SEVERAL INEQUALITIES FOR POSITIVE OPERATORS ON
HILBERT SPACES

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Abstract. In this paper, several inequalities for positive definite operators defined on Hilbert spaces will be presented under suitable assumptions, starting from some refinements of the Kittaneh-Manasrah inequality which improves the well-known inequality of Young.

1. Introduction

It is necessary to recall the following results which are given in the papers [4] and [5] and will be used below in the demonstration of inequalities from Proposition 1, Theorem 2 and Proposition 3. In these demonstrations the same method as in the paper [1] will be utilized.

Lemma 1. ([4]) Let \( a \) and \( b \) be such that \( a, b \geq 0 \) and \( 0 \leq \nu \leq 1 \). Then the following inequality holds:

\[
\nu a^2 + (1 - \nu)b^2 \leq (\nu b^{1-\nu})^2 + s_0(a - b)^2,
\]

where \( s_0 = \max\{\nu, 1 - \nu\} \).

Lemma 2. ([5]) For all \( x, y \) positive real numbers and \( \lambda \in (0, 1) \) we have the inequality

\[
2rE\left(x, y, \frac{1}{2}\right) \leq E(x, y, \lambda) \leq 2(1 - r)E\left(x, y, \frac{1}{2}\right),
\]

where

\[
E(x, y, \lambda) = \lambda \exp x + (1 - \lambda) \exp y - \exp(\lambda x + (1 - \lambda)y) - \frac{\lambda(1 - \lambda)}{2}(x - y)^2
\]

and \( r = \min\{\lambda, 1 - \lambda\} \).

Theorem 1. ([5]) For \( a, b \geq 1 \), and \( \lambda \in (0, 1) \) we have

\[
r(\sqrt{a} - \sqrt{b})^2 + A_1(\lambda) \log^2\left(\frac{a}{b}\right) \leq \lambda a + (1 - \lambda)b - a^{\lambda b^{1-\lambda}} \leq (1 - r)(\sqrt{a} - \sqrt{b})^2 + B_1(\lambda) \log^2\left(\frac{a}{b}\right)
\]

where \( r = \min\{\lambda, 1 - \lambda\} \), \( A_1(\lambda) = \frac{\lambda(1 - \lambda)}{2} - \frac{r}{4} \) and \( B_1(\lambda) = \frac{\lambda(1 - \lambda)}{2} - \frac{1 - r}{4} \).

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First, it is necessary to recall that for selfadjoint operators \( A, B \in B(H) \) we write \( A \leq B \) (or \( B \geq A \)) if \( \langle Ax, x \rangle \leq \langle Bx, x \rangle \) for every vector \( x \in H \). In this paper, we will consider \( A \) as being a selfadjoint linear operator on a complex Hilbert space \( (H; \langle \cdot, \cdot \rangle) \) as in [1] and the references therein. The Gelfand map establishes a \(*\)-isometrically isomorphism \( \Phi \) between the set \( C(Sp(A)) \) of all continuous functions defined on the spectrum of \( A \), denoted \( Sp(A) \), an the \( C^* \)- algebra \( C^*(A) \) generated by \( A \) and the identity operator \( 1_H \) on \( H \) as follows (i): For any \( f, g \in C(Sp(A)) \) and for any \( \alpha, \beta \in \mathbb{C} \) we have

(i) \( \Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g) \);
(ii) \( \Phi(fg) = \Phi(f)\Phi(g) \) and \( \Phi(f) = \Phi(f^*) \);
(iii) \( ||\Phi(f)|| = ||f|| = \sup_{t \in Sp(A)} |f(t)| \);
(iv) \( \Phi(f_0) = 1_H \) and \( \Phi(f_1) = A \), where \( f_0(t) = 1 \) and \( f_1(t) = t \) for \( t \in Sp(A) \).

Using this notation, as in [1] for example, we define

\[
f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))
\]

and we call it the continuous functional calculus for a selfadjoint operator \( A \). It is known that if \( A \) is a selfadjoint operator and \( f \) is a real valued continuous function on \( Sp(A) \), then \( f(t) \geq 0 \) for any \( t \in Sp(A) \) implies that \( f(A) \geq 0 \), i.e. \( f(A) \) is a positive operator on \( H \). In addition, if and \( f \) and \( g \) are real valued functions on \( Sp(A) \) then the following property holds:

(i) \( f(t) \geq g(t) \) for any \( t \in Sp(A) \) implies that \( f(A) \geq g(A) \)

in the operator order of \( B(H) \).

2. Main results

The following results present several inequalities for functions of positive operators.

**Proposition 1.** Let \( A \) and \( B \) be two positive definite operators on \( H \). Then we have

\[
\nu < A^2 x, x > + (1 - \nu) < B^2 y, y > \leq < A^{2 \nu} x, x > < B^{2(1 - \nu)} y, y > +
\]

\[
+ s_0 \left[ < A^2 x, x > - 2 < Ax, x > < By, y > + < B^2 y, y > \right],
\]

for each \( x, y \in H \) with \( ||x|| = ||y|| = 1 \), where \( 0 \leq \nu \leq 1 \) and \( s_0 = \max \{ \nu, 1 - \nu \} \).

**Proof.** We consider the continue function \( f(a) = (a^\nu b^{(1 - \nu)})^2 + s_0(a - b)^2 - (\nu a^2 + (1 - \nu)b^2) \), which is positive for \( a \geq 0 \) and we fix \( b \geq 0 \) and then by the property (1) for each \( x \in H \) with \( ||x|| = 1 \) we have that

\[
< (\nu A^2 + (1 - \nu)b^2 I)x, x > \leq < [A^{2 \nu} b^{2(1 - \nu)} + s_0(A^2 - 2Ab + b^2 I)]x, x >
\]

which is equivalent with

\[
\nu < A^2 x, x > + (1 - \nu)b^2 \leq
\]

\[
< b^{2(1 - \nu)} < A^{2 \nu} x, x > + s_0[< A^2 x, x > - 2b < Ax, x > + b^2 < x, x >]
\]

for each \( b > 0 \).

If we apply again the property (1) for last inequality, then for any \( y \in H \) with \( ||y|| = 1 \) we get

\[
< [\nu < A^2 x, x > + (1 - \nu)b^2]y, y > \leq
\]

\[
< [b^{2(1 - \nu)} < A^{2 \nu} x, x > + s_0(2b < Ax, x > + b^2 < x, x >)]y, y >
\]
and this inequality is equivalent with
\[ \nu < A^2 x, x > + (1 - \nu) < B^2 y, y > \leq \]
\[ \leq A^{2(1-\nu)} x, x > < B^2 (1-\nu) y, y > + s_0 < A^2 x, x > - 2 < A x, x > B y, y > + < B^2 y, y > \]
for each \( x, y \in H \) with \( \|x\| = \|y\| = 1 \).

Taking now in previous inequality \( x = y \) we obtain the desired inequality.

As an interesting application of previous result, we have the following particular cases:

**Remark 1.** (i) If we take in previous inequality \( y = x \) then we have:
\[ \nu < A^2 x, x > + (1 - \nu) < B^2 x, x > \leq A^{2(1-\nu)} x, x > + s_0 < A^2 x, x > - 2 < A x, x > B x, x > + < B^2 x, x > \]
for each \( x \in H \) with \( \|x\| = 1 \), where \( s_0 = \max \{\nu, 1 - \nu\} \).

(ii) If in addition \( A = B \) then in previous inequality we obtain:
\[ 1 - 2s_0 < A^2 x, x > - (A x, x >)^2 \leq A^{2(1-\nu)} x, x > \]
for each \( x \in H \) with \( \|x\| = 1 \), where \( s_0 = \max \{\nu, 1 - \nu\} \).

**Theorem 2.** Let \( A \) be a positive definite operator on \( H \). Then the following inequality holds:
\[ r \left[ 2 < \exp(A)x, x > - 2 \left( < \exp \left( \frac{A}{2} \right) x, x > \right)^2 - \frac{1}{2} \left( < A^2 x, x > - (A x, x >)^2 \right) \right] \leq \]
\[ \leq 1 - \nu < \exp(A)x, x > < \exp(1 - \nu)Ax, x > - \nu (1 - \nu) < A^2 x, x > - (A x, x >)^2 \]
\[ \leq 1 - r \left[ 2 < \exp(A)x, x > - 2 \left( < \exp \left( \frac{A}{2} \right) x, x > \right)^2 - \frac{1}{2} \left( < A^2 x, x > - (A x, x >)^2 \right) \right] \]
for each \( x \in H \) with \( \|x\| = 1 \), where \( r = \min \{\lambda, 1 - \lambda\} \).

**Proof.** We write and then use the inequality from Lemma 2 with \( x \) replaced by \( a \) and \( y \) replaced by \( b \) obtaining:
\[ r \left[ \exp(a) + \exp(b) - 2 \exp \left( \frac{a + b}{2} \right) - \frac{1}{4} (a - b)^2 \right] \leq \]
\[ \leq \lambda \exp(a) + (1 - \lambda) \exp(b) - \exp(\lambda a + (1 - \lambda)b) - \frac{\lambda(1 - \lambda)}{2} (a - b)^2 \leq \]
\[ \leq (1 - r) \left[ \exp(a) + \exp(b) - 2 \exp \left( \frac{a + b}{2} \right) - \frac{1}{4} (a - b)^2 \right] \cdot \]
We fix \( b > 0 \) and apply the property (1) for previous inequality obtaining:
\[ < r[\exp(A) + \exp(b)1_H - 2 \exp \left( \frac{b}{2} \right) \exp \left( \frac{A}{2} \right) - \frac{1}{4} (A^2 - 2bA + b^21_H)]x, x > \leq \]
\[ \leq [\lambda \exp(A) + (1 - \lambda) \exp(b)1_H - \exp(\lambda A) \exp((1 - \lambda)b) - \frac{\lambda(1 - \lambda)}{2} (A^2 - 2bA + b^21_H)]x, x > \]
\[ \leq (1 - r)[\exp(A) + \exp(b)1_H - 2 \exp \left( \frac{b}{2} \right) \exp \left( \frac{A}{2} \right) - \frac{1}{4} (A^2 - 2bA + b^21_H)]x, x > \]
which is equivalent with the following

\[ r[\langle \exp(A)x, x \rangle + \exp(b) - 2 \exp(\frac{b}{2}) \langle \exp(A)\frac{1}{2}x, x \rangle - \frac{1}{4}(\langle A^2x, x \rangle - 2b < Ax, x > + b^2) \leq \]

\[ \leq \lambda < \exp(A)x, x > + (1 - \lambda) \exp(b) - \exp((1 - \lambda)b) < \exp(\lambda A)x, x > - \frac{\lambda(1 - \lambda)}{2} (\langle A^2x, x \rangle - 2b < Ax, x > + b^2) \leq \]

\[ \leq (1 - r)[\langle \exp(A)x, x \rangle + \exp(b) - 2 \exp(\frac{b}{2}) \langle \exp(A)\frac{1}{2}x, x \rangle - \frac{1}{4}(\langle A^2x, x \rangle - 2b < Ax, x > + b^2)], \]

for any \( x \in H \) with \( ||x|| = 1 \).

If we apply again the property (1) for previous inequality for the variable \( b \), then we have for any \( y \in H \) with \( ||y|| = 1 \) that

\[ r[\langle \exp(A)x, x \rangle + < \exp(B)y, y > - 2 < \exp(\frac{B}{2})y, y > < \exp(A)\frac{1}{2}x, x \rangle - \frac{1}{4}(\langle A^2x, x \rangle - 2 < Ax, x > < By, y > + B^2y, y >)] \leq \]

\[ \leq \lambda < \exp(A)x, x > + (1 - \lambda) < \exp(B)y, y > - < \exp((1 - \lambda)B)y, y > < \exp(\lambda A)x, x > - \frac{\lambda(1 - \lambda)}{2} (\langle A^2x, x \rangle - 2 < Ax, x > < By, y > + B^2y, y >) \leq \]

\[ \leq (1 - r)[< \exp(A)x, x > + < \exp(B)y, y > - 2 < \exp(\frac{B}{2})y, y > < \exp(A)\frac{1}{2}x, x > - \frac{1}{4}(\langle A^2x, x \rangle - 2 < Ax, x > < By, y > + B^2y, y >)], \]

If we take now \( x = y \) in the above inequality we will obtained the desired inequality.

A multiple operator version of Proposition 1 takes place also below:

**Proposition 2.** Assume that \( A_j, j \in \{1, ..., n\} \) are positive operators on the Hilbert space \( H \). If \( 0 \leq \nu \leq 1 \) then for each \( x_j \in H, j \in \{1, ..., n\} \) with \( \sum_{j=1}^{n} ||x_j||^2 = 1 \) we have the inequality

\[ 1 \leq \sum_{j=1}^{n} < A_j^{2(1-\nu)} x_j, x_j > + \sum_{j=1}^{n} < A_j^{2\nu} x_j, x_j > + s_0 \left[ \sum_{j=1}^{n} < A_j^2 x_j, x_j > - \left( \sum_{j=1}^{n} < A_j x_j, x_j > \right)^2 \right] \]

where \( s_0 = \max\{\nu, 1 - \nu\} \).
Proof. As in the case of Theorem 2, see [1], we consider

\[
\overline{A} := \begin{pmatrix}
A_1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & A_n
\end{pmatrix} \quad \text{and} \quad \overline{x} := \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\]

having \( ||\overline{x}|| = 1 \), and \( \overline{A} \) is positive definite. Taking into account that

\[
< \overline{A}^{2\nu} \overline{x}, \overline{x} > = \sum_{j=1}^{n} < f_1(A_j)x_j, x_j > = \sum_{j=1}^{n} < A_j^{2\nu}x_j, x_j >,
\]

\[
< \overline{A}^{2(1-\nu)} \overline{x}, \overline{x} > = \sum_{j=1}^{n} < f_2(A_j)x_j, x_j > = \sum_{j=1}^{n} < A_j^{2(1-\nu)}x_j, x_j >,
\]

\[
< \overline{A} \overline{x}, \overline{x} > = \sum_{j=1}^{n} < f_3(A_j)x_j, x_j > = \sum_{j=1}^{n} < A_jx_j, x_j >,
\]

where, \( f_1, f_2, f_3 : (0, \infty) \to \mathbb{R} \) are defined by \( f_1(x) = x^{2\nu} \), \( f_2(x) = x^{2(1-\nu)} \) and \( f_3(x) = x \) respectively, and applying Remark 1 (ii) for \( \overline{A} \) and \( \overline{x} \) we find the desired inequality.

\[\square\]

Proposition 3. Let \( A \) and \( B \) be two positive definite operators on \( H \). If \( \text{Sp}(A) \subseteq [1, \infty) \), and \( \lambda \in (0, 1) \) then we have

\[
r \left( < Ax, x > + < By, y > - 2 < A^{\frac{1}{2}}x, x > < B^{\frac{1}{2}}y, y > \right) +
\]

\[
+ A_1(\lambda) \left[ < (\log^2 A)x, x > + < (\log^2 B)y, y > - 2 < (\log A)x, x > \right] \leq
\]

\[
\leq \lambda < Ax, x > + (1 - \lambda) < By, y > - 2 < A^{1-\nu}x, y > \leq
\]

\[
\leq (1 - r) \left( < Ax, x > + < By, y > - 2 < A^{\frac{1}{2}}x, x > < B^{\frac{1}{2}}y, y > \right) +
\]

\[
+ B_1(\lambda) \left[ < (\log^2 A)x, x > + < (\log^2 B)y, y > - 2 < (\log A)x, x > \right]
\]

for each \( x, y \in H \) with \( ||x|| = ||y|| = 1 \), where \( r = \min\{\lambda, 1 - \lambda\} \), \( A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{\nu}{4} \) and \( B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{\nu}{4} \).

Proof. We consider the continuous functions \( f(a) = \lambda a + (1 - \lambda)b - a^\lambda b^{1-\lambda} - r(a + b - 2a^\frac{1}{2}b^{\frac{1}{2}} - A_1(\lambda))|\log^2 a + \log^2 b - 2\log a \log b| \) and \( g(a) = (1 - r)(a + b - 2a^\frac{1}{2}b^{\frac{1}{2}}) + B_1(\lambda)|\log^2 a + \log^2 b - 2\log a \log b| - \lambda a - (1 - \lambda)b + a^\lambda b^{1-\lambda} \) which are positive for \( a \geq 1 \) and we fix \( b \geq 1 \) and then by the property (1) for each \( x \in H \) with \( ||x|| = 1 \) we have that

\[
r \left( < Ax, x > + b - 2b^\frac{1}{2} < A^{\frac{1}{2}}x, x > \right) +
\]

\[
+ A_1(\lambda) \left[ < (\log^2 A)x, x > + \log^2 b - 2\log b < (\log A)x, x > \right] \leq
\]

\[
\leq \lambda < Ax, x > + (1 - \lambda)b - b^{1-\lambda} < A^{\frac{1}{2}}x, x > \leq
\]

\[
\leq (1 - r) \left( < Ax, x > + b - 2b^\frac{1}{2} < A^{\frac{1}{2}}x, x > \right) +
\]

\[
+ B_1(\lambda) \left[ < (\log^2 A)x, x > + \log^2 b - 2\log b < (\log A)x, x > \right]
\]

for each \( b > 1 \).
If we apply again the property (1) for last inequality, then for any $y \in H$ with $||y|| = 1$ we get

$$r \left( <Ax, x> + <By, y> - 2 <B^\frac{1}{2}y, y> <A^\frac{1}{2}x, x> \right) +$$

$$A_1(\lambda) \left[ <(\log^2 A)x, x> + <(\log^2 B)y, y> - 2 <(\log B)y, y> <(\log A)x, x> \right] \leq$$

$$\leq \lambda <Ax, x> + (1 - \lambda) <By, y> - <B^{1-\lambda}y, y> <A^\lambda x, x> \leq$$

$$\leq (1 - r) \left( <Ax, x> + <By, y> - 2 <B^\frac{1}{2}y, y> <A^\frac{1}{2}x, x> \right) +$$

$$B_1(\lambda) \left[ <(\log^2 A)x, x> + <(\log^2 B)y, y> - 2 <(\log B)y, y> <(\log A)x, x> \right]$$

for each $x, y \in H$ with $||x|| = ||y|| = 1$.

Next particular case of Proposition 3 may be of interest as well:

**Remark 2.** Under previous conditions, if we consider $y = x$ and $A = B$ then the above inequality becomes:

$$2r \left[ <Ax, x> - \left( <A^\frac{1}{2}x, x> \right)^2 \right] + 2A_1(\lambda) \left[ <(\log^2 A)x, x> - <(\log A)x, x> \right]^2 \leq$$

$$\leq 1 - <A^{1-\lambda}x, x> <A^\lambda x, x> \leq$$

$$2(1 - r) \left[ <Ax, x> - \left( <A^\frac{1}{2}x, x> \right)^2 \right] + 2B_1(\lambda) \left[ <(\log^2 A)x, x> - <(\log A)x, x> \right]^2 \right].$$

**Remark 3.** Assume that $A_j, j \in \{1, \ldots, n\}$ are positive operators on the Hilbert space $H$. If $0 \leq \lambda \leq 1$ then for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^n ||x_j||^2 = 1$ we have the inequality

$$r[2 \sum_{j=1}^n <\exp (A_j)x_j, x_j> - 2 \sum_{j=1}^n <\exp (\frac{A_j}{2})x_j, x_j> +$$

$$- \frac{1}{2} (\sum_{j=1}^n <A_j^2x_j, x_j> - (\sum_{j=1}^n <A_jx_j, x_j>)^2)] \leq$$

$$\leq 1 - \sum_{j=1}^n <\exp (\lambda A_j)x_j, x_j> \sum_{j=1}^n <\exp ((1 - \lambda)A_j)x_j, x_j> -$$

$$- \lambda (1 - \lambda) \left[ \sum_{j=1}^n <A_j^2x_j, x_j> - (\sum_{j=1}^n <A_jx_j, x_j>)^2 \right] \leq$$

$$\leq (1 - r)[2 \sum_{j=1}^n <\exp (A_j)x_j, x_j> - 2 \sum_{j=1}^n <\exp (\frac{A_j}{2})x_j, x_j> +$$

$$- \frac{1}{2} (\sum_{j=1}^n <A_j^2x_j, x_j> - (\sum_{j=1}^n <A_jx_j, x_j>)^2)].$$

where $r = \min\{\lambda, 1 - \lambda\}.$
Proof. The proof will be as in Proposition 2 if we consider the following functions \( f_1, f_2, f_3, f_4, f_5 : (0, \infty) \to \mathbb{R} \) defined by 
\[
    f_1(x) = x^2, \quad f_2(x) = x, \quad f_3(x) = \exp((1 - \lambda)x), \quad f_4(x) = \exp(\lambda x) \quad \text{and} \quad f_5(x) = \exp\left(\frac{x^2}{2}\right)
\]
respectively.

References

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