Some Trace Inequalities of Čebyšev Type for Functions of Operators in Hilbert Spaces

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Abstract. Some trace operator inequalities for synchronous functions that are related to the Čebyšev inequality for sequences of real numbers are given.

1. Introduction

For \( p = (p_1, \ldots, p_n) \), \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) \( n \)-tuples of real numbers, consider the Čebyšev functional

\[
T_n(p; a, b) := P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i,
\]

where \( P_n := \sum_{i=1}^{n} p_i \).

In 1882-1883, Čebyšev [4] and [5] proved that, if \( a \) and \( b \) are monotonic in the same (opposite) sense and \( p \) is nonnegative, then

\[
T_n(p; a, b) \geq (\leq) 0.
\]

The inequality (1.2) was mentioned by Hardy, Littlewood and Polya in their book [16] in 1934 in the more general setting of synchronous sequences, i.e., if \( a, b \) are synchronous (asynchronous), this means that

\[
(a_i - a_j)(b_i - b_j) \geq (\leq) 0
\]

for each \( i, j \in \{1, \ldots, n\} \), then (1.2) holds true.

For general real weights \( p \), Mitrović and Pečarić has shown in [21] that the inequality (1.2) holds true if

\[
0 \leq P_k \leq P_n \text{ for } k \in \{1, \ldots, n - 1\},
\]

and \( a, b \) are monotonic in the same (opposite) sense.

We say that the functions \( f, g : [a, b] \rightarrow \mathbb{R} \) are synchronous (asynchronous) on the interval \([a, b]\) if they satisfy the following condition:

\[
(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0 \text{ for each } t, s \in [a, b].
\]
It is obvious that, if \( f, g \) are monotonic and have the same monotonicity on the interval \([a, b]\), then they are synchronous on \([a, b]\) while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete Čebyšev inequality for synchronous (asynchronous) sequences of vectors in an inner product space, see \([12]\) and \([13]\).

Let \( A \) be a selfadjoint linear operator on a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). The \textit{Gelfand map} establishes a \( \ast \)-isometrically isomorphism \( \Phi \) between the set \( C(\text{Sp}(A)) \) of all \textit{continuous functions} defined on the \textit{spectrum} of \( A \), denoted \( \text{Sp}(A) \), and the \( C^* \)-algebra \( C^*(A) \) generated by \( A \) and the identity operator \( 1_H \) on \( H \) as follows:

For any \( f, g \in C(\text{Sp}(A)) \) and any \( \alpha, \beta \in \mathbb{C} \) we have
\[
\begin{align*}
(i) & \quad \Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g); \\
(ii) & \quad \Phi(fg) = \Phi(f) \Phi(g) \quad \text{and} \quad \Phi(f^*) = \Phi(f)^*; \\
(iii) & \quad \|\Phi(f)\| = \|f\| := \sup_{t \in \text{Sp}(A)} |f(t)|; \\
(iv) & \quad \Phi(f_0) = 1_H \quad \text{and} \quad \Phi(f_1) = A, \text{ where } f_0(t) = 1 \text{ and } f_1(t) = t, \text{ for } t \in \text{Sp}(A).
\end{align*}
\]

With this notation we define
\[
f(A) := \Phi(f) \quad \text{for all } f \in C(\text{Sp}(A))
\]
and we call it the \textit{continuous functional calculus} for a selfadjoint operator \( A \).

If \( A \) is a selfadjoint operator and \( f \) is a real valued continuous function on \( \text{Sp}(A) \), then \( f(t) \geq 0 \) for any \( t \in \text{Sp}(A) \) implies that \( f(A) \geq 0 \), i.e., \( f(A) \) is a positive operator on \( H \). Moreover, if both \( f \) and \( g \) are real valued functions on \( \text{Sp}(A) \) then the following important property holds:

\[
\text{(P)} \quad f(t) \geq g(t) \quad \text{for any } t \in \text{Sp}(A) \quad \text{implies that } f(A) \geq g(A)
\]
in the operator order of \( B(H) \).

The following result provides an inequality of Čebyšev type for functions of one selfadjoint operator:

Let \( A \) be a selfadjoint operator on the Hilbert space \((H, \langle \cdot, \cdot \rangle)\) with \( \text{Sp}(A) \subseteq [m, M] \) for some real numbers \( m < M \). If \( f, g : [m, M] \to \mathbb{R} \) are continuous and synchronous (asynchronous) on \([m, M]\), then \([9]\)
\[
(1.5) \quad \langle f(A)g(A)x, x \rangle \geq (\leq) \langle f(A)x, x \rangle \langle g(A)x, x \rangle
\]
for any \( x \in H \) with \( \|x\| = 1 \).

As a particular case of interest we notice that if \( A \) is a positive selfadjoint operator on \( H \), then
\[
(1.6) \quad \langle A^{p+q}x, x \rangle \geq \langle A^px, x \rangle \langle A^qx, x \rangle
\]
for any \( x \in H \) with \( \|x\| = 1 \) and \( p, q > 0 \).

It is known, see for instance \([22, \text{p. 356-358}]\), that if \( A \) and \( B \) are two \textit{commuting bounded selfadjoint operators} on the complex Hilbert space \( H \), then there exists a bounded selfadjoint operator \( S \) on \( H \) and two bounded functions \( \varphi \) and \( \psi \) such that \( A = \varphi(S) \) and \( B = \psi(S) \). Moreover, if \( \{E_\lambda\} \) is the spectral family over the closed interval \([0, 1]\) for the selfadjoint operator \( S \), then \( S = \int_{0-}^1 \lambda dE_\lambda \), where the integral is taken in the Riemann-Stieltjes sense, the functions \( \varphi \) and \( \psi \) are summable with respect with \( \{E_\lambda\} \) on \([0, 1]\) and
\[
(1.7) \quad A = \varphi(S) = \int_{0-}^1 \varphi(\lambda) dE_\lambda \quad \text{and} \quad B = \psi(S) = \int_{0-}^1 \psi(\lambda) dE_\lambda.
\]
Now, if $A$ and $B$ are as above with $\text{Sp}(A), \text{Sp}(B) \subseteq J$ an interval of real numbers, then for any continuous functions $f, g : J \to \mathbb{C}$ we have the representations

\begin{equation}
\begin{aligned}
f(A) &= \int_{0^-}^{1} (f \circ \varphi)(\lambda) \, dE_{\lambda} \quad \text{and} \\
g(B) &= \int_{0^-}^{1} (g \circ \psi)(\lambda) \, dE_{\lambda}.
\end{aligned}
\end{equation}

**Definition 1.** We say that the continuous functions $f, g : J \to \mathbb{R}$ are operator synchronous (asynchronous) on $J$, if for any $A$ and $B$ two commuting bounded selfadjoint operators on the complex Hilbert space $H$ with $\text{Sp}(A), \text{Sp}(B) \subseteq J$ we have

\begin{equation}
(f(A) - f(B)) (g(A) - g(B)) \geq (\leq) 0
\end{equation}

in the operator order.

In what follows, unless specified, $H$ will be a complex Hilbert space.

In [10] we proved the following basic result:

**Theorem 1.** The continuous functions $f, g : J \to \mathbb{R}$ are synchronous (asynchronous) on $J$ if and only if they are operator synchronous (asynchronous) on $J$.

The case of monotonic functions is as follows:

**Corollary 1.** If the continuous functions $f, g : J \to \mathbb{R}$ have the same monotonicity on $J$, then for any $A$ and $B$ two commuting bounded selfadjoint operators on the Hilbert space $H$ with $\text{Sp}(A), \text{Sp}(B) \subseteq J$ we have

\begin{equation}
f(A)g(A) + f(B)g(B) \geq g(A)f(B) + f(A)g(B)
\end{equation}

in the operator order.

**Remark 1.** We observe that the above inequality (1.10) can provide numerous inequalities of interest for two commuting selfadjoint operators.

For instance, if $A$ and $B$ are positive commuting operators on $H$ then for any $p, q > 0$ we have

\begin{equation}
A^{p+q} + B^{p+q} \geq B^{p}A^{q} + A^{p}B^{q}.
\end{equation}

If the commuting operators $A$ and $B$ are positive definite on $H$, then also

\begin{equation}
A \ln(A) + B \ln(B) \geq B \ln(A) + A \ln(B).
\end{equation}

Also, if $A$ and $B$ are commuting operators on $H$ with $0 \leq A, B \leq \frac{\pi}{2}1_{H}$, then

\begin{equation}
\sin(A) \cos(A) + \sin(B) \cos(B) \leq \sin(B) \cos(A) + \sin(A) \cos(B).
\end{equation}

In order to obtain some similar results for trace of operators in Hilbert spaces we need some preliminary facts as follows.

**2. Some Facts on Trace of Operators**

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_{i}\}_{i \in I}$ an orthonormal basis of $H$. We say that $A \in B(H)$ is a Hilbert-Schmidt operator if

\begin{equation}
\sum_{i \in I} \|Ae_{i}\|^{2} < \infty.
\end{equation}
It is well known that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for $H$ and $A \in \mathcal{B}(H)$ then
\begin{equation}
\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in J} \|Af_j\|^2 = \sum_{j \in J} \|A^*f_j\|^2
\end{equation}
showing that the definition (2.1) is independent of the orthonormal basis and $A$ is a Hilbert-Schmidt operator iff $A^*$ is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define
\begin{equation}
\|A\|_2 := \left( \sum_{i \in I} \|Ae_i\|^2 \right)^{1/2}
\end{equation}
for $\{e_i\}_{i \in I}$ an orthonormal basis of $H$. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a vector space and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the Hilbert-Schmidt norm.

Denote the modulus of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\|Ax\| = \|Ax\|$ for all $x \in H$, $A$ is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \|A\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

**Theorem 2.** We have:

(i) $\mathcal{B}_2(H)$ is a Hilbert space with inner product
\begin{equation}
\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^*Ae_i, e_i \rangle
\end{equation}
and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) We have the inequalities
\begin{equation}
\|A\| \leq \|A\|_2
\end{equation}
for any $A \in \mathcal{B}_2(H)$ and
\begin{equation}
\|AT\|_2, \|TA\|_2 \leq \|T\| \|A\|_2
\end{equation}
for any $A \in \mathcal{B}_2(H)$ and $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.
\begin{equation}
\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H);
\end{equation}

(iv) $\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the algebra of compact operators on $H$.

If $\{e_i\}_{i \in I}$ an orthonormal basis of $H$, we say that $A \in \mathcal{B}(H)$ is trace class if
\begin{equation}
\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.
\end{equation}
The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.
The following proposition holds:

**Proposition 1.** If \( A \in B_1(H) \), then the following are equivalent:

(i) \( A \in B_1(H) \);

(ii) \( |A|^{1/2} \in B_2(H) \);

(ii) \( A \) (or \( |A| \)) is the product of two elements of \( B_2(H) \).

The following properties are also well known:

**Theorem 3.** With the above notations:

(i) We have

\[
\|A\|_1 = \|A^*\|_1 \quad \text{and} \quad \|A\|_2 \leq \|A\|_1
\]

for any \( A \in B_1(H) \);

(ii) \( B_1(H) \) is an operator ideal in \( B(H) \), i.e.

\[
B(H)B_1(H)B(H) \subseteq B_1(H);
\]

(iii) We have

\[
B_2(H)B_2(H) = B_1(H);
\]

(iv) We have

\[
\|A\|_1 = \sup \{ \langle A, B \rangle_2 \mid B \in B_2(H), \|B\| \leq 1 \};
\]

(v) \( B_1(H) \) is a Banach space.

(iv) We have the following isometric isomorphisms

\[
B_1(H) \cong K(H)^* \quad \text{and} \quad B_1(H)^* \cong B(H),
\]

where \( K(H)^* \) is the dual space of \( K(H) \) and \( B_1(H)^* \) is the dual space of \( B_1(H) \).

We define the **trace** of a trace class operator \( A \in B_1(H) \) to be

\[
\text{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle,
\]

where \( \{e_i\}_{i \in I} \) is an orthonormal basis of \( H \). Note that this coincides with the usual definition of the trace if \( H \) is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 4.** We have:

(i) If \( A \in B_1(H) \) then \( A^* \in B_1(H) \) and

\[
\text{tr}(A^*) = \text{tr}(A);
\]

(ii) If \( A \in B_1(H) \) and \( T \in B(H) \), then \( AT, TA \in B_1(H) \) and

\[
\text{tr}(AT) = \text{tr}(TA) \quad \text{and} \quad \| \text{tr}(AT) \| \leq \|A\|_1 \|T\|;
\]

(iii) \( \text{tr}(\cdot) \) is a bounded linear functional on \( B_1(H) \) with \( \| \text{tr} \| = 1 \);

(iv) If \( A, B \in B_2(H) \) then \( AB, BA \in B_1(H) \) and \( \text{tr}(AB) = \text{tr}(BA) \);

(v) \( B_{fin}(H) \) is a dense subspace of \( B_1(H) \).

Utilising the trace notation we obviously have that

\[
\langle A, B \rangle_2 = \text{tr}(B^*A) = \text{tr}(AB^*) \quad \text{and} \quad \|A\|_2^2 = \text{tr}(A^*A) = \text{tr}(|A|^2)
\]

for any \( A, B \in B_2(H) \).
The following Hölder’s type inequality has been obtained by Ruskai in [23]

\[(2.12) \quad |\text{tr} \,(AB)| \leq \text{tr} \,(|AB|) \leq \left[ \text{tr} \left( |A|^{1/\alpha} \right) \right]^{\alpha} \left[ \text{tr} \left( |B|^{1/(1-\alpha)} \right) \right]^{1-\alpha} \]

where \(\alpha \in (0, 1)\) and \(A, B \in \mathcal{B}(H)\) with \(|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)\).

In particular, for \(\alpha = \frac{1}{2}\) we get the Schwarz inequality

\[(2.13) \quad |\text{tr} \,(AB)| \leq \text{tr} \,(|AB|) \leq \left[ \text{tr} \left( |A|^2 \right) \right]^{1/2} \left[ \text{tr} \left( |B|^2 \right) \right]^{1/2} \]

with \(A, B \in \mathcal{B}_2(H)\).

If \(A \geq 0\) and \(P \in \mathcal{B}_1(H)\) with \(P \geq 0\), then

\[(2.14) \quad 0 \leq \text{tr} \,(PA) \leq \|A\| \text{tr} \,(P) \]

Indeed, since \(A \geq 0\), then \(|Ax, x\| \geq 0\) for any \(x \in H\). If \(\{e_i\}_{i \in I}\) is an orthonormal basis of \(H\), then

\[
0 \leq \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \leq \|A\| \left\|P^{1/2}e_i\right\|^2 = \|A\| \left\langle Pe_i, e_i \right\rangle
\]

for any \(i \in I\). Summing over \(i \in I\) we get

\[
0 \leq \sum_{i \in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \leq \|A\| \sum_{i \in I} \langle Pe_i, e_i \rangle = \|A\| \text{tr} \,(P)
\]

and since

\[
\sum_{i \in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle = \sum_{i \in I} \left\langle P^{1/2}AP^{1/2}e_i, e_i \right\rangle = \text{tr} \left( P^{1/2}AP^{1/2} \right) = \text{tr} \,(PA)
\]

we obtain the desired result (2.14).

This obviously imply the fact that, if \(A\) and \(B\) are selfadjoint operators with \(A \leq B\) and \(P \in \mathcal{B}_1(H)\) with \(P \geq 0\), then

\[(2.15) \quad \text{tr} \,(PA) \leq \text{tr} \,(PB) \]

Now, if \(A\) is a selfadjoint operator, then we know that

\[
|\langle Ax, x\rangle| \leq \|A\| |x, x\| \text{ for any } x \in H.
\]

This inequality follows by Jensen’s inequality for the convex function \(f(t) = |t|\) defined on a closed interval containing the spectrum of \(A\).

If \(\{e_i\}_{i \in I}\) is an orthonormal basis of \(H\), then

\[(2.16) \quad |\text{tr} \,(PA)| = \left| \sum_{i \in I} \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \right| \leq \sum_{i \in I} \left| \left\langle AP^{1/2}e_i, P^{1/2}e_i \right\rangle \right| \leq \sum_{i \in I} \left| \|A\| P^{1/2}e_i, P^{1/2}e_i \right| = \text{tr} \,(P |A|) \]

for any \(A\) a selfadjoint operator and \(P \in \mathcal{B}_1(H)\) with \(P \geq 0\).

For the theory of trace functionals and their applications the reader is referred to [26].

For some classical trace inequalities see [6], [8], [20] and [30], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [6], [15], [17], [18], [19], [24] and [27].
3. Trace Inequalities for Synchronous Functions

We start with the following simple result:

**Proposition 2.** Let $A$ and $B$ be two commuting bounded selfadjoint operators on the Hilbert space $H$ with $\text{Sp}(A), \text{Sp}(B) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on $J$. If $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then

\[
\text{tr} [Pf(A) g(A)] + \text{tr} [Pf(B) g(B)] \geq \text{tr} [Pg(A) f(B)] + \text{tr} [Pf(A) g(B)].
\]

(3.1)

The proof follows from the inequality (1.10) for synchronous functions and the property (2.15) for the trace functional.

**Theorem 5.** Let $A$ be a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on $J$. If $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$, then

\[
\frac{\text{tr} [Pf(A) g(A)]}{\text{tr}(P)} - \frac{\text{tr} [Pf(A)] \text{tr} [Pg(A)]}{\text{tr}(P)} \geq \left( \frac{\text{tr} [Pf(A)]}{\text{tr}(P)} - f \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right) \right) \left( g \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right) - \frac{\text{tr}[Pg(A)]}{\text{tr}(P)} \right).
\]

(3.2)
Corollary 2. With the assumptions of Theorem 5 and if one of the functions $f$ and $g$ is convex while the other is concave, then we have

$$\frac{\text{tr} [Pf (A) g (A)]}{\text{tr} (P)} - \frac{\text{tr} [Pf (A)]}{\text{tr} (P)} \frac{\text{tr} [Pg (A)]}{\text{tr} (P)} \geq \left( \frac{\text{tr} [Pf (A)]}{\text{tr} (P)} - f \left( \frac{\text{tr} (PA)}{\text{tr} (P)} \right) \right) \left( g \left( \frac{\text{tr} (PA)}{\text{tr} (P)} \right) - \frac{\text{tr} [Pg (A)]}{\text{tr} (P)} \right) \geq 0.$$  

Proof. If $f$ is convex and $g$ is concave, then by Jensen’s inequality for trace [11] we have

$$\frac{\text{tr} [Pf (A)]}{\text{tr} (P)} \geq f \left( \frac{\text{tr} (PA)}{\text{tr} (P)} \right)$$

and

$$g \left( \frac{\text{tr} (PA)}{\text{tr} (P)} \right) \geq \frac{\text{tr} [Pg (A)]}{\text{tr} (P)}$$

and the last inequality in (3.4) is proved.

The following result also holds:

Theorem 6. Let $A$ and $B$ be two selfadjoint operators on the Hilbert space $H$ with $\text{Sp} (A), \text{Sp} (B) \subseteq J$ and assume that the continuous functions $f, g : J \rightarrow \mathbb{R}$ are synchronous on $J$. If $P, Q \in \mathcal{B}_1 (H) \setminus \{0\}$ with $P, Q \geq 0$, then

$$\frac{\text{tr} [Pf (A) g (A)]}{\text{tr} (P)} + \frac{\text{tr} [Qf (B) g (B)]}{\text{tr} (Q)} \geq \frac{\text{tr} [Pf (A)]}{\text{tr} (P)} \frac{\text{tr} [Qg (B)]}{\text{tr} (Q)} + \frac{\text{tr} [Pg (A)]}{\text{tr} (P)} \frac{\text{tr} [Qf (B)]}{\text{tr} (Q)}$$

and, in particular

$$\frac{\text{tr} [Pf (A) g (A)]}{\text{tr} (P)} + \frac{\text{tr} [Pf (B) g (B)]}{\text{tr} (P)} \geq \frac{\text{tr} [Pf (A)]}{\text{tr} (P)} \frac{\text{tr} [Pg (B)]}{\text{tr} (P)} + \frac{\text{tr} [Pg (A)]}{\text{tr} (P)} \frac{\text{tr} [Pf (B)]}{\text{tr} (P)}.$$ 

Proof. We consider only the case of synchronous functions. In this case we have then

$$f (t) g (t) + f (s) g (s) \geq f (t) g (s) + f (s) g (t)$$

for each $t, s \in [a, b]$.

If we fix $s \in [a, b]$ and apply the property (P) for the inequality (1.8) then we have

$$f (A) g (A) + f (s) g (s) 1_H \geq g (s) f (A) + f (s) g (A)$$

in the operator order of $\mathcal{B} (H)$.

Utilising the property (2.15) we have

$$\text{tr} [P (f (A) g (A) + f (s) g (s) 1_H)] \geq \text{tr} [P (g (s) f (A) + f (s) g (A))],$$

which is equivalent to

$$\text{tr} [Pf (A) g (A)] + f (s) g (s) \text{tr} (P) \geq g (s) \text{tr} [Pf (A)] + f (s) \text{tr} [Pg (A)],$$
This inequality implies the following inequality in the order of $B(H)$
\[
\text{tr}[P f(A)g(A)]_H + \text{tr}(P) f(B)g(B) \geq \text{tr}[P f(A)] g(B) + \text{tr}[P g(A)] f(B).
\]
Utilising again the property (2.15) we have
\[
\text{tr}(Q) \text{tr}[P f(A)g(A)] + \text{tr}(P) \text{tr}[Q f(B)g(B)] \\
\geq \text{tr}[P f(A)] \text{tr}[Q g(B)] + \text{tr}[P g(A)] \text{tr}[Q f(B)]
\]
and the inequality (3.5) is proved.

**Corollary 3.** Let $A$ be a selfadjoint operators on the Hilbert space $H$ with $\text{Sp}(A) \subseteq J$ and assume that the continuous functions $f, g : J \rightarrow \mathbb{R}$ are synchronous on $J$. If $P, Q \in B_1(H) \setminus \{0\}$ with $P, Q \geq 0$, then
\[
\frac{\text{tr}[P f(A)g(A)]}{\text{tr}(P)} + \frac{\text{tr}[Q f(A)g(A)]}{\text{tr}(Q)} \\
\geq \frac{\text{tr}[P f(A)]}{\text{tr}(P)} \frac{\text{tr}[Q g(A)]}{\text{tr}(Q)} + \frac{\text{tr}[P g(A)]}{\text{tr}(P)} \frac{\text{tr}[Q f(A)]}{\text{tr}(Q)}
\]
and, in particular
\[
\frac{\text{tr}[P f(A)g(A)]}{\text{tr}(P)} \geq \frac{\text{tr}[P f(A)]}{\text{tr}(P)} \frac{\text{tr}[P g(A)]}{\text{tr}(P)}.
\]
The inequality (3.10) is a trace version of the Čebyšev inequality.
We can improve this inequality as follows.
Let $A$ be a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq J$ and assume that the continuous functions $f, g : J \rightarrow \mathbb{R}$ are synchronous on $J$. For $P \in B_1(H) \setminus \{0\}$ with $P \geq 0$, we can define the functional
\[
C_{(f,g)}(A, P) := \frac{\text{tr}[P f(A)g(A)]}{\text{tr}(P)} - \frac{\text{tr}[P f(A)]}{\text{tr}(P)} \frac{\text{tr}[P g(A)]}{\text{tr}(P)} \geq 0.
\]
We have the following result:

**Theorem 7.** Let $A$ be a selfadjoint operator on the Hilbert space $H$ with $\text{Sp}(A) \subseteq J$, $P \in B_1(H) \setminus \{0\}$ with $P \geq 0$ and assume that the continuous functions $f, g : J \rightarrow \mathbb{R}$ are synchronous on $J$. Then we have
\[
C_{(f,g)}(A, P) \geq \max \{ |C_{(f|f|,g)}(A, P)|, |C_{(f|g|,g)}(A, P)|, |C_{(f|f|,g)}(A, P)| \}
\]
\[
\geq 0.
\]

**Proof.** Utilising the continuity of modulus property, we have
\[
(f(t) - f(s))(g(t) - g(s)) = |(f(t) - f(s))(g(t) - g(s))| \\
\geq |(f(t)| - |f(s))|(g(t) - g(s))|
\]
for any $t, s \in J$.
This is equivalent to
\[
f(t)g(t) + f(s)g(s) - f(t)g(s) - f(s)g(t) \\
\geq ||f(t)||g(t) + ||f(s)||g(s) - ||f(t)||g(s) - ||f(s)||g(t)|
\]
for any $t, s \in J$. 
This implies in the order of $\mathcal{B}(H)$ that

\begin{equation}
\tag{3.13}
\begin{aligned}
f(A)g(A) + f(s)g(s)1_H - g(s)f(A) - f(s)g(A) \\
\geq ||f(A)||g(A) + |f(s)|g(s)1_H - g(s)|f(A)| - |f(s)||g(A)|
\end{aligned}
\end{equation}

for any $s \in J$.

Applying the property (2.15) we have

\begin{equation}
\tag{3.14}
\begin{aligned}
\text{tr}[Pf(A)g(A)] + f(s)g(s)\text{tr}(P) - g(s)\text{tr}[Pf(A)] - f(s)\text{tr}[Pg(A)] \\
\geq \text{tr}[P||f(A)||g(A)] + |f(s)|g(s)1_H - g(s)|f(A)| - |f(s)||g(A)|
\end{aligned}
\end{equation}

for any $s \in J$.

Using the property (2.16) we also have

\begin{equation}
\tag{3.15}
\begin{aligned}
\text{tr}[P||f(A)||g(A)] + |f(s)|g(s)1_H - g(s)|f(A)| - |f(s)||g(A)| \\
\geq \text{tr}[P||f(A)||g(A)] + |f(s)|g(s)\text{tr}(P) \\
- g(s)\text{tr}[P||f(A)||] - |f(s)|\text{tr}[Pg(A)]
\end{aligned}
\end{equation}

for any $s \in J$.

By (3.14) and (3.15) we have

\begin{equation}
\begin{aligned}
\text{tr}[Pf(A)g(A)] + f(s)g(s)\text{tr}(P) \\
- g(s)\text{tr}[Pf(A)] - f(s)\text{tr}[Pg(A)] \\
\geq \text{tr}[P||f(A)||g(A)] + |f(s)|g(s)\text{tr}(P) \\
- g(s)\text{tr}[P||f(A)||] - |f(s)|\text{tr}[Pg(A)]
\end{aligned}
\end{equation}

for any $s \in J$.

This inequality implies in the order of $\mathcal{B}(H)$ that

\begin{equation}
\tag{3.16}
\begin{aligned}
\text{tr}[Pf(A)g(A)]1_H + \text{tr}(P)f(A)g(A) \\
- \text{tr}[Pf(A)]g(A) - \text{tr}[Pg(A)]f(A) \\
\geq \text{tr}[P||f(A)||g(A)]1_H + \text{tr}(P)||f(A)||g(A) \\
- \text{tr}[P||f(A)||]g(A) - \text{tr}[Pg(A)]||f(A)||
\end{aligned}
\end{equation}

Taking the trace and repeating the reason, we deduce

\begin{equation}
\tag{3.17}
\begin{aligned}
\text{tr}(P)\text{tr}[Pf(A)g(A)] + \text{tr}(P)\text{tr}[Pf(A)g(A)] \\
- \text{tr}[Pf(A)]\text{tr}[Pg(A)] - \text{tr}[Pg(A)]\text{tr}[Pf(A)] \\
\geq \text{tr}[P||f(A)||g(A)]\text{tr}(P) + \text{tr}(P)\text{tr}[||f(A)||g(A)] \\
- \text{tr}[P||f(A)||]\text{tr}[Pg(A)] - \text{tr}[Pg(A)]\text{tr}[P||f(A)||]
\end{aligned}
\end{equation}

which is equivalent to

\[ C_{(f,g)}(A,P) \geq |C_{(||f||,g)}(A,P)|. \]

The other inequalities follow in a similar way and the details are omitted. \qed
4. Some Examples

If we take the functions $f, g : [0, \infty) \to [0, \infty)$, $f(t) = t^p$ and $g(t) = t^q$ with $p, q > 0$ then by (3.4) we have

\begin{equation}
\frac{\text{tr}(PA^{p+q})}{\text{tr}(P)} - \frac{\text{tr}(PA^p) \text{tr}(PA^q)}{\text{tr}(P)} \geq \left( \frac{\text{tr}(PA^p)}{\text{tr}(P)} - \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right)^p \right) \left( \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right)^q - \frac{\text{tr}(PA^q)}{\text{tr}(P)} \right),
\end{equation}

for any $A \geq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

If $p > 0$ and $q \in (0, 1)$, then we have a better result:

\begin{equation}
\frac{\text{tr}(PA^{p+q})}{\text{tr}(P)} - \frac{\text{tr}(PA^p) \text{tr}(PA^q)}{\text{tr}(P)} \geq \left( \frac{\text{tr}(PA^p)}{\text{tr}(P)} - \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right)^p \right) \left( \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right)^q - \frac{\text{tr}(PA^q)}{\text{tr}(P)} \right)
\end{equation}

\geq 0,

for any $A \geq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

By using (3.5) we have for $p, q > 0$ that

\begin{equation}
\frac{\text{tr}(PA^{p+q})}{\text{tr}(P)} + \frac{\text{tr}(QB^{p+q})}{\text{tr}(Q)} \geq \frac{\text{tr}(PA^p) \text{tr}(QA^q)}{\text{tr}(P)} + \frac{\text{tr}(PA^q) \text{tr}(QB^p)}{\text{tr}(Q)}
\end{equation}

for any $A, B \geq 0$ and $P, Q \in \mathcal{B}_1(H) \setminus \{0\}$ with $P, Q \geq 0$.

In particular, we have

\begin{equation}
\frac{\text{tr}(PA^{p+q})}{\text{tr}(P)} + \frac{\text{tr}(QB^{p+q})}{\text{tr}(Q)} \geq \frac{\text{tr}(PA^p) \text{tr}(QA^q)}{\text{tr}(P)} + \frac{\text{tr}(PA^q) \text{tr}(QB^p)}{\text{tr}(Q)}
\end{equation}

for any $A \geq 0$ and $P, Q \in \mathcal{B}_1(H) \setminus \{0\}$ with $P, Q \geq 0$.

Also

\begin{equation}
\frac{\text{tr}(PA^{p+q})}{\text{tr}(P)} \geq \frac{\text{tr}(PA^p) \text{tr}(PA^q)}{\text{tr}(P)} \text{tr}(P)
\end{equation}

for any $A \geq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

Moreover, if in (4.5) we choose $A = P$, then from (4.5) we get

\begin{equation}
\frac{\text{tr}(P^{p+q+1})}{\text{tr}(P)} \geq \frac{\text{tr}(P^{p+1}) \text{tr}(P^{q+1})}{\text{tr}(P)} \text{tr}(P)
\end{equation}

for $p, q > 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

If we take the functions $f, g : [0, \infty) \to [0, \infty)$, $f(t) = t^p$ and $g(t) = \ln t$ with $p \geq 1$ then by (3.4) we have

\begin{equation}
\frac{\text{tr}(PA^p \ln A)}{\text{tr}(P)} - \frac{\text{tr}(PA^p) \text{tr}(P \ln A)}{\text{tr}(P)} \geq \left( \frac{\text{tr}(PA^p)}{\text{tr}(P)} - \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right)^p \right) \left( \ln \left( \frac{\text{tr}(PA)}{\text{tr}(P)} \right) - \frac{\text{tr}(P \ln A)}{\text{tr}(P)} \right)
\end{equation}

\geq 0,

for any positive definite operators $A$ and $P$ with $P \in \mathcal{B}_1(H) \setminus \{0\}$. 
If we use (3.9), then we have for \( p > 0 \)

\[
\frac{\text{tr} \left( PA^p \ln A \right)}{\text{tr} (P)} + \frac{\text{tr} \left( QA^p \ln A \right)}{\text{tr} (Q)} \geq \frac{\text{tr} \left( PA^p \right)}{\text{tr} (P)} \frac{\text{tr} \left( Q \ln A \right)}{\text{tr} (Q)} + \frac{\text{tr} \left( P \ln A \right)}{\text{tr} (P)} \frac{\text{tr} \left( QA^p \right)}{\text{tr} (Q)}
\]

(4.8)

for any positive definite operators \( A, P \) and \( Q \) with \( P, Q \in B_1 (H) \setminus \{0\} \).

In particular

\[
\frac{\text{tr} \left( PA^p \ln A \right)}{\text{tr} (P)} \geq \frac{\text{tr} \left( PA^p \right)}{\text{tr} (P)} \frac{\text{tr} \left( P \ln A \right)}{\text{tr} (P)},
\]

(4.9)

for any positive definite operators \( A \) and \( P \) with \( P \in B_1 (H) \setminus \{0\} \).

If we apply the inequality (3.11), then we have an improvement of (4.9) as follows

\[
\frac{\text{tr} \left( PA^p \ln A \right)}{\text{tr} (P)} \geq \frac{\text{tr} \left( PA^p \right)}{\text{tr} (P)} \frac{\text{tr} \left( P \ln A \right)}{\text{tr} (P)} - \frac{\text{tr} \left( PA^p \ln A \right)}{\text{tr} (P)} \geq 0,
\]

(4.10)

for any positive definite operators \( A \) and \( P \) with \( P \in B_1 (H) \setminus \{0\} \).

If we use the inequality (3.10) for the \( f, g : [0, \infty) \rightarrow [0, \infty) \), \( f (t) = t \) and \( g (t) = t^{-1} \), then we get

\[
1 \leq \frac{\text{tr} \left( PA \right)}{\text{tr} (P)} \frac{\text{tr} \left( PA^{-1} \right)}{\text{tr} (P)},
\]

(4.11)

for any positive definite operators \( A \) and \( P \) with \( P \in B_1 (H) \setminus \{0\} \).

References

SOME TRACE INEQUALITIES OF ČEBYŠEV TYPE


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