Received 15/01/14

INEQUALITIES OF HERMITE-HADAMARD TYPE FOR λ-CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Some inequalities of Hermite-Hadamard type for λ-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([38]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [28], [29], [31], [44], [47] and [48]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([31]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [31] and [45] while for quasi convex functions, the reader can consult [30].

If $f : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (1.2) (or (1.3)) holds true for $x, y \in C$ and $t \in [0,1]$.

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Convex functions, Integral inequalities, λ-Convex functions.

Definition 3 ([7])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [26], [27], [39], [41] and [50].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([53])**.** *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t) h(s) \text{ for any } t, s \in J. \tag{1.6}$$

If the inequality (1.6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (1.6) then h is said to be a multiplicative function on J.

In [53] it has been noted that if $h : [0, \infty) \to [0, \infty)$ with $h(t) = (x + c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

We can prove now the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces.

Theorem 1. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0, 1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0, 1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq [f(x) + f(y)] \int_0^1 h(t)\, dt. \tag{1.7}$$

Proof. By the h-convexity of f we have

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{1.8}$$

for any $t \in [0, 1]$.

Integrating (1.8) on $[0, 1]$ over t, we get

$$\int_0^1 f(tx + (1-t)y)\, dt \leq f(x) \int_0^1 h(t)\, dt + f(y) \int_0^1 h(1-t)\, dt$$

and since $\int_0^1 h(t)\, dt = \int_0^1 h(1-t)\, dt$, we get the second part of (1.7).

From the h-convexity of f we have

$$f\left(\frac{z+w}{2}\right) \leq h\left(\frac{1}{2}\right) [f(z) + f(w)] \tag{1.9}$$

for any $z, w \in C$.

If we take in (1.9) $z = tx + (1-t)y$ and $w = (1-t)x + ty$, then we get

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right) [f(tx + (1-t)y) + f((1-t)x + ty)] \tag{1.10}$$

for any $t \in [0, 1]$.

Integrating (1.10) on $[0, 1]$ over t and taking into account that

$$\int_0^1 f(tx + (1-t)y)\, dt = \int_0^1 f((1-t)x + ty)\, dt$$

we get the first inequality in (1.7). $\square$

Remark 1. *If $f : I \to [0, \infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0, 1]$ and $f \in L[a, b]$ with $a, b \in I, a < b$, then from (1.7) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in* [49]

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \int_a^b f(u)\, du \leq [f(a) + f(b)] \int_0^1 h(t)\, dt.$$

If we write (1.7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write (1.7) for the case of P-type functions $f : C \to [0, \infty)$, i.e., $h(t) = 1, t \in [0, 1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq f(x) + f(y), \tag{1.11}$$

that has been obtained for functions of real variable in [31].

If f is Breckner s-convex on C, for $s \in (0, 1)$, then by taking $h(t) = t^s$ in (1.7) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq \frac{f(x) + f(y)}{s+1}, \tag{1.12}$$

that was obtained for functions of a real variable in [26].

Since the function $g(x) = \|x\|^s$ is Breckner s-convex on on the normed linear space X, $s \in (0, 1)$, then for any $x, y \in X$ we have

$$\frac{1}{2} \|x + y\|^s \leq \int_0^1 \|(1-t)x + ty\|^s\, dt \leq \frac{\|x\|^s + \|x\|^s}{s+1}. \tag{1.13}$$

If $f : C \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq \frac{f(x) + f(y)}{1-s}. \tag{1.14}$$

We notice that for $s = 1$ the first inequality in (1.14) still holds, i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt. \tag{1.15}$$

The case for functions of real variables was obtained for the first time in [31].

2. λ-Convex Functions

We start with the following definition:

Definition 6. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if*

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)} \tag{2.1}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0, 1]$.

If $f : C \to [0, \infty)$ is h-convex function with h supermultiplicative on $[0, \infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &\leq h\left(\frac{\alpha}{\alpha + \beta}\right) f(x) + h\left(\frac{\beta}{\alpha + \beta}\right) f(y) \\ &\leq \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha + \beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions.

Proposition 1. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \geq 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0, \infty)$.

(iii) If there exists $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha + \beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0, \infty)$.

Proof. (i) For every $\beta > 0$ and $x, y \in C$ we can state

$$f\left(\frac{0x + \beta y}{0 + \beta}\right) \leq \frac{\lambda(0) f(x) + \lambda(\beta) f(y)}{\lambda(\beta)}$$

from where we get

$$f(y) \leq \frac{\lambda(0)}{\lambda(\beta)} f(x) + f(y)$$

and since $\lambda(0) > 0$ we get that $f(x) \geq 0$ for all $x \in C$.

(ii) For all $\alpha, \beta > 0$ we have

$$f\left(\frac{\alpha x_0 + \beta x_0}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x_0) + \lambda(\beta) f(x_0)}{\lambda(\alpha + \beta)}$$

from where we get

$$f(x_0) \leq \frac{\lambda(\alpha) + \lambda(\beta)}{\lambda(\alpha + \beta)} f(x_0)$$

and since $f(x_0) > 0$, then we get that $\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$ for all $\alpha, \beta > 0$.

(iii) If we write the inequality for y_0 we also have

$$f(y_0) \leq \frac{\lambda(\alpha) + \lambda(\beta)}{\lambda(\alpha + \beta)} f(y_0)$$

and since $f(y_0) < 0$ we get that

$$\lambda(\alpha + \beta) \geq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$. □

We have the following result providing many examples of subadditive functions $\lambda : [0, \infty) \to [0, \infty)$.

Theorem 2. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. If $r \in (0, R)$ then the function $\lambda_r : [0, \infty) \to [0, \infty)$ given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r \exp(-t))}\right] \tag{2.2}$$

is nonnegative, increasing and subadditive on $[0, \infty)$.

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}}$ and nonnegative weights $(p_i)_{i \in \mathbb{N}}$, namely

$$\sum_{i=0}^{n} p_i \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i, \tag{2.3}$$

for any $n \in \mathbb{N}$.

Let $t, s \in (0,1)$ and define the sequences $c_i := t^i$, $b_i := s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i := a_i r^i \geq 0$ we get

$$\sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rts)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \tag{2.4}$$

for any $n \in \mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty} a_i r^i, \quad \sum_{i=0}^{\infty} a_i (rts)^i, \quad \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i$$

are convergent, then by letting $n \to \infty$ in (2.4) we get

$$h(r)\, h(rts) \geq h(rt)\, h(rs)$$

which can be written as

$$\frac{h(r)}{h(rts)} \leq \frac{h(r)}{h(rt)} \cdot \frac{h(r)}{h(rs)}$$

for any $t, s \in (0,1)$.

Let $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Then

$$\begin{aligned} \lambda_r(\alpha+\beta) &= \ln\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right] = \ln\left[\frac{h(r)}{h(r\exp(-\alpha)\exp(-\beta))}\right] \\ &\leq \ln\left[\frac{h(r)}{h(r\exp(-\alpha))} \cdot \frac{h(r)}{h(r\exp(-\beta))}\right] \\ &= \ln\left[\frac{h(r)}{h(r\exp(-\alpha))}\right] + \ln\left[\frac{h(r)}{h(r\exp(-\beta))}\right] \\ &= \lambda_r(\alpha) + \lambda_r(\beta). \end{aligned} \tag{2.5}$$

Since $h(r) \geq h(r\exp(-t))$ for any $t \in [0,\infty)$ we deduce that λ_r is nonnegative and subadditive on $[0,\infty)$.

Now, observe that λ_r is differentiable on $(0,\infty)$ and

$$\begin{aligned} \lambda_r'(t) \quad : &= -(\ln[h(r\exp(-t))])' \\ &= -\frac{h'(r\exp(-t))\,(r\exp(-t))'}{h(r\exp(-t))} \\ &= \frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))} \geq 0 \end{aligned} \tag{2.6}$$

for $t \in (0,\infty)$, where

$$h'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

This proves the monotonicity of λ_r. □

We have the following fundamental examples of power series with positive coefficients

$$h(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1) \tag{2.7}$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=\exp(z)\qquad z\in\mathbb{C},$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1); \tag{2.8}$$

$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1);$$

$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

Remark 2. *Now, if we take* $h(z)=\frac{1}{1-z}$, $z\in D(0,1)$, *then*

$$\lambda_r(t)=\ln\left[\frac{1-r\exp(-t)}{1-r}\right] \tag{2.9}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r\in(0,1)$.

If we take $h(z)=\exp(z)$, $z\in\mathbb{C}$, *then*

$$\lambda_r(t)=r[1-\exp(-t)] \tag{2.10}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r>0$.

Corollary 1. *Let* $h(z)=\sum_{n=0}^{\infty}a_nz^n$ *a power series with nonnegative coefficients* $a_n\geq 0$ *for all* $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$ *and* $r\in(0,R)$. *For a mapping* $f:C\to\mathbb{R}$ *defined on convex subset* C *of a linear space* X, *the following statements are equivalent:*

(i) The function f *is* λ_r*-convex with* $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \leq \left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \tag{2.11}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}} \leq [h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{2.12}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

Proof. We have that

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\lambda_r(\alpha+\beta)\leq\lambda_r(\alpha)f(x)+\lambda_r(\beta)f(y)$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$, is equivalent to

$$\begin{aligned}\ln\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} &\leq \ln\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}+\ln\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \\ &= \ln\left\{\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)}\right\}\end{aligned} \tag{2.13}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

The inequality (2.13) is equivalent to (2.11) and the proof of the equivalence " $(i)\Leftrightarrow(ii)$" is concluded. The last part is obvious. □

Remark 3. *We observe that, in the case when*

$$\lambda_r(t)=r[1-\exp(-t)],\ t\geq 0,$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)} \tag{2.14}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We observe that this definition is independent of $r>0$.

The inequality (2.14) is equivalent to

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\exp(\beta)[\exp(\alpha)-1]f(x)+\exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1} \tag{2.15}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

3. Hermite-Hadamard Type Inequalities

For an arbitrary mapping $f : C \subset X \to \mathbb{R}$ where C is a convex subset of the linear space X, we can define the mapping

$$g_{x,y} : [0,1] \to \mathbb{R},\ g_{x,y}(t) := f(tx + (1-t)y),$$

where x, y are two distinct fixed elements in C.

Proposition 2. *With the above assumptions, the following statements are equivalent:*
(i) f is λ-convex on C;
(ii) For every $x, y \in C$, the mapping $g_{x,y}$ is λ-convex on $[0,1]$.

Proof. "$(i) \Rightarrow (ii)$". Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Then we have

$$\begin{aligned} g_{x,y}\left(\frac{\alpha t_1 + \beta t_2}{\alpha + \beta}\right) & \\ &= f\left[\left(\frac{\alpha t_1 + \beta t_2}{\alpha + \beta}\right) x + \left(1 - \frac{\alpha t_1 + \beta t_2}{\alpha + \beta}\right) y\right] \\ &= f\left[\frac{\alpha(t_1 x + (1-t_1) y) + \beta(t_2 x + (1-t_2) y)}{\alpha + \beta}\right] \\ &\leq \frac{\lambda(\alpha) f(t_1 x + (1-t_1) y) + \lambda(\beta) f(t_2 x + (1-t_2) y)}{\lambda(\alpha + \beta)} \\ &= \frac{\lambda(\alpha) g_{x,y}(t_1) + \lambda(\beta) g_{x,y}(t_2)}{\lambda(\alpha + \beta)} \end{aligned} \tag{3.1}$$

and the implication is proved.

"$(ii) \Rightarrow (i)$". Let $x, y \in C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. Then we have

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) = g_{x,y}\left(\frac{\alpha}{\alpha + \beta}\right) &= g_{x,y}\left(\frac{\alpha \cdot 1 + \beta \cdot 0}{\alpha + \beta}\right) \\ &\leq \frac{\lambda(\alpha) g_{x,y}(1) + \lambda(\beta) g_{x,y}(0)}{\lambda(\alpha + \beta)} \\ &= \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)} \end{aligned}$$

and the implication is thus proved. □

We also can introduce the following mapping $k_{x,y} : [0,1] \to \mathbb{R}$

$$k_{x,y}(t) := \frac{1}{2}\left[f(tx + (1-t)y) + f((1-t)x + ty)\right]$$

for $x, y \in C$, $x \neq y$.

Theorem 3. *Let $f : C \to [0, \infty)$ be a λ-convex function on C. Assume that $x, y \in C$ with $x \neq y$.*
(i) We have the equality

$$k_{x,y}(1-t) = k_{x,y}(t) \text{ for all } t \in [0,1];$$

(ii) The mapping $k_{x,y}$ is λ-convex on $[0,1]$;
(iii) One has the inequalities

$$k_{x,y}(t) \leq \frac{\lambda(t) + \lambda(1-t)}{\lambda(1)} \cdot \frac{f(x) + f(y)}{2} \tag{3.2}$$

and

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)}\cdot f\left(\frac{x+y}{2}\right)\leq k_{x,y}(t) \tag{3.3}$$

for all $t\in[0,1]$ *and* $\alpha>0$.

(iv) Let $y,x\in C$ *with* $y\neq x$ *and assume that the mappings* $[0,1]\ni t\mapsto f[(1-t)x+ty]$ *and* λ *are Lebesgue integrable on* $[0,1]$, *then we have the Hermite-Hadamard type inequalities*

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)}\cdot f\left(\frac{x+y}{2}\right)\leq\int_0^1 f((1-t)x+ty)\,dt\leq\frac{f(x)+f(y)}{\lambda(1)}\int_0^1\lambda(t)\,dt \tag{3.4}$$

for any $\alpha>0$.

Proof. The statements (i) and (ii) are obvious.

(iii). By the λ-convexity of f we have:

$$f(tx+(1-t)y)\leq\frac{\lambda(t)f(x)+\lambda(1-t)f(y)}{\lambda(1)}$$

and

$$f((1-t)x+ty)\leq\frac{\lambda(1-t)f(x)+\lambda(t)f(y)}{\lambda(1)},$$

which gives by addition the inequality (3.2).

We also have

$$\frac{\lambda(\alpha)f(z)+\lambda(\alpha)f(u)}{\lambda(2\alpha)}\geq f\left(\frac{\alpha z+\alpha u}{\alpha+\alpha}\right)=f\left(\frac{z+u}{2}\right)$$

i.e.,

$$\frac{\lambda(\alpha)}{\lambda(2\alpha)}[f(z)+f(u)]\geq f\left(\frac{z+u}{2}\right)$$

for all $z,u\in C$.

If we write this inequality for $z=tx+(1-t)y$ and $u=(1-t)x+ty$ we get

$$\frac{\lambda(\alpha)}{\lambda(2\alpha)}[f(tx+(1-t)y)+f((1-t)x+ty)]\geq f\left(\frac{x+y}{2}\right),$$

which is equivalent to (3.3).

Integrating (3.3) and (3.4) over t on $[0,1]$ we get

$$\begin{aligned}\frac{2\lambda(\alpha)}{\lambda(2\alpha)}\cdot f\left(\frac{x+y}{2}\right)&\leq\frac{1}{2}\int_0^1[f(tx+(1-t)y)+f((1-t)x+ty)]\,dt\\&\leq\frac{f(x)+f(y)}{2}\int_0^1\frac{\lambda(t)+\lambda(1-t)}{\lambda(1)}dt.\end{aligned} \tag{3.5}$$

Since

$$\int_0^1 f(tx+(1-t)y)\,dt=\int_0^1 f((1-t)x+ty)\,dt$$

and

$$\int_0^1\lambda(t)\,dt=\int_0^1\lambda(1-t)\,dt$$

then by (3.5) we get the desired result (3.4). $\square$

Remark 4. *Since λ is subadditive, then*

$$\frac{\lambda(2\alpha)}{2\lambda(\alpha)} \leq 1 \text{ for any } \alpha > 0.$$

From (3.4) we have the best inequality

$$\sup_{\alpha>0}\left\{\frac{\lambda(2\alpha)}{2\lambda(\alpha)}\right\} \cdot f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{\lambda(1)}\int_0^1 \lambda(t)\,dt. \tag{3.6}$$

If the right limit

$$k = \lim_{s\to 0+}\frac{\lambda(s)}{s}$$

exists and is finite with $k>0$, then

$$\lim_{\alpha\to 0+}\frac{\lambda(2\alpha)}{2\lambda(\alpha)} = \lim_{\alpha\to 0+}\frac{\left(\frac{\lambda(2\alpha)}{2\alpha}\right)}{\left(\frac{\lambda(\alpha)}{\alpha}\right)} = \frac{\lim_{\alpha\to 0+}\left(\frac{\lambda(2\alpha)}{2\alpha}\right)}{\lim_{\alpha\to 0+}\left(\frac{\lambda(\alpha)}{\alpha}\right)} = \frac{k}{k} = 1$$

and by (3.4) we get

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{\lambda(1)}\int_0^1 \lambda(t)\,dt. \tag{3.7}$$

Corollary 2. *Assume that the function $f: C \to [0,\infty)$ is λ_r-convex with $\lambda_r : [0,\infty) \to [0,\infty)$,*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

and h is as in Corollary 1.

If $y,x \in C$ with $y \neq x$ and the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$, then we have the Hermite-Hadamard type inequalities

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{\ln\left[\frac{h(r)}{h(re^{-1})}\right]}\int_0^1 \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]dt. \tag{3.8}$$

Proof. We know that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t) := \frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t \in (0,\infty)$, where

$$h'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Since $\lambda_r(0) = 0$, then

$$k = \lim_{s\to 0+}\frac{\lambda(s)}{s} = \lambda_+'(0) = \frac{rh'(r)}{h(r)} > 0 \text{ for } r \in (0,R)$$

and by (3.7) we get (3.8). $\square$

Further on, we observe that the following elementary inequality holds:

$$(\alpha+\beta)^p \geq (\leq)\, \alpha^p+\beta^p \tag{3.9}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0<p<1)$.

Indeed, if we consider the function $f_p : [0,\infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$ we have $f_p'(t) = p\left[(t+1)^{p-1} - t^{p-1}\right]$. Observe that for $p>1$ and $t>0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0,\infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (3.9).

For $p \in (0,1)$ we have that f_p is strictly decreasing on $[0,\infty)$ which proves the second case in (3.9).

If we consider the power function $\hat{\lambda}_q(t) = t^q$ with $q \in (0,1)$, then $\hat{\lambda}_q$ is subadditive and by (3.4) we have

$$\frac{1}{2^{1-q}} \cdot f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{q+1}, \tag{3.10}$$

therefore we recapture the inequality (1.12) that was obtained from (1.7).

For $q \geq 1$ and if we consider the function $\check{\lambda}_q(t) = \frac{1}{t^q}$, then for any $t, s > 0$ we have

$$\check{\lambda}_q(t+s) = \frac{1}{(t+s)^q} \leq \frac{1}{t^s+s^q} \leq \frac{1}{t^s} + \frac{1}{s^q} = \check{\lambda}_q(t) + \check{\lambda}_q(s)$$

which shows that $\check{\lambda}_q$ is subadditive.

If $f : C \to [0,\infty)$ is a $\check{\lambda}_q$-convex function on C, i.e.

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\alpha^{-q} f(x) + \beta^{-q} f(y)}{(\alpha+\beta)^{-q}} \tag{3.11}$$

for all $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$, where $q \geq 1$, then we observe that the inequality (3.11) is equivalent to

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \left(\frac{\alpha+\beta}{\alpha\beta}\right)^q \left[\beta^q f(x) + \alpha^q f(y)\right] \tag{3.12}$$

for all $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$, where $q \geq 1$.

Since $\check{\lambda}_q$ is not integrable on $[0,1]$ we cannot apply the second inequality from (3.4). However, from the first inequality we get

$$\frac{1}{2^{q+1}} \cdot f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \tag{3.13}$$

provided that f is $\check{\lambda}_q$-convex and the integral $\int_0^1 f((1-t)x+ty)\,dt$ exists for some $x, y \in C$.

Moreover, if we assume that $f : C \to [0,\infty)$ is a λ-convex function on C with $\lambda(t) = 1-\exp(-t)$, $t \geq 0$, i.e.

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\exp(\beta)\left[\exp(\alpha)-1\right] f(x) + \exp(\alpha)\left[\exp(\beta)-1\right] f(y)}{\exp(\alpha+\beta)-1} \tag{3.14}$$

for any $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$, then by (3.7) we have

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x+ty)\,dt \leq \frac{f(x)+f(y)}{1-e^{-1}} \int_0^1 \left[1-\exp(-t)\right] dt,$$

that is equivalent to

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f\left((1-t)x+ty\right)dt \le \frac{f(x)+f(y)}{e-1}, \tag{3.15}$$

provided the integral $\int_0^1 f\left((1-t)x+ty\right)dt$ exists for some $x,y \in C$.

4. Inequalities for Double Integrals

We have the following result:

Theorem 4. *Let $f : C \to [0,\infty)$ be a λ-convex function on C. Let $y,x \in C$ with $y \neq x$ and assume that the mappings $[0,1] \ni t \mapsto f[(1-t)x+ty]$ and λ are Lebesgue integrable on $[0,1]$, then for $0 \le a < b$ we have the Hermite-Hadamard type inequalities*

$$\begin{aligned} &\frac{\lambda(2\eta)}{2\lambda(\eta)} \cdot f\left(\frac{x+y}{2}\right)(b-a)^2 \\ &\le \frac{1}{2}\int_a^b \int_a^b \left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta \\ &\le [f(x)+f(y)]\int_a^b\int_a^b \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}d\alpha d\beta \end{aligned} \tag{4.1}$$

for any $\eta > 0$.

Proof. By the λ-convexity of f we have

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \le \frac{\lambda(\alpha)f(x)+\lambda(\beta)f(y)}{\lambda(\alpha+\beta)}$$

and

$$f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) \le \frac{\lambda(\beta)f(x)+\lambda(\alpha)f(y)}{\lambda(\alpha+\beta)}$$

for all $\alpha,\beta \ge 0$ with $\alpha+\beta > 0$.

By adding these inequalities we obtain

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) + f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) \le \frac{\lambda(\alpha)+\lambda(\beta)}{\lambda(\alpha+\beta)}[f(x)+f(y)] \tag{4.2}$$

for all $\alpha,\beta \ge 0$ with $\alpha+\beta > 0$.

Since the mappings $[0,1] \ni t \mapsto f[(1-t)x+ty]$ and λ are Lebesgue integrable on $[0,1]$, then the integrals

$$\int_a^b\int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta \text{ and } \int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta$$

exist and by integrating the inequality (4.2) on the square $[a,b]^2$ we get

$$\begin{aligned} &\int_a^b\int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta + \int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta \\ &\le [f(x)+f(y)]\int_a^b\int_a^b \frac{\lambda(\alpha)+\lambda(\beta)}{\lambda(\alpha+\beta)}d\alpha d\beta \\ &= 2[f(x)+f(y)]\int_a^b\int_a^b \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}d\alpha d\beta \end{aligned}$$

and the second inequality in (4.1) is proved.

We know from the proof of Theorem 3 that

$$\frac{\lambda(\eta)}{\lambda(2\eta)}[f(z)+f(u)] \geq f\left(\frac{z+u}{2}\right)$$

for all $z, u \in C$ and $\eta > 0$.

Taking

$$z = \frac{\alpha x + \beta y}{\alpha + \beta} \text{ and } u = \frac{\beta x + \alpha y}{\alpha + \beta}$$

we get

$$\frac{\lambda(\eta)}{\lambda(2\eta)}\left[f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right)\right] \geq f\left(\frac{x+y}{2}\right) \tag{4.3}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $\eta > 0$.

Integrating the inequality (4.3) on the square $[a,b]^2$ we get the first part of (4.1). □

Remark 5. *If we write the inequality (4.1) for $f : C \to [0,\infty)$ a $\check{\lambda}_q$-convex function on C, then we get the inequality*

$$\begin{aligned} &\frac{1}{2^{q+1}} \cdot f\left(\frac{x+y}{2}\right)(b-a)^2 \\ &\leq \frac{1}{2}\int_a^b \int_a^b \left[f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) d\alpha d\beta + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right)\right] d\alpha d\beta \\ &\leq [f(x) + f(y)] \int_a^b \int_a^b \left(\frac{\alpha + \beta}{\alpha}\right)^q d\alpha d\beta, \end{aligned} \tag{4.4}$$

provided that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$.

For $q = 1$ we have

$$\begin{aligned} \int_a^b \int_a^b \frac{\alpha + \beta}{\alpha} d\beta d\alpha &= \int_a^b \int_a^b \left(1 + \frac{\beta}{\alpha}\right) d\beta d\alpha \\ &= (b-a)^2 + (\ln b - \ln a)\frac{b^2 - a^2}{2} \\ &= (b-a)^2 \left(1 + \frac{\ln b - \ln a}{b-a} \cdot \frac{a+b}{2}\right) \\ &= (b-a)^2 \left[1 + \frac{A(a,b)}{L(a,b)}\right] \end{aligned}$$

where

$$L(a,b) := \frac{b-a}{\ln b - \ln a}$$

is the logarithmic mean.

Then from (4.4) we get

$$\begin{aligned}
&\frac{1}{4}\cdot f\left(\frac{x+y}{2}\right) \\
&\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta \\
&\leq [f(x)+f(y)]\left[1+\frac{A(a,b)}{L(a,b)}\right],
\end{aligned}\tag{4.5}$$

provided that $f : C \to [0,\infty)$ *is a* $\check{\lambda}_1$*-convex function on* C *and the mapping* $[0,1] \ni t \mapsto f[(1-t)x+ty]$ *is Lebesgue integrable on* $[0,1]$*.*

For $q = 2$ *we have*

$$\begin{aligned}
\int_a^b\int_a^b\left(\frac{\alpha+\beta}{\alpha}\right)^2 d\beta d\alpha &= \int_a^b\int_a^b\left(1+\frac{\beta}{\alpha}\right)^2 d\beta d\alpha \\
&= \int_a^b\int_a^b\left(1+\frac{2\beta}{\alpha}+\frac{\beta^2}{\alpha^2}\right)d\beta d\alpha \\
&= (b-a)^2\left(1+2\frac{\ln b-\ln a}{b-a}\cdot\frac{a+b}{2}+\frac{a^2+ab+b^2}{3ab}\right) \\
&= \left(2\frac{\ln b-\ln a}{b-a}\cdot\frac{a+b}{2}+\frac{a^2+4ab+b^2}{3ab}\right) \\
&= 2(b-a)^2\left[\frac{1}{3}+\frac{2}{3}\cdot\frac{A(a,b)}{G(a,b)}+\frac{A(a,b)}{L(a,b)}\right],
\end{aligned}$$

where $G(a,b) := \sqrt{ab}$ *is the geometric mean.*

Then from (4.4) we get

$$\begin{aligned}
&\frac{1}{8}\cdot f\left(\frac{x+y}{2}\right) \\
&\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta \\
&\leq 2[f(x)+f(y)]\left[\frac{1}{3}+\frac{2}{3}\cdot\frac{A(a,b)}{G(a,b)}+\frac{A(a,b)}{L(a,b)}\right],
\end{aligned}\tag{4.6}$$

provided that $f : C \to [0,\infty)$ *is a* $\check{\lambda}_2$*-convex function on* C *and the mapping* $[0,1] \ni t \mapsto f[(1-t)x+ty]$ *is Lebesgue integrable on* $[0,1]$*.*

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