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WEIGHTED GENERALIZATION OF SOME INEQUALITIES FOR DIFFERENTIABLE CO-ORDINATED CONVEX FUNCTIONS WITH APPLICATIONS TO $2D$ WEIGHTED MIDPOINT FORMULA AND MOMENTS OF RANDOM VARIABLES

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Abstract. In this paper, a new weighted identity for differentiable functions of two variables defined on a rectangle from the plane is established. By using the obtained identity and analysis, some new weighted integral inequalities for the classes of co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi-convex functions on the rectangle from the plane are established which provide weighted generalization of some recent results proved for co-ordinated convex functions. Some applications of our results to random variables and $2D$ weighted quadrature formula are given as well.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$.

The most celebrated inequality for convex functions is the Hermite-Hadamard's inequality (see for instance [7]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, a, $b \in I$ with $a < b$. The inequalities in (1.1) are reversed if f is a concave function.

The inequalities (1.1) have various applications for generalized means, information measures, quadrature rules etc., and there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 4, 5, 6, 9, 21, 22] and the references therein.

Let us consider now a bidimensional interval $[a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on $[a,b] \times [c,d]$ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w),$$

holds for all $(x,y), (z,w) \in [a,b] \times [c,d]$ and $\lambda \in [0,1]$.

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A modification for convex functions on $[a,b]\times[c,d]$, which are also known as co-ordinated convex functions, was initiated by Dragomir [4, 6] as follows:

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1. [13] *A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on the co-ordinates on $[a,b]\times[c,d]$ if the inequality*

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)w)\\ &\leq tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w),\end{aligned}$$

holds for all $(t,s)\in[0,1]\times[0,1]$ and $(x,u),(y,w)\in[a,b]\times[c,d]$.

Clearly, every convex mapping $f:[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [4, 6]).

The following Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [4]:

Theorem 1. [4] *Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is co-ordinated convex on $[a,b]\times[c,d]$. Then one has the inequalities:*

$$\begin{aligned}(1.2)\quad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ &\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\ &\leq\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\ &\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}$$

The above inequalities are sharp.

Latif and Dragomir [15], proved the following Hermite-Hadamard type inequalities.

Theorem 2. [15] *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on the co-ordinates on*

$[a,b]\times[c,d]$, *then one has the inequalities:*

$$\begin{aligned}(1.3)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\ &\left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right|\\ &\leq\frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|}{4}\right].\end{aligned}$$

The next two results from [15] involve powers of the absolute value of $\frac{\partial^2 f}{\partial s\partial t}$.

Theorem 3. [15] *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q\geq 1$, is convex on the co-ordinates on $[a,b]\times[c,d]$, then one has the inequalities:*

$$\begin{aligned}(1.4)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\ &\left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right|\\ &\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. [15] *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q>1$, is convex on the co-ordinates on $[a,b]\times[c,d]$, then one has the inequalities:*

$$\begin{aligned}(1.5)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\ &\left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right|\\ &\leq\frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.\end{aligned}$$

In a recent paper [22], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

Definition 2. [20] *A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be quasi-convex on $[a,b]\times[c,d]$ if the inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\max\{f(x,y),f(z,w)\},$$

holds for all $(x, y), (z, w) \in [a, b] \times [c, d]$ *and* $\lambda \in [0, 1]$ *.*

A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a, b] \times [c, d]$ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are quasi-convex where defined for all $x \in [a, b], y \in [c, d]$.

The definition of co-ordinated quasi-convex functions may be stated as follows.

Definition 3. [16] *A function* $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be quasi-convex on the co-ordinates on* $[a, b] \times [c, d]$ *if*

$$f(tx + (1 - t)z, sy + (1 - s)w) \leq \max\{f(x, y), f(x, w), f(z, y), f(z, w)\},$$

for all $(x, y), (z, w) \in [a, b] \times [c, d]$ *and* $(s, t) \in [0, 1] \times [0, 1]$ *.*

The class of co-ordinated quasi-convex functions on $[a, b] \times [c, d]$ is denoted by $QC([a, b] \times [c, d])$. It has also been proved in [20] that every quasi-convex functions on $[a, b] \times [c, d]$ is quasi-convex on the co-ordinates on $[a, b] \times [c, d]$. The following example reveals that there exists quasi-convex function on the co-ordinates which is not quasi-convex.

Example 1. [16] *The function* $f : [-2, 2]^2 \to \mathbb{R}$*, defined by* $f(x, y) = \lfloor x \rfloor \lfloor y \rfloor$*, where* $\lfloor . \rfloor$ *is the floor function. This function is quasi-convex on the co-ordinates on* $[-2, 2]^2$ *but is not quasi-convex on* $[0, 1]^2$*.*
For example, take $(x, y) = (-2, 1)$*,* $(z, w) = (1, -1)$ *and* $\lambda = \frac{1}{2}$*, then*

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) = f\left(-\frac{1}{2}, 0\right) = 0,$$

on the other hand

$$\max\{f(x, y), f(z, w)\} = \max\{f(-2, 1), f(1, -1)\} = -1,$$

which shows that $f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) > \max\{f(x, y), f(z, w)\}$*.*

Another generalization of the notion of the co-ordinated convex functions is the concept of wright-convex functions which is given in the definition below.

Definition 4. [20] *A function* $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be wright-convex on* $[a, b] \times [c, d]$ *if the inequality*

$$\begin{aligned} &f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) + f((1 - \lambda)x + \lambda z, (1 - \lambda)y + \lambda w) \\ &\leq \max\{f(x, z), f(y, w)\}, \end{aligned}$$

holds for all $(x, z), (y, w) \in [a, b] \times [c, d]$ *and* $\lambda \in [0, 1]$ *.*

A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be wright-convex on the co-ordinates on $[a, b] \times [c, d]$ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are wright-convex where defined for all $x \in [a, b], y \in [c, d]$.

Definition 5. [20] *A function* $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be wright-convex on the co-ordinates on* $[a, b] \times [c, d]$ *if*

$$\begin{aligned} &f(tx + (1 - t)z, sy + (1 - s)w) + f((1 - t)x + tz, (1 - s)y + sw) \\ &\leq f(x, y) + f(z, y) + f(x, w) + f(z, w) \end{aligned}$$

for all $(x, z), (y, w) \in [a, b] \times [c, d]$ *and* $(s, t) \in [0, 1] \times [0, 1]$ *.*

The class of co-ordinated wright-convex functions on $[a,b]\times[c,d]$ is represented by $W([a,b]\times[c,d])$. It has also been proved in [20] that every wright-convex functions on $[a,b]\times[c,d]$ is wright-convex on the co-ordinates on $[a,b]\times[c,d]$.

For more recent results on co-ordinated convex, co-ordinated quasi-convex, co-ordinated m-convex, co-ordinated $(\alpha\ ,m)$-convex and co-ordinated s-convex functions on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we refer the readers to [1, 5, 8], [10]-[20].

In the present paper, we establish a new weighted identity for differentiable mappings defined on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$ and by using the obtained identity and analysis, some new weighted integral inequalities for differentiable co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi convex functions are proved. The results proved in the paper provide a weighted generalization of the results given in Theorem 2, Theorem 3 and Theorem 4. Applications of our results to random variables and $2D$ weighted midpoint formula are provided as well.

2. Main Results

We need the following lemma to establish our main results of this section.

Lemma 1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial s\partial t}\in L([a,b]\times[c,d])$, then*

$$\begin{aligned}(2.1)\quad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\\ &-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\\ &=\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left[\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right.\\ &\left.-\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))-\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))+\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right]dsdt,\end{aligned}$$

where

$$\begin{aligned}U_1(t)&=\frac{1-t}{2}a+\frac{1+t}{2}b, L_1(t)=\frac{1+t}{2}a+\frac{1-t}{2}b,\\ U_2(s)&=\frac{1-s}{2}c+\frac{1+s}{2}d, L_2(s)=\frac{1+s}{2}c+\frac{1-s}{2}d.\end{aligned}$$

Proof. Let

$$\begin{aligned}I=&\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left[\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right.\\ &\left.-\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))-\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))+\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right]dsdt\end{aligned}$$

and

$$\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy=q(t,s).$$

then

$$I = \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)\left[\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s)) - \frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right.$$
$$\left.-\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s)) + \frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right]dsdt.$$

Now by integration by parts and by using the symmetry of $p(x,y)$ about $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have

$$\begin{aligned}
(2.2)\quad & \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\,dsdt \\
&= \frac{(b-a)(d-c)}{4}\int_0^1\left[\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\,ds\right]dt \\
&= \frac{(b-a)(d-c)}{4}\int_0^1\left[\frac{2}{d-c}q(t,s)\frac{\partial}{\partial t}f(U_1(t),U_2(s))\Big|_0^1\right. \\
&\quad \left.-\frac{2}{d-c}\int_0^1\frac{\partial}{\partial s}q(t,s)\frac{\partial}{\partial t}f(U_1(t),U_2(s))\,ds\right]dt \\
&= \frac{(b-a)}{2}\int_0^1\left[-\frac{\partial}{\partial t}f\left(U_1(t),\frac{c+d}{2}\right)\left(\int_{\frac{c+d}{2}}^d\int_{U_1(t)}^b p(x,y)\,dxdy\right)\right. \\
&\quad \left.+\frac{(d-c)}{2}\int_0^1\left(\int_{U_1(t)}^b p(x,U_2(s))\,dx\right)\frac{\partial}{\partial t}f(U_1(t),U_2(s))\,ds\right]dt \\
&= -\frac{(b-a)}{2}\int_0^1\frac{\partial}{\partial t}f\left(U_1(t),\frac{c+d}{2}\right)\left(\int_{\frac{c+d}{2}}^d\int_{U_1(t)}^b p(x,y)\,dxdy\right)dt \\
&\quad +\frac{(b-a)}{2}\int_{\frac{c+d}{2}}^d\int_0^1\left(\int_{U_1(t)}^b p(x,y)\,dx\right)\frac{\partial}{\partial t}f(U_1(t),y)\,dtdy \\
&= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b p(x,y)\,dxdy - \int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy \\
&\quad -\int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy + \int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b p(x,y)f(x,y)\,dxdy.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(2.3)\quad & -\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\,dsdt \\
&= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b p(x,y)\,dxdy - \int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy \\
&\quad -\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy + \int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b p(x,y)f(x,y)\,dxdy,
\end{aligned}$$

$$
\begin{aligned}
(2.4)\quad & -\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\,dsdt \\
&= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{c+d}{2}}^{d}\int_a^{\frac{a+b}{2}} p(x,y)\,dxdy - \int_{\frac{c+d}{2}}^{d}\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy \\
&\quad +\frac{1}{2}\int_{\frac{c+d}{2}}^{d}\int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy + \int_{\frac{c+d}{2}}^{d}\int_a^{\frac{a+b}{2}} p(x,y)f(x,y)\,dxdy
\end{aligned}
$$

and

$$
\begin{aligned}
(2.5)\quad & \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\,dsdt \\
&= f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} p(x,y)\,dxdy - \int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy \\
&\quad -\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy + \int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} p(x,y)f(x,y)\,dxdy.
\end{aligned}
$$

Adding (2.2)-(2.5), we get the desired result. □

Remark 1. *If we take* $p(x,y)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$ *in Lemma 1, we get Lemma 1 from* [15, page 13].

Now by using lemma 1, we present the main results of this section.

Theorem 5. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a twice differentiable mapping on* Δ° *and* $p:[a,b]\times[c,d]\to[0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for* $[a,b]\times[c,d]\subset\Delta^\circ$ *with* $a<b$, $c<d$. *If* $\frac{\partial^2 f}{\partial s\partial t}\in L([a,b]\times[c,d])$ *and* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$, *then*

$$
\begin{aligned}
(2.6)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy + \int_c^d\int_a^b f(x,y)p(x,y)\,dxdy \right.\\
&\left. -\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy - \int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy \right| \\
&\le \frac{(b-a)(d-c)}{4}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right] \\
&\qquad\qquad \times\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdydtds.
\end{aligned}
$$

Proof. Taking absolute value on both sides of (2.1) and using the properties of absolute value, we have

$$\begin{aligned}(2.7)\quad &\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\ &\left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\ &\leq\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left[\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|\right.\\ &\left.+\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|+\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|+\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|\right]dsdt.\end{aligned}$$

By the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\begin{aligned}(2.8)\quad &\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|\\ &\leq\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\ &\quad+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\end{aligned}$$

$$\begin{aligned}(2.9)\quad &\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|\\ &\leq\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\ &\quad+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\end{aligned}$$

$$\begin{aligned}(2.10)\quad &\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|\\ &\leq\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\ &\quad+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\end{aligned}$$

and

$$\begin{aligned}(2.11)\quad &\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|\\ &\leq\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\ &\quad+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|.\end{aligned}$$

Using (2.8)-(2.11) in (2.7), we get (2.6). $\square$

Remark 2. *If we take* $p(x,y)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$ *in Theorem 5, we get Theorem 2 from* [15].

A more general result is given in the following theorem.

Theorem 6. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a twice differentiable mapping on* Δ° *and* $p:[a,b]\times[c,d]\to[0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for* $[a,b]\times[c,d]\subset\Delta^\circ$ *with* $a<b$, $c<d$. *If* $\frac{\partial^2 f}{\partial s\partial t}\in L([a,b]\times[c,d])$ *and* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\geq1$, *then*

$$\begin{aligned}(2.12)\quad &\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\ &\left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\ &\leq(b-a)(d-c)\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}\\ &\qquad\times\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdydsdt.\end{aligned}$$

Proof. Taking absolute value on both sides of (2.1), by using the properties of absolute value and the Hölder inequality, we have

$$\begin{aligned}(2.13)\quad &\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\ &\left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\ &\leq\frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]dsdt\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right.\\ &+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\ &+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\ &\left.+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right].\end{aligned}$$

By the power-mean inequality $(a_1^r+a_2^r+a_3^r+a_4^r\leq4^{1-r}(a_1+a_2+a_3+a_4)^r$ for a_1, a_2, a_3, $a_4>0$ and $r<1)$ and using the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on the co-ordinates on

$[a,b]\times[c,d]$ for $q\geq 1$, we have

$$\begin{aligned}
(2.14)\quad & \left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\Bigg] \\
&\leq 4^{1-\frac{1}{q}}\Bigg\{\int_0^1\int_0^1\left(\int_c^{L_2(s)}\int_a^{L_1(t)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdy\right) \\
&\times\Bigg[\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q \\
&+\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q\Bigg]dsdt\Bigg\}^{\frac{1}{q}} \\
&\leq 4^{1-\frac{1}{q}}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right]^{\frac{1}{q}} \\
&\times\left(\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdydsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

A usage of (2.14) in (2.13) yields the desired result. □

Remark 3. *If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Theorem 6, we get Theorem 4.*

A different approach leads to the following result.

Theorem 7. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial s\partial t}\in L([a,b]\times[c,d])$ and $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on*

the co-ordinates on $[a,b]\times[c,d]$ *for* $q>1$*, then*

$$\begin{aligned}
(2.15)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\
& \left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\
& \le (b-a)(d-c)\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}\\
& \qquad\times\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]^p dsdt\right)^{\frac{1}{p}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and the Hölder inequality, we have

$$\begin{aligned}
(2.16)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\
& \left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\
& \le \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left[\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right]^p dsdt\right)^{\frac{1}{p}}\\
& \quad\times\left[\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right.\\
& \quad+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
& \quad+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
& \quad\left.+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By the power-mean inequality ($a_1^r+a_2^r+a_3^r+a_4^r\le 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for a_1, a_2, a_3, $a_4>0$ and $r<1$) and using the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on the co-ordinates on

$[a,b]\times[c,d]$ for $q>1$, we have

$$\begin{aligned}
(2.17)\quad & \left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1(t),U_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1(t),L_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1(t),U_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1(t),L_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}} \\
&\le 4^{1-\frac{1}{q}}\left[\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1(t),U_2(s)\right)\right|^q dsdt+\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1(t),L_2(s)\right)\right|^q dsdt\right. \\
&\left.\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1(t),U_2(s)\right)\right|^q dsdt\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1(t),L_2(s)\right)\right|^q dsdt\right]^{\frac{1}{q}} \\
&\le 4\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.
\end{aligned}$$

From (2.16) and (2.17), we get (2.15). □

Remark 4. *If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Theorem 7, we get Theorem 3.*

Remark 5. *Theorem 5-Theorem 7 continue to hold true if in their statements we replace the condition "convex on the co-ordinates" with the condition "wright-convex on the co-ordinates". However, the details are left to the interested reader.*

In what follows we give our results for the quasi-convex mappings on the co-ordinates on $[a,b]\times[c,d]$.

Theorem 8. *Suppose the assumptions of Theorem 5 are satisfied. If the mapping $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is quasi-convex on the co-ordinates on $[a,b]\times[c,d]$, then the following* *inequality holds*

$$\begin{aligned}
(2.18)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy \right.\\
&\left. -\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\
&\le \frac{(b-a)(d-c)}{4}\left[\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\\
&+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\\
&+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\\
&+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\right.\\
&\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\right]\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdydtds.
\end{aligned}$$

Proof. We continue inequality (2.7) in the proof of Theorem 1. Now, by the quasi-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$on $[a,b]\times[c,d]$, we obtain

$$\begin{aligned}
(2.19)\quad \left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right| &\le \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},
\end{aligned}$$

$$\begin{aligned}
(2.20)\quad \left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right| &\le \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},
\end{aligned}$$

$$\begin{aligned}
(2.21)\quad \left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right| &\le \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},
\end{aligned}$$

and

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),L_2\left(s\right)\right)\right|\leq\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\tag{2.22}$$

for all $(t,s)\in[0,1]\times[0,1]$. A combination of (2.19)-(2.22) and (2.7)gives the required inequality (2.18). □

Corollary 1. *Suppose the assumptions of Theorem 8 are fulfilled and if $p\left(x,y\right)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$, then the following inequality holds valid*

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(x,y\right)dxdy\right.$$
$$\left.-\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right|$$
$$\leq\frac{\left(b-a\right)\left(d-c\right)}{4}\left[\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\right].\tag{2.23}$$

Corollary 2. *Suppose the assumptions of Theorem 8 are satisfied and additionally*

(1) *If f is non-decreasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.24)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy \right.\\ & \left. -\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+d}{2},y\right)p(x,y)\,dxdy\right| \\ & \leq \frac{(b-a)(d-c)}{4}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|\right. \\ & \left. +\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdydtds.\end{aligned}$$

(2) *If f is non-increasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.25)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy \right.\\ & \left. -\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+d}{2},y\right)p(x,y)\,dxdy\right| \\ & \leq \frac{(b-a)(d-c)}{4}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|\right. \\ & \left. +\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdydtds.\end{aligned}$$

Corollary 3. *If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Corollary 2 and additionally*

(1) *If f is non-decreasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.26)\quad & \left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,dxdy \right.\\ & \left. -\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right| \\ & \leq \frac{(b-a)(d-c)}{4}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|\right. \\ & \left. +\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right].\end{aligned}$$

(2) *If f is non-increasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,dxdy \right. \\
&\left. -\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right| \\
&\leq \frac{(b-a)(d-c)}{4}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|\right. \\
&\left.+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right].
\end{aligned}
\tag{2.27}
$$

Theorem 9. *Suppose the assumptions of Theorem 5 are satisfied. If the mapping $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is quasi-convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, then the following inequality holds*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy \right. \\
&\left. -\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right| \\
&\leq \frac{(b-a)(d-c)}{4}\left[\left(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\right.\right.\right. \\
&\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\left(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\right.\right. \\
&\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\left(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\right.\right. \\
&\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&+\left(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\right.\right. \\
&\left.\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right] \\
&\times\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdydtds.
\end{aligned}
\tag{2.28}
$$

Proof. We continue inequality (2.13) in the proof of Theorem **??**. Now, by the quasi-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on $[a,b]\times[c,d]$ for $q\geq 1$ and the power-mean inequality, we obtain

$$\begin{aligned}(2.29)\quad \left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),U_2\left(s\right)\right)\right|^q &\leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}$$

$$\begin{aligned}(2.30)\quad \left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),U_2\left(s\right)\right)\right|^q &\leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}$$

$$\begin{aligned}(2.31)\quad \left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),L_2\left(s\right)\right)\right|^q &\leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\},\end{aligned}$$

and

$$\begin{aligned}(2.32)\quad \left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q &\leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}\end{aligned}$$

for all $(t,s)\in[0,1]\times[0,1]$. Using (2.29)-(2.32) in (2.13) we get the desired result. $\square$

Corollary 4. *Suppose the assumptions of Theorem 9 are fulfilled and if* $p(x,y) = \frac{1}{(b-a)(d-c)}$ *for all* $(x,y) \in [a,b] \times [c,d]$, *then the following inequality holds valid*

$$\begin{aligned}
(2.33)\quad & \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\,dxdy \right. \\
& \left. -\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy \right| \\
& \le \frac{(b-a)(d-c)}{4}\left[\left(\max\left\{\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(b, \frac{c+d}{2}\right)\right|^q,\right.\right.\right. \\
& \left.\left.\left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, d\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
& + \left(\max\left\{\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(a, \frac{c+d}{2}\right)\right|^q,\right.\right. \\
& \left.\left.\left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, d\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
& + \left(\max\left\{\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(b, \frac{c+d}{2}\right)\right|^q,\right.\right. \\
& \left.\left.\left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, c\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
& + \left(\max\left\{\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(a, \frac{c+d}{2}\right)\right|^q,\right.\right. \\
& \left.\left.\left.\left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, c\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 6. *Suppose the assumptions of Theorem 9 are satisfied and additionally*

(1) *If* f *is non-decreasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.24) holds valid.*
(2) *If* f *is non-increasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.25) holds true.*

Remark 7. *In Corollary 4*

(1) *If* f *is non-decreasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.26) holds valid.*
(2) *If* f *is non-increasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.27) holds true.*

3. Applications to Random Variables

Let $0 < a < b$, $0 < c < d$, $\alpha, \beta \in \mathbb{R}$ and let X and Y be two independent continuous random variables having the bi-variate continuous probability density function $p : [a,b] \times [c,d] \to [0,\infty)$ which is symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ the α-moment of X and the β-moment of Y about the origin are respectively defined as follows

$$E_\alpha(X) = \int_c^b t^\alpha p_1(t)\,dt, E_\beta(Y) = \int_c^b s^\beta p_2(s)\,ds$$

which are assumed to be finite, here $p_1 : [a,b] \to [0,\infty)$ and $p_2 : [c,d] \to [0,\infty)$ are the marginal probability density functions of X and Y. Since X and Y are independent random variables, we have

$$p(t,s) = p_1(t)\, p_2(s)$$

for all $(t,s) \in [a,b] \times [c,d]$.

Now we give some applications of our result to random variables.

Theorem 10. *The inequality*

$$\left|\left(E_\alpha(X) - \left(\frac{a+b}{2}\right)^\alpha\right)\left(E_\beta(Y) - \left(\frac{c+d}{2}\right)^\beta\right)\right| \le \frac{(b-a)(d-c)}{4}\alpha\beta\left(\frac{a^{\alpha-1}+b^{\alpha-1}}{2}\right)\left(\frac{c^{\beta-1}+d^{\beta-1}}{2}\right). \tag{3.1}$$

holds holds for $0 < a < b$, $0 < c < d$ *and* $\alpha, \beta \ge 2$.

Proof. Let $f(t,s) = t^\alpha s^\beta$ on $[a,b] \times [c,d]$ for $\alpha, \beta \ge 2$, we observe that $\left|\frac{\partial^2 f(t,s)}{\partial s \partial t}\right| = \alpha\beta t^{\alpha-1} s^{\beta-1}$ is convex on the co-ordinates on $[a,b] \times [c,d]$. Since

$$\begin{aligned}&\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right| \\ &= \alpha\beta\left(a^{\alpha-1}+b^{\alpha-1}\right)\left(c^{\beta-1}+d^{\beta-1}\right),\end{aligned}$$

$$\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdy \le \int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} p(x,y)\,dxdy = \frac{1}{4}$$

and hence

$$\int_0^1\int_0^1\int_c^{L_2(s)}\int_a^{L_1(t)} p(x,y)\,dxdydtds \le \frac{1}{4}.$$

Also

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy = \left(\frac{a+b}{2}\right)^\alpha\left(\frac{c+d}{2}\right)^\beta,$$

$$\begin{aligned}&\int_c^d\int_a^b f\left(x, \frac{c+d}{2}\right)p(x,y)\,dxdy + \int_c^d\int_a^b f\left(\frac{a+b}{2}, y\right)p(x,y)\,dxdy \\ &= \left(\frac{c+d}{2}\right)^\beta E_\alpha(X) + \left(\frac{a+b}{2}\right)^\alpha E_\beta(Y)\end{aligned}$$

and

$$\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy = E_\alpha(X)\,E_\beta(Y).$$

The result follows immediately from the inequality (2.6). □

Theorem 11. *The inequality*

$$\left|\left(E_\alpha(X) - \left(\frac{a+b}{2}\right)^\alpha\right)\left(E_\beta(Y) - \left(\frac{c+d}{2}\right)^\beta\right)\right| \le \frac{(b-a)(d-c)}{16}\alpha\beta\left(b^{\alpha-1} + \left(\frac{a+b}{2}\right)^{\alpha-1}\right)\left(d^{\beta-1} + \left(\frac{c+d}{2}\right)^{\beta-1}\right). \tag{3.2}$$

holds holds for $0 < a < b$, $0 < c < d$ *and* $\alpha, \beta \ge 1$.

Proof. Let $f(t,s)=t^{\alpha}s^{\beta}$ on $[a,b]\times[c,d]$ for $\alpha,\beta\geq 1$, we observe that $\left|\frac{\partial^2 f(t,s)}{\partial s\partial t}\right|=\alpha\beta t^{\alpha-1}s^{\beta-1}$ is non-decreasing and quasi-convex on the co-ordinates on $[a,b]\times[c,d]$. The proof is similar to that of Theorem 10 by using the inequality (2.24) we obtain the required result. □

Remark 8. *For $\alpha=\beta=1$, we have from Theorem 11 that*

$$\left|\left(E\left(X\right)-\frac{a+b}{4}\right)\left(E\left(Y\right)-\frac{c+d}{2}\right)\right|\leq\frac{\left(b-a\right)\left(d-c\right)}{4},\tag{3.3}$$

where $E_1\left(X\right)=E\left(X\right)$ and $E_1\left(Y\right)=E\left(Y\right)$ are the expectation of the random variables X and Y respectively.

4. Applications to 2D weighted Midpoint formula

Let $[a,b]\times[c,d]$ be a rectangle from the plane $\mathbb{R}^2$. Suppose d_1 and d_2 are the divisions $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ and $c=y_0<y_1<\cdots<y_{m-1}<y_m=b$ of the intervals $[a,b]$ and $[c,d]$ respectively and let $d=\{[x_i,x_{i+1}]\times[y_j,y_{j+1}]:0\leq i\leq n-1,0\leq j\leq m-1\}$ be a corresponding division of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$.

Consider the following $2D$ weighted quadrature formula

$$\int_c^d\int_a^b f\left(x,y\right)p\left(x,y\right)dxdy=T\left(f,p,d\right)+E\left(f,p,d\right),\tag{4.1}$$

where

$$\begin{aligned}T\left(f,p,d\right)=-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}&\left[f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right.\\ &\times\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}p\left(x,y\right)dxdy+\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}f\left(x,\frac{y_j+y_{j+1}}{2}\right)p\left(x,y\right)dxdy\\ &\left.+\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}f\left(\frac{x_i+x_{i+1}}{2},y\right)p\left(x,y\right)dxdy\right]\end{aligned}\tag{4.2}$$

for the midpoint version and $E(f,p,d)$ denotes the associated approximation error.

The following results provide some estimates of the remainder term $E(f,p,d)$.

Theorem 12. *Suppose the assumptions of Theorem 6 are satisfied. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, then in (4.1), for every division d of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, the following holds*

$$\begin{aligned}\left|E\left(f,p,d\right)\right|&\leq\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(x_{i+1}-x_i\right)\left(y_{j+1}-y_j\right)\\ &\times\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,y_j\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,y_{j+1}\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},y_j\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},y_{j+1}\right)\right|^q}{4}\right]\\ &\times\int_0^1\int_0^1\int_{y_j}^{L_2\left(y_j,y_{j+1},s\right)}\int_{x_i}^{L_1\left(x_i,x_{i+1},t\right)}p\left(x,y\right)dxdydsdt,\end{aligned}\tag{4.3}$$

where

$$L_1\left(x_i,x_{i+1},t\right)=\frac{1+t}{2}x_i+\frac{1-t}{2}x_{i+1}$$

and

$$L_2\left(y_j,y_{j+1},s\right)=\frac{1+s}{2}y_j+\frac{1-s}{2}y_{j+1}.$$

Proof. Applying Theorem 6 on the rectangles $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ $(0\le i\le n-1, 0\le j\le m-1)$ of the division d of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we get

$$\begin{aligned}(4.4)\quad &\Bigg| f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} p\left(x,y\right)dxdy\\
&-\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} f\left(x,\frac{y_j+y_{j+1}}{2}\right)p\left(x,y\right)dxdy\\
&-\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} f\left(\frac{x_i+x_{i+1}}{2},y\right)p\left(x,y\right)dxdy\\
&+\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} f\left(x,y\right)p\left(x,y\right)dxdy\Bigg|\le\left(x_{i+1}-x_i\right)\left(y_{j+1}-y_j\right)\\
&\times\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,y_j\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,y_{j+1}\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},y_j\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},y_{j+1}\right)\right|^q}{4}\right]^{\frac{1}{q}}\\
&\times\int_0^1\int_0^1\int_{y_j}^{L_2(y_j,y_{j+1},s)}\int_{x_i}^{L_1(x_i,x_{i+1},t)} p\left(x,y\right)dxdydsdt.\end{aligned}$$

Summing over i from 0 to $n-1$ and j over 0 to $m-1$, we deduce, by the triangle inequality, that (4.3) holds. □

Remark 9. *The inequality (4.3) holds if the condition of convexity of* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *on the co-ordinates on* $[a,b]\times[c,d]$ *is replaced with the condition of wright-convexity of* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\ge 1$.

Theorem 13. *Suppose the assumptions of Theorem 6 are satisfied. If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\ge 1$*, then in (4.1), for every division*

d of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, the following holds

$$
\begin{aligned}
(4.5)\quad |E(f,p,d)| &\le \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j)\\
&\times\Bigg[\Bigg(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_{j+1})\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right|^q,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\right\}\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_{j+1})\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,\frac{y_j+y_{j+1}}{2}\right)\right|^q,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\right\}\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_j)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right|^q,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_j\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\right\}\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_j)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,\frac{y_j+y_{j+1}}{2}\right)\right|^q,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_j\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\right\}\Bigg)^{\frac{1}{q}}\Bigg]\\
&\times\int_0^1\int_0^1\int_{y_j}^{L_2(y_j,y_{j+1},s)}\int_{x_i}^{L_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt.
\end{aligned}
$$

Proof. The proof follows from (2.28) by using the similar arguments as that of the proof of Theorem 12. □

Remark 10. *If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is non-decreasing in Theorem 13, then the following inequality holds*

$$
\begin{aligned}
(4.6)\quad |E(f,p,d)| &\le \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j)\\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_{j+1})\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|\right.\\
&\left.+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right|\right]\\
&\times\int_0^1\int_0^1\int_{y_j}^{L_2(y_j,y_{j+1},s)}\int_{x_i}^{L_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt.
\end{aligned}
$$

and if $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is non-increasing in Theorem 13, then the following inequality holds*

$$
\begin{aligned}
(4.7) \quad |E(f,p,d)| \le{}& \frac{1}{4} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (x_{i+1}-x_i)(y_{j+1}-y_j) \\
&\times \left[\left| \frac{\partial^2 f}{\partial s \partial t}(x_i, y_j) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}\left(x_i, \frac{y_j + y_{j+1}}{2}\right) \right| \right. \\
&\left. + \left| \frac{\partial^2 f}{\partial s \partial t}\left(\frac{x_i + x_{i+1}}{2}, y_j\right) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right) \right| \right] \\
&\times \int_0^1 \int_0^1 \int_{y_j}^{L_2(y_j, y_{j+1}, s)} \int_{x_i}^{L_1(x_i, x_{i+1}, t)} p(x,y)\, dxdydsdt.
\end{aligned}
$$

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