Approximation by Perturbed Neural Network Operators

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Abstract
This article deals with the determination of the rate of convergence to the unit of each of three newly introduced here perturbed normalized neural network operators of one hidden layer. These are given through the modulus of continuity of the involved function or its high order derivative and that appears in the right-hand side of the associated Jackson type inequalities. The activation function is very general, especially it can derive from any sigmoid or bell-shaped function. The right hand sides of our convergence inequalities do not depend on the activation function. The sample functionals are of Stancu, Kantorovich and Quadrature types. We give applications for the first derivative of the involved function.

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1 Introduction

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma \left( \langle a_j \cdot x \rangle + b_j \right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental network
models, the activation function is the sigmoidal function of logistic type or other sigmoidal function or bell-shaped function.

It is well known that FNNs are universal approximators. Theoretically, any continuous function defined on a compact set can be approximated to any desired degree of accuracy by increasing the number of hidden neurons. It was proved by Cybenko [14] and Funahashi [16], that any continuous function can be approximated on a compact set with uniform topology by a network of the form $N_n(x)$, using any continuous, sigmoidal activation function. Hornik et al. in [19], have shown that any measurable function can be approached with such a network. Furthermore, these authors proved in [20], that any function of the Sobolev spaces can be approached with all derivatives. A variety of density results on FNN approximations to multivariate functions were later established by many authors using different methods, for more or less general situations: [21] by Leshno et al., [25] by Mhaskar and Micchelli, [11] by Chui and Li, [10] by Chen and Chen, [17] by Hahm and Hong, etc.

Usually these results only give theorems about the existence of an approximation. A related and important problem is that of complexity: determining the number of neurons required to guarantee that all functions belonging to a space can be approximated to the prescribed degree of accuracy $\varepsilon$.

Barron [6] shows that if the function is supposed to satisfy certain conditions expressed in terms of its Fourier transform, and if each of the neurons evaluates a sigmoidal activation function, then at most $O(\varepsilon^{-2})$ neurons are needed to achieve the order of approximation $\varepsilon$. Some other authors have published similar results on the complexity of FNN approximations: Mhaskar and Micchelli [26], Suzuki [29], Maiorov and Meir [22], Makovoz [23], Ferrari and Stengel [15], Xu and Cao [30], Cao et al. [7], etc.

P. Cardaliaguet and G. Euvrard were the first, see [8], to describe precisely and study neural network approximation operators to the unit operator. Namely they proved: be given $f : \mathbb{R} \to \mathbb{R}$ a continuous bounded function and $b$ a centered bell-shaped function, then the functions

$$F_n(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I n^\alpha} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $I := \int_{-\infty}^{\infty} b(t) \, dt$, $0 < \alpha < 1$, converge uniformly on compacta to $f$.

You see above that the weights $\frac{f(k/n)}{I n^\alpha}$ are explicitly known, for the first time shown in [8].

Still the work [8] is qualitative and not quantitative.

The author in [1], [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates, that is quantitative works, by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of
the engaged function or its high order derivative, and producing very tight
Jackson type inequalities. He treats there both the univariate and multivariate
cases. The defining these operators ”bell-shaped” and ”squashing” function are
assumed to be of compact support. Also in [3] he gives the Nth order asymptotic
expansion for the error of weak approximation of these two operators to a special
natural class of smooth functions, see chapters 4-5 there.

Though the work in [1], [2], [3], was quantitative, the rate of convergence
was not precisely determined.

Finally the author in [4], [5], by normalizing his operators he achieved to
determine the exact rates of convergence.

In this article the author continuous and completes his related work, by intro-
ducing three new perturbed neural network operators of Cardaliaguet-Euvrard
type.

The sample coefficients $f \left( \frac{x}{n} \right)$ are replaced by three suitable natural pertur-
bations, what is actually happens in reality of a neural network operation.

The calculation of $f \left( \frac{x}{n} \right)$ at the neurons many times are not calculated as
such, but rather in a distored way.

Next we justify why we take here the activation function to be of compact
support, of course it helps us to conduct our study.

The activation function, same as transfer function or learning rule, is con-
nected and associated to firing of neurons. Firing, which sends electric pulses
or an output signal to other neurons, occurs when the activation level is above
the threshold level set by the learning rule.

Each Neural Network firing is essentially of finite time duration. Essentially
the firing in time decays, but in practice sends positive energy over a finite time
interval.

Thus by using an activation function of compact support, in practice we do
not alter much of the good results of our approximation.

To be more precise, we may take the compact support to be a large symmetric
to the origin interval. This is what happens in real time with the firing of
neurons.

For more about neural networks in general we refer to [9], [12], [13], [18],
[24], [27].

2 Basics

Here the activation function $b : \mathbb{R} \to \mathbb{R}_+$ is of compact support $[-T, T]$, $T > 0$.
That is $b(x) > 0$ for any $x \in [-T, T]$, and clearly $b$ may have jump discontinu-
ties. Also the shape of the graph of $b$ could be anything. Typically in
neural networks approximation we take $b$ as a sigmoidal function or bell-shaped
function, of course here of compact support $[-T, T]$, $T > 0$. 
Example 1 (i) $b$ can be the characteristic function on $[-1, 1]$, (ii) $b$ can be that hat function over $[-1, 1]$, i.e.,

$$b(x) = \begin{cases} 
1 + x, & -1 \leq x \leq 0, \\
1 - x, & 0 < x \leq 1, \\
0, & \text{elsewhere},
\end{cases}$$

(iii) the truncated sigmoidals

$$b(x) = \begin{cases} 
\frac{1}{1+e^{-x}} \text{ or } \tanh x \text{ or } \text{erf}(x), & x \in [-T, T], \text{ with large } T > 0, \\
0, & x \in \mathbb{R} - [-T, T],
\end{cases}$$

(iv) the truncated Gompertz function

$$b(x) = \begin{cases} 
e^{-\alpha e^{-\beta x}}, & x \in [-T, T]; \alpha, \beta > 0; \text{ large } T > 0, \\
0, & x \in \mathbb{R} - [-T, T].
\end{cases}$$

The Gompertz functions are also sigmoidal functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

So the general function $b$ we will be using here covers all kinds of activation functions in neural network approximations.

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

Let here the parameters $\mu, \nu \geq 0; \mu_i, \nu_i \geq 0, i = 1, \ldots, r \in \mathbb{N}; w_i \geq 0$:

$$\sum_{i=1}^{r} w_i = 1; 0 < \alpha < 1, x \in \mathbb{R}, n \in \mathbb{N}.$$

We use here the first modulus of continuity

$$\omega_1(f, \delta) := \sup_{x, y \in \mathbb{R}} |f(x) - f(y)|, \quad |x - y| \leq \delta$$

and given that $f$ is uniformly continuous we get $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$.

In this article mainly we study the pointwise convergence with rates over $\mathbb{R}$, to the unit operator, of the following one hidden layer normalized neural network perturbed operators,

(i)

$$(H_n^*(f))(x) = \frac{\sum_{k=-n^2}^{n^2} \left( \sum_{i=1}^{r} w_i f \left( \frac{k + \mu_i}{n + \nu_i} \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \right)}{\sum_{k=-n^2}^{n^2} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}, (1)$$

(ii) the Kantorovich type

$$(K_n^*(f))(x) = \frac{\sum_{k=-n^2}^{n^2} \left( \sum_{i=1}^{r} w_i (n + \nu_i) \int_{k + \mu_i + 1}^{n + \nu_i} f(t) \, dt \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{\sum_{k=-n^2}^{n^2} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}, (2)$$

4
and

(iii) the quadrature type

\[
(M_{n}^{*}(f))(x) = \frac{\sum_{k=-n}^{n} \left( \sum_{r=1}^{r} w_{r} \left( \frac{k}{n} + \frac{r}{n} \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \right)}{\sum_{k=-n}^{n} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}.
\]  

(3)

Similar operators defined for bell-shaped functions and sample coefficients \( f \left( \frac{k}{n} \right) \) were studied initially in [8], [1], [2], [3], [4], [5], etc.

Here we study the generalized perturbed cases of these operators.

Operator \( K_{n}^{*} \) in the corresponding Signal Processing context, represents the natural called "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of \( k + n \) rather than exactly at the node \( k \).

The perturbed sample coefficients \( f \left( \frac{k + \mu}{n + \nu} \right) \) with \( 0 \leq \mu \leq \nu \), were first used by D. Stancu [28], in a totally different context, generalizing Bernstein operators approximation on \( C \left( [0, 1] \right) \).

The terms in the ratio of sums (1), (2), (3) are nonzero, iff

\[
\left| n^{1-\alpha} \left( x - \frac{k}{n} \right) \right| \leq T, \text{ i.e. } \left| x - \frac{k}{n} \right| \leq \frac{T}{n^{1-\alpha}}
\]

(4)

iff

\[
x - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}.
\]

(5)

In order to have the desired order of the numbers

\[-n^{2} \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^{2},
\]

(6)

it is sufficiently enough to assume that

\[n \geq T + |x|.
\]

(7)

When \( x \in [-T, T] \) it is enough to assume \( n \geq 2T \), which implies (6).

**Proposition 2 ([1])** Let \( a \leq b \), \( a, b \in \mathbb{R} \). Let \( \text{card} \ (k) \ (\geq 0) \) be the maximum number of integers contained in \( [a, b] \). Then

\[
\max \left( 0, (b - a) - 1 \right) \leq \text{card} \ (k) \leq (b - a) + 1.
\]

(8)

**Note 3** We would like to establish a lower bound on \( \text{card} \ (k) \) over the interval \([nx - Tn^{\alpha}, nx + Tn^{\alpha}] \). From Proposition 2 we get that

\[
\text{card} \ (k) \geq \max \left( 2Tn^{\alpha} - 1, 0 \right).
\]

(9)

We obtain \( \text{card} \ (k) \geq 1 \), if

\[2Tn^{\alpha} - 1 \geq 1 \text{ iff } n \geq T^{-\frac{1}{\alpha}}.
\]

(10)
So to have the desired order (6) and \( \text{card}(k) \geq 1 \) over \([nx - Tn^\alpha, nx + Tn^\alpha]\), we need to consider

\[
\frac{n}{n+\nu} - x \leq \frac{k}{n+\nu} - x + \frac{\mu}{n+\nu},
\]

Also notice that \( \text{card}(k) \to +\infty, \) as \( n \to +\infty \).

Denote by \([\cdot]\) the integral part of a number and by \( [\cdot] \) its ceiling.

So under assumption (11), the operators \( H_n^\alpha, K_n^\alpha, M_n^\alpha \), collapse to

\[
(H_n^\alpha(f))(x) = \frac{\sum_{k=[nx - Tn^\alpha]}^{[nx + Tn^\alpha]} \left( \sum_{j=1}^{r} w_i f \left( \frac{k+\mu_j}{n+\nu_i} \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \right)}{\sum_{k=[nx - Tn^\alpha]}^{[nx + Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)},
\]

(ii)

\[
(K_n^\alpha(f))(x) = \frac{\sum_{k=[nx - Tn^\alpha]}^{[nx + Tn^\alpha]} \left( \sum_{i=1}^{r} w_i (n+\nu_i) \int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\nu_i+1}{n+\nu_i}} f(t) \, dt \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{\sum_{k=[nx - Tn^\alpha]}^{[nx + Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)},
\]

and

(iii)

\[
(M_n^\alpha(f))(x) = \frac{\sum_{k=[nx - Tn^\alpha]}^{[nx + Tn^\alpha]} \left( \sum_{i=1}^{r} w_i f \left( \frac{k+i}{n} \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{\sum_{k=[nx - Tn^\alpha]}^{[nx + Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}.
\]

We make

**Remark 4** Let \( k \) as in (5). We observe that

\[
\left| \frac{k+\mu}{n+\nu} - x \right| \leq \left| \frac{k}{n+\nu} - x \right| + \frac{\mu}{n+\nu}.
\]

Next we see

\[
\left| \frac{k}{n+\nu} - x \right| \leq \left| \frac{k}{n+\nu} - \frac{k}{n} \right| + \left| \frac{k}{n} - x \right| \leq \frac{\nu |k|}{n(n+\nu)} + \frac{T}{n^{1-\alpha}}\tag{14}
\]

(by \( |k| \leq \max(|nx - Tn^\alpha|, |nx + Tn^\alpha|) \leq n |x| + Tn^\alpha \))

\[
\leq \left( \frac{\nu}{n+\nu} \right) \left( |x| + \frac{T}{n^{1-\alpha}} \right) + \frac{T}{n^{1-\alpha}}\tag{16}
\]

Consequently it holds

\[
\left| \frac{k+\mu}{n+\nu} - x \right| \leq \left( \frac{\nu}{n+\nu} \right) \left( |x| + \frac{T}{n^{1-\alpha}} \right) + \frac{T}{n^{1-\alpha}} + \frac{\mu}{n+\nu} = \tag{17}
\]
\[ \left( \frac{\nu |x| + \mu}{n + \nu} \right) + \left( 1 + \frac{\nu}{n + \nu} \right) \frac{T}{n^{1-\alpha}}. \]

Hence we obtain

\[ \omega_1 \left( f, \left| \frac{k + \mu}{n + \nu} - x \right| \right)^{(17)} \leq \omega_1 \left( f, \left( \frac{\nu |x| + \mu}{n + \nu} \right) + \left( 1 + \frac{\nu}{n + \nu} \right) \frac{T}{n^{1-\alpha}} \right), \tag{18} \]

where \( \mu, \nu \geq 0, 0 < \alpha < 1, \) so that the dominant speed above is \( \frac{1}{n^{1-\alpha}}. \)

Also, by change of variable method, the operator \( K_n^* \) could be written conveniently as follows:

\[ (K_n^* (f)) (x) = \]

\[ \frac{\sum_{k=0}^{n T n \alpha \left| x \right|}}{\sum_{k=0}^{n T n \alpha \left| x \right|}} \left( \sum_{i=1}^{r} w_i \left( n + \nu_i \right) \int_{0}^{\frac{1}{n T n \alpha \left| x \right|}} f \left( t + \frac{k + \mu_i}{n + \nu_i} \right) dt \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \sum_{k=0}^{n T n \alpha \left| x \right|} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right). \tag{19} \]

\section{Main Results}

We present our first approximation result.

\textbf{Theorem 5} Let \( x \in \mathbb{R}, T > 0 \) and \( n \in \mathbb{N} \) such that \( n \geq \max \left( T + |x|, T^{-\frac{1}{n}} \right). \) Then

\[ |(H_n^* (f)) (x) - f (x)| \leq \sum_{i=1}^{r} w_i \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \]

\[ \leq \max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}. \tag{20} \]

\textbf{Proof.} We notice that

\[ (H_n^* (f)) (x) - f (x) = \frac{\sum_{k=0}^{n T n \alpha \left| x \right|} \left( \sum_{i=1}^{r} w_i f \left( \frac{k + \mu_i}{n + \nu_i} \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) - f (x)}{\sum_{k=0}^{n T n \alpha \left| x \right|} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)} \]

\[ = \sum_{k=0}^{n T n \alpha \left| x \right|} \left( \sum_{i=1}^{r} w_i f \left( \frac{k + \mu_i}{n + \nu_i} \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) - f (x) \sum_{k=0}^{n T n \alpha \left| x \right|} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \sum_{k=0}^{n T n \alpha \left| x \right|} \left( \sum_{i=1}^{r} w_i f \left( \frac{k + \mu_i}{n + \nu_i} \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \sum_{k=0}^{n T n \alpha \left| x \right|} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) = \tag{21} \]

\[ \sum_{k=0}^{n T n \alpha \left| x \right|} \left( \sum_{i=1}^{r} w_i f \left( \frac{k + \mu_i}{n + \nu_i} \right) \right) - f (x) \sum_{k=0}^{n T n \alpha \left| x \right|} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \sum_{k=0}^{n T n \alpha \left| x \right|} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) = \tag{22} \]
\[
\sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} \left( \sum_{i=1}^{r} w_i \left( f \left( \frac{k+\mu_i}{n+v_i} \right) - f \left( x \right) \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)
\]

Hence it holds
\[
\frac{\sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} \left( \sum_{i=1}^{r} w_i \left( f \left( \frac{k+\mu_i}{n+v_i} \right) - f \left( x \right) \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{\sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)} \leq (23)
\]
\[
\frac{\sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} \left( \sum_{i=1}^{r} w_i \omega_1 f \left( \frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{\sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)} \leq (24)
\]
proving the claim. \(\blacksquare\)

**Corollary 6** Let \(x \in [-T^*, T^*] \), \(T^* > 0 \), \(n \in \mathbb{N} : n \geq \max \left( T + T^*, T^{-\frac{1}{\alpha}} \right) \), \(T > 0 \). Then
\[
\|H_n^\alpha(f) - f\|_{\infty, [-T^*, T^*]} \leq \sum_{i=1}^{r} w_i \omega_1 f \left( \frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \leq \max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 f \left( \frac{\nu_i T^* + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right\}.
\]
\[
\leq \max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 f \left( \frac{\nu_i T^* + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right\}.
\]

**Proof.** By (20). \(\blacksquare\)

We continue with
Therefore by (19) and (28) we get
\[
\max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.
\]

Hence it holds
\[
\delta_{n,k} (f) = \sum_{i=1}^{r} w_i (n + \nu_i) \int_{0}^{\frac{x}{n + \nu_i}} f \left( t + \frac{k + \mu_i}{n + \nu_i} \right) dt.
\]

We observe that
\[
\delta_{n,k} (f) - f (x) = \sum_{i=1}^{r} w_i (n + \nu_i) \int_{0}^{\frac{x}{n + \nu_i}} f \left( t + \frac{k + \mu_i}{n + \nu_i} \right) dt - f (x) = \sum_{i=1}^{r} w_i (n + \nu_i) \int_{0}^{\frac{x}{n + \nu_i}} \left( f \left( t + \frac{k + \mu_i}{n + \nu_i} \right) - f (x) \right) dt.
\]

Hence it holds
\[
|\delta_{n,k} (f) - f (x)| \leq \sum_{i=1}^{r} w_i (n + \nu_i) \int_{0}^{\frac{x}{n + \nu_i}} \left| f \left( t + \frac{k + \mu_i}{n + \nu_i} \right) - f (x) \right| dt \leq \sum_{i=1}^{r} w_i (n + \nu_i) \int_{0}^{\frac{x}{n + \nu_i}} \omega_1 \left( f, \left| t \right| + \left| \frac{k + \mu_i}{n + \nu_i} - x \right| \right) dt \leq \sum_{i=1}^{r} w_i \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \leq \max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.
\]

We proved that
\[
|\delta_{n,k} (f) - f (x)| \leq \max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.
\]

Therefore by (19) and (28) we get
\[
(K_n^\ast (f)) (x) - f (x) = \sum_{k=[nx+Tn^\alpha]}^{[nx+Tn^\alpha]} \delta_{n,k} (f) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) - f (x) = \sum_{k=[nx+Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) - f (x).
\]
\[
\sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \delta_{n,k} \left( f \left( (1 - \alpha) \left( x - \frac{k}{n} \right) \right) - f(x) \sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \delta_{n,k} \left( f \left( (1 - \alpha) \left( x - \frac{k}{n} \right) \right) \right) \right)
\]

\[
= \sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \left( \delta_{n,k} \left( f - f(x) \right) \right) \sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \left( b \left( (1 - \alpha) \left( x - \frac{k}{n} \right) \right) \right)
\]

Consequently we obtain

\[
\left\| (K_n^* (f))(x) - f(x) \right\| \leq \frac{\sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \left( \delta_{n,k} \left( f - f(x) \right) \right) \sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \left( b \left( (1 - \alpha) \left( x - \frac{k}{n} \right) \right) \right)}{\sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \left( b \left( (1 - \alpha) \left( x - \frac{k}{n} \right) \right) \right)}
\]

\[
\leq \left( \sum_{k=\lfloor nx - T_n^* \rfloor}^{\lfloor nx + T_n^* \rfloor} \left( b \left( (1 - \alpha) \left( x - \frac{k}{n} \right) \right) \right) \right)^{\frac{1}{2}}
\]

\[
\max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.
\]

proving the claim. ■

**Corollary 8** Let \( x \in [-T^*, T^*] \), \( T^* > 0 \), \( n \in \mathbb{N} \) : \( n \geq \max \left( T + T^*, T^* \right) \), \( T > 0 \). Then

\[
\| K_n^* (f) - f \|_{\infty, [-T^*, T^*]} \leq \max_{i \in \{1, \ldots, r\}} \left\{ \omega_1 \left( f, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.
\]

**Proof.** By (27). ■

We also give

**Theorem 9** Let \( x \in \mathbb{R} \), \( T > 0 \) and \( n \in \mathbb{N} \) such that \( n \geq \max \left( T + |x|, T^* \right) \). Then

\[
\| M_n^* (f)(x) - f(x) \| \leq \omega_1 \left( f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right).
\]

**Proof.** Let \( k \) as in (5). Set

\[
\lambda_{nk} (f) = \sum_{i=1}^{r} w_i f \left( \frac{k}{n} + \frac{i}{nr} \right),
\]

then

\[
\lambda_{nk} (f) - f(x) = \sum_{i=1}^{r} w_i \left( f \left( \frac{k}{n} + \frac{i}{nr} \right) - f(x) \right).
\]

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then

$$|\lambda_{nk}(f) - f(x)| \leq \sum_{i=1}^{r} w_i \left| f \left( \frac{k}{n} + \frac{i}{nr} \right) - f(x) \right| \leq \sum_{i=1}^{r} w_i \omega_1 \left( f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right).$$

(39)

Hence it holds

$$|\lambda_{nk}(f) - f(x)| \leq \omega_1 \left( f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right).$$

(40)

By (14) we can write and use next

$$(M_n^*(f))(x) = \frac{\sum_{k=\lfloor nx + Tn^\alpha \rfloor}^{\lfloor nx + Tn^\alpha \rfloor} \lambda_{nk}(f) b \left( n^{1-\alpha} (x - \frac{k}{n}) \right)}{\sum_{k=\lfloor nx - Tn^\alpha \rfloor}^{\lfloor nx - Tn^\alpha \rfloor} b \left( n^{1-\alpha} (x - \frac{k}{n}) \right)}. \quad (41)$$

That is we have

$$M_n^*(f)(x) - f(x) = \frac{\sum_{k=\lfloor nx + Tn^\alpha \rfloor}^{\lfloor nx + Tn^\alpha \rfloor} \lambda_{nk}(f) - f(x) b \left( n^{1-\alpha} (x - \frac{k}{n}) \right)}{\sum_{k=\lfloor nx - Tn^\alpha \rfloor}^{\lfloor nx - Tn^\alpha \rfloor} b \left( n^{1-\alpha} (x - \frac{k}{n}) \right)}. \quad (42)$$

Hence we easily derive by (40), as before, that

$$|M_n^*(f)(x) - f(x)| \leq \omega_1 \left( f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right), \quad (43)$$

proving the claim.  

**Corollary 10** Let $x \in [-T^*,T^*], T^* > 0, n \in \mathbb{N} : n \geq \max \left( T + T^*, T^{-\frac{1}{\alpha}} \right), T > 0$. Then

$$\|M_n^*(f) - f\|_{\infty,[-T^*,T^*]} \leq \omega_1 \left( f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right). \quad (44)$$

**Proof.** By (37).
Theorem 11  Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max \left( T + |x|, T^{-\frac{1}{2}} \right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then

$$\sum_{j=1}^{N} \left| f^{(j)}(x) \right| \leq \sum_{i=1}^{r} w_i \left[ \left( \nu_i |x| + \mu_i \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^{j} +$$

$$\sum_{i=1}^{r} w_i \omega_i \left( f^{(N)}(\nu_i |x| + \mu_i) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right).$$

Inequality (45) implies the pointwise convergence with rates of $(H_n^*(f))(x)$ to $f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. Let $k$ as in (5). We observe that

$$w_i f \left( \frac{k + \mu_i}{n + \nu_i} \right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} w_i \left( \frac{k + \mu_i}{n + \nu_i} - x \right)^{j} +$$

$$w_i \int_{x}^{\frac{k + \mu_i}{n + \nu_i}} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{(k + \mu_i - t)^{N-1}}{(N-1)!} dt, \quad i = 1, \ldots, r.$$

Call

$$V(x) = \sum_{k=\lfloor nx + Tn^\alpha \rfloor}^{\lfloor nx + Tn^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right).$$

Hence

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left( \sum_{i=1}^{r} w_i \left( \frac{k + \mu_i}{n + \nu_i} - x \right)^{j} \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left( \sum_{i=1}^{r} w_i \left( \frac{k + \mu_i}{n + \nu_i} - x \right)^{j} \right) \frac{b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V(x)} +$$

$$\frac{b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V(x)} \left( \sum_{i=1}^{r} \int_{x}^{\frac{k + \mu_i}{n + \nu_i}} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{(k + \mu_i - t)^{N-1}}{(N-1)!} dt \right).$$

Therefore it holds (see (12))

$$(H_n^*(f))(x) - f(x) =$$

(49)
Furthermore we see

\[ \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \left( \sum_{k=\lfloor n x - T n^\alpha \rfloor}^{\lfloor n x + T n^\alpha \rfloor} \left\{ \left( \sum_{i=1}^{r} w_i \frac{k + \mu_i}{n + \nu_i} - x \right)^j b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \right\} \right) + R, \]

where

\[ R = \frac{\sum_{k=\lfloor n x - T n^\alpha \rfloor}^{\lfloor n x + T n^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V(x)}. \]  

(50)

So that

\[ \left| (H_n^*(f))(x) - f(x) \right| \leq \left( \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left( \sum_{k=\lfloor n x - T n^\alpha \rfloor}^{\lfloor n x + T n^\alpha \rfloor} \left\{ \left( \sum_{i=1}^{r} w_i \frac{k + \mu_i}{n + \nu_i} - x \right)^j b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \right\} \right) + |R| \leq \right. \]

\[ \left. \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left( \sum_{k=\lfloor n x - T n^\alpha \rfloor}^{\lfloor n x + T n^\alpha \rfloor} \left\{ \left( \sum_{i=1}^{r} w_i \left[ \left( \nu_i |x| + \mu_i \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^j \right\} \right) + |R| = \right. \]

\[ \sum_{j=0}^{N} \frac{|f^{(j)}(x)|}{j!} \left( \sum_{i=1}^{r} w_i \left[ \left( \nu_i |x| + \mu_i \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^j \right) + |R|. \]  

(52)

So that thus far we have

\[ \left| (H_n^*(f))(x) - f(x) \right| \leq \left( \sum_{j=0}^{N} \frac{|f^{(j)}(x)|}{j!} \left( \sum_{i=1}^{r} w_i \left[ \left( \nu_i |x| + \mu_i \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^j \right) + |R|. \]  

(54)

Furthermore we see

\[ |R| \leq \frac{\sum_{k=\lfloor n x - T n^\alpha \rfloor}^{\lfloor n x + T n^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V(x)}. \]  

(55)

\[ \left( \sum_{i=1}^{r} w_i \int_{x}^{\frac{k + \mu_i}{n + \nu_i}} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{k + \mu_i}{n + \nu_i} - t \right)^{N-1} (N-1)! dt \right) \leq \]

\[ \frac{\sum_{k=\lfloor n x - T n^\alpha \rfloor}^{\lfloor n x + T n^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V(x)} \gamma, \]  

(56)
where
\[
\gamma := \sum_{i=1}^{r} w_i \left| \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) - f^{(N)}(x) \left| \frac{k+\mu_i - t}{(N-1)!} \right| dt \right|. \tag{57}
\]

(i) Let \( x \leq \frac{k+\mu_i}{n+\nu_i} \), then
\[
\varepsilon_i := \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) - f^{(N)}(x) \left| \frac{k+\mu_i - t}{(N-1)!} \right| dt \leq \tag{58}
\]
\[
\omega_1 \left( f^{(N)}, \left| \frac{k+\mu_i}{n+\nu_i} - x \right| \right) \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} \left( \frac{k+\mu_i - t}{(N-1)!} \right)^{N-1} dt = \tag{17}
\]
\[
\omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left( 1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^{N} \left( \frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left( 1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^{N} \tag{59}
\\
\]
So when \( x \leq \frac{k+\mu_i}{n+\nu_i} \), we got
\[
\varepsilon_i \leq \omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left( 1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^{N} \left( \frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left( 1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^{N} \tag{60}
\\
\]

(ii) Let \( x > \frac{k+\mu_i}{n+\nu_i} \), then
\[
\rho_i := \int_{\frac{k+\mu_i}{n+\nu_i}}^{x} f^{(N)}(t) - f^{(N)}(x) \left| \frac{t - \frac{k+\mu_i}{n+\nu_i}}{(N-1)!} \right| dt \leq \tag{61}
\]
\[
\omega_1 \left( f^{(N)}, \left| \frac{k+\mu_i}{n+\nu_i} - x \right| \right) \left( \frac{t - \frac{k+\mu_i}{n+\nu_i}}{N!} \right)^{N} = \\tag{61}
\]
\[
\omega_1 \left( f^{(N)}, \left| \frac{\nu_i |x| + \mu_i}{n+\nu_i} - x \right| \right) \left( \frac{\nu_i |x| + \mu_i - x}{N!} \right)^{N} \leq \tag{61}
\]
\[
\omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left( 1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^{N} \]
\[
\left( \frac{\nu_i \|x\| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1 - \alpha}} \right)^N \frac{N!}{n^N}. \tag{62}
\]

Hence when \( x > k + \mu_i \), then
\[
\rho_i \leq \omega_1 \left( f^{(N)}, \left( \frac{\nu_i \|x\| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1 - \alpha}} \right)^N \frac{N!}{n^N}. \tag{63}
\]

Notice in (60) and (63) we obtained the same upper bound. Hence it holds
\[
\gamma \leq \sum_{i=1}^{r} w_i \omega_1 \left( f^{(N)}, \left( \frac{\nu_i \|x\| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1 - \alpha}} \right)^N \frac{N!}{n^N} =: E. \tag{64}
\]

Thus
\[
|\mathcal{R}| \leq E, \tag{65}
\]
proving the claim. ■

**Corollary 12** All as in Theorem 11, plus \( f^{(j)}(x) = 0, j = 1, ..., N \). Then
\[
|\mathcal{H}_n^{(i)}(f)(x) - f(x)| \leq \sum_{i=1}^{r} w_i \omega_1 \left( f^{(N)}, \left( \frac{\nu_i \|x\| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1 - \alpha}} \right)^N \frac{N!}{n^N}. \tag{66}
\]

Proof. By (49), (50), (64) and (65). ■

In (66) notice the extremely high speed of convergence \( \frac{1}{n^{(1 - \alpha)(N + 1)}} \).

The uniform convergence with rates follows from

**Corollary 13** Let \( x \in [-T^*, T^*] \), \( T^* > 0 \); \( T > 0 \) and \( n \in \mathbb{N} \) such that \( n \geq \max \left( T + T^*, T^{-\frac{1}{2}} \right) \). Let \( f \in C^N(\mathbb{R}) \), \( N \in \mathbb{N} \), such that \( f^{(N)} \) is uniformly continuous or is continuous and bounded. Then
\[
\|\mathcal{H}_n^{(i)}(f) - f\|_{\infty, [-T^*, T^*]} \leq \sum_{j=1}^{N} \left\{ \sum_{i=1}^{r} w_i \left[ \left( \frac{\nu_i T^* + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1 - \alpha}} \right]^j \right\}. \tag{67}
\]
\[
\sum_{i=1}^{r} w_i \omega_1 \left( f^{(N)} \left( \frac{\nu_i T^* + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right).
\]

\[
\left( \frac{\nu_i T^* + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N.
\]

**Proof.** By (45). \(\blacksquare\)

**Corollary 14** All as in Theorem 11, case of \(N = 1\). It holds

\[
| (H_n^* (f) (x) - f (x) | \leq \sum_{i=1}^{r} w_i \omega_1 \left( f' \left( \frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right).
\]

We continue with

**Theorem 15** Same assumptions as in Theorem 11, with \(0 < \alpha < 1\). Then

\[
| (K_n^* (f) (x) - f (x) | \leq \sum_{j=1}^{N} \frac{|f^{(j)} (x)|}{j!}.
\]

\[
\left( \sum_{i=1}^{r} w_i \left( \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^j \right) +
\]

\[
\sum_{i=1}^{r} w_i \left\{ \omega_1 \left( f^{(N)} \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right. \left( \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N \right\}.
\]

Inequality (69) implies the pointwise convergence with rates of \((K_n^* (f) (x))\) to \(f (x)\), as \(n \to \infty\), at the speed \(\frac{1}{n^{\alpha}}\).

**Proof.** Let \(k\) as in (5). We observe that

\[
f \left( t + \frac{k + \mu_i}{n + \nu_i} \right) = \sum_{j=0}^{N} \frac{f^{(j)} (x)}{j!} \left( t + \frac{k + \mu_i}{n + \nu_i} - x \right)^j +
\]

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\begin{align*}
&\int_{x}^{t+\frac{k+\mu_{i}}{\nu_{i}}+\nu_{i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{(t + k + \mu_{i})^{N-1}}{(N - 1)!} \, dz, \\
&\quad i = 1, \ldots, r.
\end{align*}

That is
\begin{align*}
&\int_{0}^{\frac{t}{\nu_{i}}} f \left( t + \frac{k + \mu_{i}}{n + \nu_{i}} \right) \, dt = \\
&\quad \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \int_{0}^{\frac{t}{\nu_{i}}} \left( t + \frac{k + \mu_{i}}{n + \nu_{i}} - x \right)^{j} \, dt + \\
&\quad \int_{0}^{\frac{1}{\nu_{i}}} \left( \int_{x}^{t+\frac{k+\mu_{i}}{\nu_{i}}+\nu_{i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{(t + k + \mu_{i})^{N-1}}{(N - 1)!} \, dz \right) \, dt,
\end{align*}

\begin{align*}
&\quad i = 1, \ldots, r.
\end{align*}

Furthermore we have
\begin{align*}
&\quad \sum_{i=1}^{r} w_{i} (n + \nu_{i}) \int_{0}^{\frac{t}{\nu_{i}}} f \left( t + \frac{k + \mu_{i}}{n + \nu_{i}} \right) \, dt = \\
&\quad \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left( \sum_{i=1}^{r} w_{i} (n + \nu_{i}) \int_{0}^{\frac{t}{\nu_{i}}} \left( t + \frac{k + \mu_{i}}{n + \nu_{i}} - x \right)^{j} \, dt \right) + \\
&\quad \sum_{i=1}^{r} w_{i} (n + \nu_{i}) \int_{0}^{\frac{1}{\nu_{i}}} \left( \int_{x}^{t+\frac{k+\mu_{i}}{\nu_{i}}+\nu_{i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{(t + k + \mu_{i})^{N-1}}{(N - 1)!} \, dz \right) \, dt.
\end{align*}

Call
\begin{align*}
V(x) = \sum_{k=[nx+Tn^{\alpha}]}^{[nx+Tn^{\alpha}]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right), \\
\quad (72)
\end{align*}

Consequently we get
\begin{align*}
&\quad (K_{n}^{\alpha}(f))(x) = \\
&\quad \frac{\sum_{k=[nx+Tn^{\alpha}]}^{[nx+Tn^{\alpha}]} \left( \sum_{i=1}^{r} w_{i} (n + \nu_{i}) \int_{0}^{\frac{t}{\nu_{i}}} f \left( t + \frac{k + \mu_{i}}{n + \nu_{i}} \right) \, dt \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V(x)} \\
&\quad = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=[nx+Tn^{\alpha}]}^{[nx+Tn^{\alpha}]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \\
&\quad \left( \sum_{i=1}^{r} w_{i} (n + \nu_{i}) \int_{0}^{\frac{t}{\nu_{i}}} \left( t + \frac{k + \mu_{i}}{n + \nu_{i}} - x \right)^{j} \, dt \right) + \\
&\quad \left( \sum_{i=1}^{r} w_{i} (n + \nu_{i}) \int_{0}^{\frac{1}{\nu_{i}}} \left( \int_{x}^{t+\frac{k+\mu_{i}}{\nu_{i}}+\nu_{i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{(t + k + \mu_{i})^{N-1}}{(N - 1)!} \, dz \right) \, dt \right),
\end{align*}

\begin{align*}
&\quad \text{17}
\end{align*}
Above we used

We derive that

Therefore it holds

\[ (K_n^a (f)) (x) - f (x) = \sum_{j=1}^{N} \frac{f^{(j)} (x) \sum_{k=[nx-Tn^a]}^{[nx+Tn^a]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V (x)} \cdot \left( \sum_{i=1}^{r} w_i (n + \nu_i) \right) \cdot \int_{0}^{\frac{1}{n+\nu_i}} \left( t + \frac{k + \mu_i}{n + \nu_i} - x \right)^j \, dt + R, \tag{75} \]

where

\[ R = \frac{\sum_{k=[nx-Tn^a]}^{[nx+Tn^a]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V (x)} \cdot \left( \sum_{i=1}^{r} w_i (n + \nu_i) \right) \cdot \int_{0}^{\frac{1}{n+\nu_i}} \left( f^{(N)} (z) - f^{(N)} (x) \right) \left( t + \frac{k + \mu_i}{n + \nu_i} - z \right)^{N-1} \left( \frac{N-1!}{(N-1)!} \right) \, dz \, dt. \tag{76} \]

We derive that

\[ |(K_n^a (f)) (x) - f (x)| \leq \sum_{j=1}^{N} \frac{f^{(j)} (x) \sum_{k=[nx-Tn^a]}^{[nx+Tn^a]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V (x)} \cdot \left( \sum_{i=1}^{r} w_i (n + \nu_i) \right) \cdot \int_{0}^{\frac{1}{n+\nu_i}} |t + \frac{k + \mu_i}{n + \nu_i} - x|^j \, dt + |R| \leq \tag{77} \]

\[ \sum_{j=1}^{N} \frac{|f^{(j)} (x)| \sum_{k=[nx-Tn^a]}^{[nx+Tn^a]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right)}{V (x)} \cdot \left( \sum_{i=1}^{r} w_i \left( \nu_i \left| x + \frac{\mu_i + 1}{n + \nu_i} \right| + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right)^j \cdot \left( \frac{N-1!}{(N-1)!} \right) \cdot \left( \frac{k + \mu_i}{n + \nu_i} - x \right)^j + |R|. \tag{78} \]

Above we used

\[ |t + \frac{k + \mu_i}{n + \nu_i} - x| \leq \left( \frac{\nu_i \left| x + \frac{\mu_i + 1}{n + \nu_i} \right| + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}}}{n^{1-\alpha}} \right), \tag{79} \]
We have found that
\[ |(K_n^+ (f)) (x) - f (x)| \leq \sum_{j=1}^{N} \frac{|f^{(j)} (x)|}{j!} \sum_{i=1}^{r} w_i \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \]
\[ + |R|. \] (80)

Notice that
\[ \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \to 0, \text{ as } n \to \infty, \ 0 < \alpha < 1. \] (81)

We observe that
\[ |R| \leq \sum_{k=0}^{N} \frac{(n^\alpha - T n^\alpha)}{V(x)} b \left( n^{1-\alpha} \frac{x - \frac{k}{n}}{n^\alpha} \right) \left( \sum_{i=1}^{r} w_i (n + \nu_i) \right) \] (82)
\[ \int_{0}^{n} \left| \int_{x}^{t + \frac{k+\mu_i}{n+\nu_i}} \left| f^{(N)} (z) - f^{(N)} (x) \right| \frac{t + \frac{k+\mu_i}{n+\nu_i} - z}{(N-1)!} dz \right| dt =: (\xi). \]

We distinguish the cases:
(i) if \( t + \frac{k+\mu_i}{n+\nu_i} \geq x \), then
\[ \theta_i := \int_{x}^{t + \frac{k+\mu_i}{n+\nu_i}} \left| f^{(N)} (t) - f^{(N)} (x) \right| \frac{t + \frac{k+\mu_i}{n+\nu_i} - z}{(N-1)!} dz \] (83)
\[ \int_{x}^{t + \frac{k+\mu_i}{n+\nu_i}} \left| f^{(N)} (t) - f^{(N)} (x) \right| \frac{(t + \frac{k+\mu_i}{n+\nu_i} - z)^{N-1}}{(N-1)!} dz \leq \] \[ \omega_1 \left( f^{(N)}, \left| t \right| + \frac{\left| k + \mu_i \right|}{n + \nu_i} - x \right) \left( \frac{\left| t \right| + \frac{\left| k + \mu_i \right|}{n + \nu_i} - x}{N!} \right)^N \leq \] \[ \omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N \] (84)

That is, if \( t + \frac{k+\mu_i}{n+\nu_i} \geq x \), we proved that
\[ \theta_i \leq \omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right). \]
\[ \left( \frac{\mu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \] 

(85)

ii) if \( t + \frac{k + \mu_i}{n + \nu_i} < x \), then

\[
\theta_i := \int_{t + \frac{k + \mu_i}{n + \nu_i}}^{x} \left| f^{(N)}(z) - f^{(N)}(x) \right| \frac{(z - \left( t + \frac{k + \mu_i}{n + \nu_i} \right))^{N-1}}{(N-1)!} dz \leq
\]

\[
\omega_1 \left( f^{(N)}, \left| t + \frac{k + \mu_i}{n + \nu_i} - x \right| + \left| \frac{k + \mu_i}{n + \nu_i} - x \right| \right) \frac{N}{N!} \leq \]

(17)

\[
\omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \frac{N}{N!} \]

(86)

same estimate as in (85).

Therefore we derive (see (82))

\[
(\xi) \leq \sum_{k=\lceil nx - Tn^\alpha \rceil}^{\lceil nx + Tn^\alpha \rceil} b \left( \frac{n^{1-\alpha}(x - \frac{k}{n})}{V(x)} \right),
\]

\[
\left( \sum_{i=1}^{\tau} w_i \left\{ \omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\} \right),
\]

(87)

Clearly we have found the estimate

\[
|\mathcal{R}| \leq \sum_{i=1}^{\tau} w_i \left\{ \omega_1 \left( f^{(N)}, \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left( 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\},
\]

(88)

Based on (80) and (88) we derive (69). ■

**Corollary 16** All as in Theorem 15, plus \( f^{(j)}(x) = 0, j = 1, \ldots, N; 0 < \alpha < 1 \).

Then

\[ |(K_n^\alpha(f))(x) - f(x)| \leq \]
Corollary 18
All as in Theorem 15, case of $\max$.

Proof. By (69). We also present

The uniform convergence with rates follows from

Corollary 17
Let $x \in [-T^*, T^*]$, $T^* > 0$; $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max(T + T^*, T^{-\frac{1}{\alpha}})$, $0 < \alpha < 1$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then

$$\|K_n^*(f) - f\|_{\infty, [-T^*, T^*]} \leq \sum_{j=1}^{N} \frac{\|f^{(j)}\|_{\infty, [-T^*, T^*]}}{j!}.$$  \hspace{1cm} (90)

Proof. By (69).

Corollary 18
All as in Theorem 15, case of $N = 1$. It holds

$$|f'(x)| \left( \sum_{i=1}^{r} w_i \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} + 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \leq \left( \sum_{i=1}^{r} w_i \left( \frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} + 1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right).$$  \hspace{1cm} (91)

Proof. By (69). We also present
Theorem 19 Let all as in Theorem 11. Then

\[ |(M_n^\alpha (f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^j + \]  

(92)

\[ \omega_1 \left( f^{(N)} \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N \frac{N!}{N} . \]

Inequality (92) implies the pointwise convergence with rates of \((M_n^\alpha (f))(x)\) to \(f(x)\), as \(n \to \infty\), at the speed \(\frac{1}{n^{1-\alpha}}\).

Proof. Let \(k\) as in (5). Again by Taylor’s formula we have that

\[ \sum_{i=1}^{r} w_i f \left( \frac{k}{n} + \frac{i}{nr} \right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \sum_{i=1}^{r} w_i \left( \frac{k}{n} + \frac{i}{nr} - x \right)^j + \]  

(93)

\[ \sum_{i=1}^{r} w_i \int_x^{\frac{k}{n} + \frac{i}{nr}} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left( \frac{k}{n} + \frac{i}{nr} - t \right)^{N-1}}{(N-1)!} dt. \]

Call

\[ V(x) = \sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right). \]

Then

\[ (M_n^\alpha (f))(x) = \sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} \left( \sum_{i=1}^{r} w_i f \left( \frac{k}{n} + \frac{i}{nr} \right) \right) b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \frac{V(x)}{V(x)} . \]

(94)

\[ = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \frac{V(x)}{V(x)} . \]

(95)

\[ \sum_{i=1}^{r} w_i \left( \frac{k}{n} + \frac{i}{nr} - x \right)^j + \sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \frac{V(x)}{V(x)} . \]

\[ \sum_{i=1}^{r} \int_x^{\frac{k}{n} + \frac{i}{nr}} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left( \frac{k}{n} + \frac{i}{nr} - t \right)^{N-1}}{(N-1)!} dt. \]

Therefore we get

\[ (M_n^\alpha (f))(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=[nx-Tn^\alpha]}^{[nx+Tn^\alpha]} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \]

\[ \frac{V(x)}{V(x)} . \]

(96)
where
\[ R = \sum_{k=\lfloor nx - T_n^\alpha \rfloor}^{\lfloor nx + T_n^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \frac{V(x)}{V(x)}. \]

\[ \sum_{i=1}^{r} w_i \int_{x}^{x + \frac{k}{n} + \frac{i}{n \tau}} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left( \frac{k}{n} + \frac{i}{n \tau} - t \right)^{N-1}}{(N-1)!} dt. \] (97)

Hence it holds
\[ |(M^*_n(f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=\lfloor nx - T_n^\alpha \rfloor}^{\lfloor nx + T_n^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \frac{V(x)}{V(x)}. \] (98)

\[ \sum_{i=1}^{r} w_i \left[ \left| \frac{k}{n} - x + \frac{i}{n \tau} \right| \right]^{j} + |R| \leq \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left[ \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right]^{j} + |R|. \] (99)

We have proved that
\[ |(M^*_n(f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \left[ \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right]^{j} + |R|. \] (100)

Next we observe it holds
\[ |R| \leq \sum_{k=\lfloor nx - T_n^\alpha \rfloor}^{\lfloor nx + T_n^\alpha \rfloor} b \left( n^{1-\alpha} \left( x - \frac{k}{n} \right) \right) \frac{V(x)}{V(x)}. \] (101)

\[ \sum_{i=1}^{r} w_i \int_{x}^{x + \frac{k}{n} + \frac{i}{n \tau}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left( \frac{k}{n} + \frac{i}{n \tau} - t \right)^{N-1}}{(N-1)!} dt. \]

Call
\[ \varepsilon_i := \int_{x}^{x + \frac{k}{n} + \frac{i}{n \tau}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left( \frac{k}{n} + \frac{i}{n \tau} - t \right)^{N-1}}{(N-1)!} dt. \] (102)

We distinguish the cases:
(i) if \( \frac{k}{n} + \frac{i}{n \tau} \geq x \), then
\[ \varepsilon_i := \int_{x}^{x + \frac{k}{n} + \frac{i}{n \tau}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left( \frac{k}{n} + \frac{i}{n \tau} - t \right)^{N-1}}{(N-1)!} dt \leq \omega_1 \left( f^{(N)}, \frac{k}{n} - x + \frac{1}{n} \right) \frac{\left( \frac{k}{n} + \frac{i}{n \tau} - x \right)^{N}}{N!} \leq \] (103)
\[ \omega_1 \left( f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N. \] (104)

Thus
\[ \varepsilon_i \leq \omega_1 \left( f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N. \] (105)

ii) if \( \frac{k}{n} + \frac{i}{nr} < x \), then
\[
\varepsilon_i := \int_{\frac{k}{n} + \frac{i}{nr}}^{x} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{(t - \left( \frac{k}{n} + \frac{i}{nr} \right))^{N-1}}{(N-1)!} dt \leq \omega_1 \left( f^{(N)}, \left( x - \left( \frac{k}{n} + \frac{i}{nr} \right) \right) \right) \left( x - \left( \frac{k}{n} + \frac{i}{nr} \right) \right)^N \leq \omega_1 \left( f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N. \] (106)

So we obtain again (105).

Clearly now by (101) we derive that
\[ |R| \leq \omega_1 \left( f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N, \] (107)
proving the claim. ■

**Corollary 20** All as in Theorem 19, plus \( f^{(j)}(x) = 0, j = 1, \ldots, N \). Then
\[ |(M^*_n(f))(x) - f(x)| \leq \omega_1 \left( f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N. \] (108)

**Proof.** By (92). ■

In (108) notice the extremely high speed of convergence \( \frac{1}{n^{1-\alpha-n+1}} \).
Uniform convergence estimate follows

**Corollary 21** All here as in Corollary 13. Then
\[ \|M^*_n(f) - f\|_{\infty,[1-T^*, T^*]} \leq \sum_{j=1}^{N} \frac{\|f^{(j)}\|_{\infty,[-T^*, T^*]}}{j!} \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^j \] (109)

\[ + \omega_1 \left( f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right)^N. \]

**Proof.** By (92). ■
Corollary 22 All as in Theorem 19, \( N = 1 \) case. It holds
\[
\| (M_n^a (f)) (x) - f (x) \| \leq \| f' (x) \| \\
+ \omega_1 \left( f', \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \left( \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right).
\] (110)

Proof. By (92). \( \blacksquare \)

Note 23 We also observe that all the right hand sides of convergence inequalities (45), (66), (67), (68), (69), (89), (90), (91), (92), (108), (109), (110), are independent of \( b \).

Note 24 We observe that
\[
H_n^* (1) = K_n^* (1) = M_n^* (1) = 1,
\] (111)
thus unitary operators.

Also, given that \( f \) is bounded, we get
\[
\| H_n^* (f) \|_{\infty, \mathbb{R}} \leq \| f \|_{\infty, \mathbb{R}},
\] (112)
\[
\| K_n^* (f) \|_{\infty, \mathbb{R}} \leq \| f \|_{\infty, \mathbb{R}},
\] (113)
and
\[
\| M_n^* (f) \|_{\infty, \mathbb{R}} \leq \| f \|_{\infty, \mathbb{R}}.
\] (114)

Operators \( H_n^* \), \( K_n^* \), \( M_n^* \) are positive linear operators, and of course bounded operators directly by (112)-(114).

References


