{D_{f'}(p, q) - m} \right) - m \cdot \frac{D_{\Phi_*}(p, q)}{D_{f'}(p, q) - m},
\]
where \( \Phi_*(x) := xf'(x) \), \( \Phi_+(x) := (x - 1)f'(x) \) for \( x \in [0, \infty] \) and \( D_{f'}(p, q) \neq m \).

**Proof.** Consider the auxiliary function \( f_m(x) = f(x) - m(x - 1) \), \( x \in [0, \infty] \). Since \( f'_m(x) = f'(x) - m \), \( x \in (r, R) \), it follows that \( f_m \) is differentiable, convex and monotonic nondecreasing on \((r, R)\), and we may apply Lemma 1 to get
\[
\sum_{i=1}^{n} q_i f_m\left( \frac{p_i}{q_i} \right) \leq f_m\left( \frac{\sum_{i=1}^{n} q_i p_i f'_m\left( \frac{p_i}{q_i} \right)}{\sum_{i=1}^{n} q_i f'_m\left( \frac{p_i}{q_i} \right)} \right).
\]
It is easy to see that
\[
\sum_{i=1}^{n} q_i f_m \left( \frac{p_i}{q_i} \right) = \sum_{i=1}^{n} q_i \left[ f \left( \frac{p_i}{q_i} \right) - m \left( \frac{p_i}{q_i} - 1 \right) \right] = D_f (p, q)
\]
and
\[
\sum_{i=1}^{n} q_i p_i f'_m \left( \frac{p_i}{q_i} \right) = \sum_{i=1}^{n} p_i \left[ f' \left( \frac{p_i}{q_i} \right) - m \right] = \sum_{i=1}^{n} p_i f' \left( \frac{p_i}{q_i} \right) - m = D_{\Phi_*} (p, q) - m
\]
where \( \Phi_* (x) \) has been defined above.

Also, one may observe that
\[
\sum_{i=1}^{n} q_i f'_m \left( \frac{p_i}{q_i} \right) = \sum_{i=1}^{n} q_i \left[ f' \left( \frac{p_i}{q_i} \right) - m \right] = D_{f'} (p, q) - m
\]
and
\[
f_m \left( \frac{D_{\Phi_*} (p, q) - m}{D_{f'} (p, q) - m} \right) = f \left( \frac{D_{\Phi_*} (p, q) - m}{D_{f'} (p, q) - m} \right) - m \left( \frac{D_{\Phi_*} (p, q) - m}{D_{f'} (p, q) - m} - 1 \right)
\]
\[
= f \left( \frac{D_{\Phi_*} (p, q) - m}{D_{f'} (p, q) - m} \right) - m \cdot \frac{D_{\Phi_*} (p, q) - D_{f'} (p, q)}{D_{f'} (p, q) - m}
\]
\[
= f \left( \frac{D_{\Phi_*} (p, q) - m}{D_{f'} (p, q) - m} \right) - m \cdot \frac{D_{\Phi_*} (p, q)}{D_{f'} (p, q) - m},
\]
which gives, by (3.4), the desired inequality (3.3). \( \square \)

If one would like to drop the assumption of lower boundedness for the derivative \( f' \) (see (3.1)), one may need to impose another condition as described in the following theorem:

**Theorem 4.** Let \( f : [0, \infty) \to \mathbb{R} \) be a differentiable, convex and normalized function and \( 0 \leq r \leq 1 \leq R \leq \infty \). As above, consider \( \Phi_* (x) = xf' (x) \) and assume that for two probabilities \( p \) and \( q \) satisfying (3.2) one has \( D_{f'} (p, q) \neq 0 \) and
\[
D_{\Phi_*} (p, q) \geq 0.
\]
Then one has the inequality
\[
0 \leq D_f (p, q) \leq f \left( \frac{D_{\Phi_*} (p, q)}{D_{f'} (p, q)} \right).
\]

The proof follows in a similar way as the one in Theorem 3 by utilizing Lemma 2. We omit the details.

Now we can point out another result for \( f \)-divergences when bounds for the likelihood ratio \( \frac{p}{q} \) are available:

**Theorem 5.** Let \( f : [0, \infty] \to \mathbb{R} \) be a differentiable convex and normalized function and \( 0 \leq r \leq 1 \leq R \leq \infty \) and let \( K(r, R) \) be as stated in Lemma 4. If there exists a real number \( m \) so that
\[
-\infty < m \leq f' (x) \text{ for any } x \in (r, R)
\]
then for all probability distributions \( p, q \in \mathcal{P} \) satisfying
\[
    r \leq \frac{p_i}{q_i} \leq R \quad \text{for each} \quad i \in \{1, \ldots, n\},
\]
one has the inequality
\[
    D_f(p, q) \leq f(K^2(r, R)) - m(K^2(r, R) - 1).
\]

**Proof.** As in Theorem 3, the function \( f_m(x) = f(x) - m(x - 1) \) is differentiable, convex and monotonic nondecreasing on \( (r, R) \). If we apply Lemma 4 we get
\[
    \sum_{i=1}^{n} q_i f_m \left( \frac{p_i}{q_i} \right) \leq f_m \left( K^2(r, R) \cdot \sum_{i=1}^{n} q_i \frac{p_i}{q_i} \right) = f \left( K^2(r, R) \right) - m \left( K^2(r, R) - 1 \right),
\]
which completes the proof. \( \square \)

4. Applications for Particular Divergences

We consider the Kullback-Leibler distance
\[
    KL(p, q) = \sum_{i=1}^{n} p_i \log \left( \frac{p_i}{q_i} \right)
\]
that is the \( f \)-divergence for the convex function \( f : (0, \infty) \to \mathbb{R}, \ f(t) = t \log t \).

If we take the convex function \( f(t) = \log t \), then the corresponding \( f \)-divergence is
\[
    D_f(p, q) := \sum_{i=1}^{n} q_i f \left( \frac{p_i}{q_i} \right) = \sum_{i=1}^{n} q_i \left[ -\log \left( \frac{p_i}{q_i} \right) \right] = \sum_{i=1}^{n} q_i \log \left( \frac{q_i}{p_i} \right) = KL(q, p)
\]
for all probability distributions \( p, q \in \mathcal{P} \).

For the function \( f(t) = -\log t \) we have
\[
    \Phi_\ast(t) := tf'(t) = -1 \quad \text{and} \quad \Phi_\ast(t) := (t-1)f'(t) = \frac{1-t}{t}, \quad t > 0.
\]
Now for \( 0 \leq r \leq 1 \leq R < \infty \) and \( m = -\frac{1}{R} \) we have
\[
    m \leq f'(t) = -\frac{1}{t} \quad \text{for any} \quad t \in (r, R)
\]
and the condition (3.1) is satisfied.

We also have
\[
    D_{\Phi_\ast}(p, q) = -1 \quad \text{and} \quad D_{f'}(p, q) = -\sum_{i=1}^{n} q_i^2 + 1 - 1 = -1 - D_{\chi^2}(q, p)
\]
where
\[
    D_{\chi^2}(p, q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1
\]
is the \( \chi^2 \)-distance between \( p \) and \( q \).

We also have
\[
    D_{\Phi_\ast}(p, q) = \sum_{i=1}^{n} q_i \left( 1 - \frac{p_i}{q_i} \right) = \sum_{i=1}^{n} q_i^2 - 1 = D_{\chi^2}(q, p).
\]
Therefore, for any probability distribution $p, q \in \mathcal{P}$ with
\[ r \leq \frac{p_i}{q_i} \leq R \quad \text{for any} \quad i \in \{1, \ldots, n\} \]
we have by (3.3) the inequality
\[
0 \leq KL(q, p) \leq -\ln \left( \frac{-1 + \frac{1}{R}}{-1 - D_{\chi^2}(q, p) + \frac{1}{R}} \right) + \frac{1}{R} \frac{D_{\chi^2}(q, p)}{-1 - D_{\chi^2}(q, p) + \frac{1}{R}},
\]
which is equivalent to
\[
0 \leq KL(q, p) \leq \ln \left( \frac{R \left( D_{\chi^2}(q, p) + 1 \right) - 1}{R - 1} \right) - \frac{D_{\chi^2}(q, p)}{R \left( D_{\chi^2}(q, p) + 1 \right) - 1}.
\]
Observe that
\[
D_{\Phi_*}(p, q) = \frac{-1}{-1 - D_{\chi^2}(q, p)} = \frac{1}{D_{\chi^2}(q, p) + 1} > 0,
\]
then by the inequality (3.6) we have
\[
0 \leq KL(q, p) \leq \ln \left( D_{\chi^2}(q, p) + 1 \right)
\]
for any $p, q \in \mathcal{P}$.

We notice that the inequality (4.2) can be obtained from (4.1) by letting $R \to \infty$.

Now, for the function $f(t) = t \log t$, we have
\[
\Phi_*(t) := tf'(t) = t \log t + t.
\]
Then
\[
D_{\Phi_*}(p, q) = \sum_{i=1}^{n} q_i \left( \frac{p_i}{q_i} \log \frac{p_i}{q_i} + \frac{p_i}{q_i} \right) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} + \sum_{i=1}^{n} p_i = KL(p, q) + 1
\]
and
\[
D_{f'}(p, q) = \sum_{i=1}^{n} q_i \left( \log \frac{p_i}{q_i} + 1 \right) = 1 - KL(q, p).
\]
Then, if we take $p, q \in \mathcal{P}$ with $1 > KL(q, p)$, by utilizing the inequality (3.6) we get
\[
0 \leq KL(p, q) \leq \frac{1 + KL(p, q)}{1 - KL(q, p)} \ln \left( \frac{1 + KL(p, q)}{1 - KL(q, p)} \right).
\]
For $\alpha > 0$ consider $\alpha$-order entropy
\[
D_\alpha(p, q) := \sum_{i=1}^{n} p_i^\alpha q_i^{-\alpha},
\]
which is an $f$-divergence for the convex function $f(t) = t^\alpha$.

We have
\[
K(r, R) = \frac{\sqrt{rf'(r)} + \sqrt{Rf'(R)}}{\sqrt{rf'(r)} + \sqrt{Rf'(R)}} = \frac{r^{\frac{\alpha}{2}} + R^{\frac{\alpha}{2}}}{r^{\frac{\alpha}{2}} R^{\frac{\alpha-1}{2}} + R^{\frac{\alpha}{2}} r^{\frac{\alpha}{2}}},
\]
We have
\[ f'(t) = \alpha t^{\alpha - 1} \geq \alpha r^{\alpha - 1}. \]

If we apply Theorem 5, then for all probability distributions \( p, q \in \mathcal{P} \) satisfying
\[ 0 < r \leq \frac{p_i}{q_i} \leq R < \infty \text{ for each } i \in \{1, \ldots, n\}, \]
we have the inequality
\begin{equation}
D_\alpha (p, q) \leq \left( \frac{r^{-\frac{\alpha}{2}} + R^{-\frac{\alpha}{2}}}{r^{-\frac{1}{2}} R^{-\frac{1}{2}} + R^{-\frac{1}{2}} r^{\frac{1}{2}}} \right)^{2\alpha}
- \alpha^{\alpha - 1} \left[ \left( \frac{r^{-\frac{1}{2}} + R^{\frac{1}{2}}}{r^{-\frac{1}{2}} R^{\frac{1}{2}} + R^{\frac{1}{2}} r^{-\frac{1}{2}}} \right)^2 - 1 \right].
\end{equation}

References


1Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

2School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa