Some new integral inequalities via variant of Pompeiu’s mean value theorem

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Abstract

The main of this paper is to establish an inequality providing some better bounds for integral mean by using a mean value theorem. Our results are generalization the results of Farooq et. al in [8].

1 Introduction

The inequality of Ostrowski [7] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if \( f : [a, b] \to \mathbb{R} \) is a differentiable function with bounded derivative, then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \frac{(x-a+b)^2}{2 (b-a)^2} \right] (b-a) \| f' \|_{\infty}
\]

for every \( x \in [a, b] \). Moreover the constant \( 1/4 \) is the best possible.

For a differentiable function \( f : [a, b] \to \mathbb{R} \), \( a \cdot b > 0 \), Dragomir has in [2] proved, using Pompeiu’s mean value theorem [5], the following Ostrowski type inequality:

\[
\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq D(x) \| f - \ell f' \|_{\infty}
\]

where \( \ell(t) = t \), \( t \in [a, b] \), and

\[
D(x) = \frac{(b-a)}{|x|} \left[ \frac{1}{4} + \frac{(x-a+b)^2}{2 (b-a)^2} \right].
\]
In [4], Pecaric and Ungar proved a general estimate with the $p$-norm, $1 < p < \infty$, which will for $p = 1$ give the Dragomir [2] result. In [8], for a twice differentiable function $f : [a, b] \to \mathbb{R}$, $a \cdot b > 0$ Farooq et. al gave the following integral inequality:

$$\left|\frac{a + b}{2} \left(\frac{2f(x)}{3x} - \frac{f'(x)}{2}\right) + \frac{1}{3} \left(\frac{bf(b) - af(a)}{b-a}\right) - \frac{1}{b-a} \int_a^b f(t) dt\right| \leq \frac{(b-a)}{3} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right] \left\|2f - 2\ell f' + \ell^2 f''\right\|_\infty$$

where $\ell(t) = t$, $t \in [a, b]$.

The interested reader is also referred to ([2]-[4], [6], [8], [9]) for integral inequalities by using Pompeiu’s mean value theorem.

In this paper, we establish a general form with the $p$-norm, $1 \leq p \leq \infty$, which will give the Farooq et. al result for $p = \infty$. Our results are generalization the results of Farooq et. al in [8].

### 2 Main Results

Before stating the main results, we will give the following lemma proved by Pecaric and Ungar in [4]:

**Lemma 1** For $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, and $0 < a \leq x \leq b$, denote

$$A(x, q) := \left(\int_a^x \left(\int_t^x \frac{t^q du}{u^{2q}}\right) dt\right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_t^x \frac{t^q du}{u^{2q}}\right) dt\right)^{\frac{1}{q}}$$

where for $p = 1$, i.e. $q = \infty$, the integrals are to be interpreted as the $\infty$-norms, i.e. as maxima of the function $(u, t) \mapsto \frac{1}{u^q}$ on the corresponding domains of integration. Then,

$$A(x, q) = \left(\frac{x^{2-q} - x^{-2+q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q} \cdot \frac{1}{q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}$$

$$+ \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q} \cdot \frac{1}{q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}},$$

for $1 < p, q < \infty$, $p, q \neq 2$;

$$A(x, 2) = \frac{1}{3} \left[\left(\ln\left(\frac{x}{a}\right)^3 + \frac{a^3}{x^3} - 1\right)^{\frac{1}{2}} + \left(\ln\left(\frac{x}{b}\right)^3 + \frac{b^3}{x^3} - 1\right)^{\frac{1}{2}}\right] = \lim_{q \to 2} A(x, q);$$
\[ A(x, \infty) = \frac{a^2 + b^2}{2x} + x - a - b = \lim_{q \to \infty} A(x, q); \]

\[ A(x, 1) = \frac{1}{a} + \frac{b}{x^2} = \lim_{q \to 1} A(x, 1). \]

To prove our theorems, we need the following lemma:

**Lemma 2** \( f : [a, b] \to \mathbb{R} \) be continuous function on \([a, b]\) and twice order differentiable function on \((a, b)\) with \(0 < a < b\). Then for any \(t, x \in [a, b]\), we have

\[
\frac{t f(x)-xf(t)+xt f'(t)-f'(x)}{2} = \frac{xt}{2} \int_x^t \left[ 2uf'(u) - 2uf''(u) + u^2 f'''(u) \right] \frac{1}{u^2} du.
\]

**Proof.** Define \( \Psi : \left[ \frac{1}{b}, \frac{1}{a} \right] \to \mathbb{R} \) by \( \Psi(t) := t^2 f \left( \frac{1}{t} \right) \). The function \( \Psi \) is continuously differentiable on \( \left( \frac{1}{b}, \frac{1}{a} \right) \), and for all \( x_1, x_2 \in \left[ \frac{1}{b}, \frac{1}{a} \right] \), we get

\[
\Psi(x_1) - \Psi(x_2) = \int_{x_2}^{x_1} \Psi''(t) dt
\]

\[
= \int_{x_2}^{x_1} \left[ 2tf \left( \frac{1}{t} \right) - 2t f' \left( \frac{1}{t} \right) + \frac{1}{t^3} f'' \left( \frac{1}{t} \right) \right] dt.
\]

Using the change of the variable in last integrals with \( u = \frac{1}{t} \), we get

\[
\Psi(x_1) - \Psi(x_2) = - \int_{\frac{1}{x_1}}^{\frac{1}{x_2}} \left[ 2uf'(u) - 2uf''(u) + u^2 f'''(u) \right] \frac{1}{u^2} du. \tag{2}
\]

Denote \( x_1 = \frac{1}{x} \) and \( x_2 = \frac{1}{t} \). Then for all \( x, t \in [a, b] \) from (2), we have

\[
\frac{2}{x} f(x) - f'(x) - 2 \frac{1}{t} f(t) + f'(t) = \int_x^t \left[ 2uf'(u) - 2uf''(u) + u^2 f'''(u) \right] \frac{1}{u^2} du
\]

which gives (3) and completes the proof. \( \blacksquare \)

**Theorem 3** \( f : [a, b] \to \mathbb{R} \) be continuous function on \([a, b]\) and twice order differentiable function on \((a, b)\) with \(0 < a < b\). Then for \( \frac{1}{p} + \frac{1}{q} = 1 \),
with \(1 \leq p, q \leq \infty\), and all \(x \in [a, b]\), we have

\[
\left| \frac{a+b}{2} \left( \frac{2f(x)}{3x} - \frac{f'(x)}{2} \right) + \frac{1}{3} \left( \frac{bf(b) - af(a)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{(b-a)^{\frac{3}{2}}}{3} \| 2tf - 2tf' + t^2 f'' \|_p A(x, q)
\]

where \(l(t) = t\), \(t \in [a, b]\).

**Proof.** From Lemma 2, we have

\[
tf(x) - xf(t) + xt \frac{f'(t) - f'(x)}{2} = \frac{xt}{2} \int_x^t \left[ 2uf(u) - 2uf'(u) + u^2 f''(u) \right] \frac{1}{u^2} \, du.
\]  

(4)

Integrating with respect to \(t\) on \([a, b]\) and dividing by \(\frac{3x}{2}\), we get

\[
\left( \frac{b^2 - a^2}{2} \right) \left( \frac{2f(x)}{3x} - \frac{f'(x)}{2} \right) + \frac{bf(b) - af(a)}{3} - \int_a^b f(t) \, dt
\]

\[
= \int_a^b \frac{1}{3} \left( \int_x^t \left[ 2uf(u) - 2uf'(u) + u^2 f''(u) \right] \frac{1}{u^2} \, du \right) \, dt
\]

and therefore

\[
\left| \frac{(b^2 - a^2)}{2} \left( \frac{2f(x)}{3x} - \frac{f'(x)}{2} \right) + \frac{bf(b) - af(a)}{3} - \int_a^b f(t) \, dt \right| \leq \int_a^b \left( \int_x^t \left[ 2uf(u) - 2uf'(u) + u^2 f''(u) \right] \frac{t}{3u^2} \, du \right) \, dt
\]

\[
= \int_a^x \left( \int_x^t \left[ 2uf(u) - 2uf'(u) + u^2 f''(u) \right] \frac{t}{3u^2} \, du \right) \, dt
\]

\[
+ \int_x^b \left( \int_x^t \left[ 2uf(u) - 2uf'(u) + u^2 f''(u) \right] \frac{t}{3u^2} \, du \right) \, dt.
\]

4
Firstly, we consider the case $1 < p, q < \infty$. By using Hölder’s inequality, the sum in the last line (5) is

$$
\leq \left( \int_a^b \left( \int_x^t \left| 2uf(u) - 2uf'(u) + u^2f''(u) \right|^p \, du \right) \, dt \right)^{\frac{1}{p}} \left( \int_a^b \left( \int_x^t \frac{t^q du}{u^{2q/3}} \, dt \right)^{\frac{1}{q}} \right) \quad (6)
$$

$$
+ \left( \int_x^b \left( \int_x^t \left| 2uf(u) - 2uf'(u) + u^2f''(u) \right|^p \, du \right) \, dt \right)^{\frac{1}{p}} \left( \int_x^b \left( \int_x^t \frac{t^q du}{u^{2q/3}} \, dt \right)^{\frac{1}{q}} \right)
$$

$$
\leq \frac{1}{3} \left( \int_a^b \left( \int_a^b \left| 2uf(u) - 2uf'(u) + u^2f''(u) \right|^p \, du \right) \, dt \right)^{\frac{1}{p}}
$$

$$
\times \left[ \left( \int_a^b \left( \int_t^a \frac{t^q du}{u^{2q}} \right) \, dt \right)^{\frac{1}{q}} + \left( \int_x^b \left( \int_x^t \frac{t^q du}{u^{2q}} \right) \, dt \right)^{\frac{1}{q}} \right].
$$

The first factor in (6) equals

$$
\left( \int_a^b \left( \int_a^b \left| 2uf(u) - 2uf'(u) + u^2f''(u) \right|^p \, du \right) \, dt \right)^{\frac{1}{p}}
$$

$$
= (b - a)^{\frac{1}{p}} \|2lf - 2Lf' + \ell^2f''\|_p. \quad (7)
$$

and, by Lemma 1, the second factor equals $A(x, q)$. Thus, putting (7) into (5) and dividing $b - a$ gives the required inequality (3). □

**Theorem 4** Let $f : \Delta \to \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t, s) \in \Delta$ with $0 < a < b$, $0 < c < d$, and let $w : \Delta \to \mathbb{R}$ be a nonnegative integrable function. Then for $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \leq p, q \leq \infty$ any $(t, s), (x, y) \in \Delta$, we have

$$
\left| \frac{2f(x) - xf(x)}{2x} \right| \int_a^b tw(t) \, dt - \int_a^b w(t) f(t) \, dt + \frac{1}{2} \int_a^b tw(t) f'(t) \, dt \quad (8)
$$

$$
\leq \frac{(b - a)^{\frac{1}{p}}}{(1 - 2q)^{\frac{1}{q}}} \|2lf - 2Lf' + \ell^2f''\|_p
$$

$$
\times \left[ \left( \int_a^b x^{1-2q} y - t^{1-q} \, w^q \right) ds dt \right]^{\frac{1}{q}} + \left( \int_x^b \left[ t^{1-q} - x^{1-2q} t^q \right] w^q \, ds dt \right)^{\frac{1}{q}}.
$$
Proof. Multiplying (4) by \( \frac{w(t)}{x} \) and integrating with respect to \( t \) on \([a, b]\), we have

\[
\left( \frac{2f(x) - xf(x)}{2x} \right) \int_a^b tw(t) \, dt - \int_a^b w(t) f(t) \, dt + \frac{1}{2} \int_a^b tw(t) f'(t) \, dt
\]

\[
= \frac{1}{2} \int_a^b tw(t) \left( \int_x^t [2uf(u) - 2uf'(u) + u^2 f''(u)] \frac{1}{u^2} \, du \right) \, dt
\]

and as in the proof of Theorem 3, we get

\[
\left| \left( \frac{2f(x) - xf(x)}{2x} \right) \int_a^b tw(t) \, dt - \int_a^b w(t) f(t) \, dt + \frac{1}{2} \int_a^b tw(t) f'(t) \, dt \right|
\]

\[
\leq \frac{1}{2} \int_a^b \int_x^t \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right| \frac{tw(t)}{u^2} \, du \, dt
\]

\[
= \int_a^b \left( \int_x^t \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right| \frac{tw(t)}{u^2} \, du \right) \, dt
\]

\[
+ \int_a^b \left( \int_x^t \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right| \frac{tw(t)}{u^2v^2} \, dv \right) \, dt
\]

\[
\leq \left[ \int_a^b \left( \int_a^c \left( \int_t^s \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right|^p \, du \right) \, ds \right) \, dt \right]^{\frac{1}{p}} \left[ \int_a^b \left( \int_t^b \left| \frac{w^q(t)}{u^2} \right| \, du \right) \, dt \right]^{\frac{1}{q}}
\]

\[
+ \left[ \int_a^b \left( \int_a^c \left( \int_t^s \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right|^p \, du \right) \, ds \right) \, dt \right]^{\frac{1}{p}} \left[ \int_a^b \left( \int_t^b \left| \frac{w^q(t)}{u^2} \right| \, du \right) \, dt \right]^{\frac{1}{q}}
\]

\[
\times \left[ \int_a^b \left( \int_t^b \left| \frac{w^q(t)}{u^2} \right| \, du \right) \, dt \right]^{\frac{1}{p}} + \left[ \int_a^b \left( \int_t^b \left| \frac{w^q(t)}{u^2} \right| \, du \right) \, dt \right]^{\frac{1}{q}}
\]

which gives (8).  ■
References


