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GENERALIZED HERMITE -HADAMARD TYPE INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE 3RD DERIVATIVES ARE S-CONVEX

MEHMET ZEKI SARIKAYA AND HÜSEYIN BUDAK

Abstract. In this paper, we have established Hermite-Hadamard type inequalities for functions whose 3rd derivatives are s-convex depending on a parameter. These results have generalized some relationships with [4].

1. Introduction

Definition 1. *The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[6],[10, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1.1}$$

We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 6, 7, 10]) and the references cited therein.

Definition 2. [3] *Let s be a real numbers, $s \in (0, 1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if f*

$$f(\alpha x + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.

An $s-$convex function was introduced in Breckner's paper [3] and a number of properties and connections with $s-$convexity in the first sense are discussed in paper [8]. Of course, $s-$convexity means just convexity when $s = 1$.

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Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on I° with $a, b \in I$ and $a < b$. If $f''' \in L[a, b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]$$

$$= \frac{(b-a)^3}{12}\int_0^1 t(1-t)(2t-1) f'''[tb+(1-t)a]\,dt. \tag{1.2}$$

In [4], *Chun and Qi establish the following inequalities:*

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a three times differentiable mapping on (a, b) with $0 \le a < b$. If $|f'''|^q$ is s-convex on $[a, b]$ for same fixed $s \in (0, 1]$ and $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right|$$

$$\le \frac{(b-a)^3}{192}\left(\frac{2^{2-s}\left(6+s+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\left[|f'''(a)|^q + |f'''b|^q\right]\right)^{\frac{1}{q}}. \tag{1.3}$$

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a three times differentiable mapping on (a, b) with $0 \le a < b$. If $|f'''|^q$ is s-convex on $[a, b]$ for same fixed $s \in (0, 1]$ and $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right|$$

$$\le \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{1-s}\left(s2^s+1\right)}{(s+1)(s+2)}\left[|f'''(a)|^q + |f'''b|^q\right]\right)^{\frac{1}{q}} \tag{1.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a three times differentiable mapping on (a, b) with $0 \le a < b$. If $|f'''|^q$ is s-convex on $[a, b]$ for same fixed $s \in (0, 1]$ and $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}\left[f'(b) - f'(a)\right]\right|$$

$$\le \frac{(b-a)^3}{24}\left(\frac{1}{(p+1)(p+3)}\right)^{\frac{1}{p}}\left(\frac{2}{(s+2)(s+3)}\left[|f'''(a)|^q + |f'''b|^q\right]\right)^{\frac{1}{q}} \tag{1.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For more information and recent developments on this topic, please refer to [4, 5, 9, 11, 12].

The aim of this paper is to establish generalized Hermite-Hadamard's inequalities for function whose 3rd derivatives in absolute value at certain powers are s-convex functions and these results have generalized some relationships with [4].

2. Main Results

We give a important identity for three times differentiable convex functions:

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a three times differentiable mapping on (a,b) with $a<b$. If $f'''\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}(2.1)\quad & \frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right]\\ & -\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\\ = & \frac{(1-2\lambda)^4(b-a)^3}{12}\int_0^1 t(1-t)(2t-1)f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 t(1-t)(2t-1)f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\ &= -2\int_0^1 t^3 f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\ &\quad +3\int_0^1 t^2 f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\ &\quad -\int_0^1 t f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\ &= -2I_1+3I_2-I_3.\end{aligned}$$

Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 t^3 f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\\ &= \frac{f''(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}-\frac{3f'(\lambda a+(1-\lambda)b)}{(1-2\lambda)^2(b-a)^2}\\ &\quad +\frac{6f(\lambda a+(1-\lambda)b)}{(1-2\lambda)^3(b-a)^3}-\frac{6}{(1-2\lambda)^4(b-a)^4}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx,\end{aligned}$$

similarly,

$$\begin{aligned}
I_2 &= \int_0^1 t^2 f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= \frac{f''(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)} - \frac{2f'(\lambda a+(1-\lambda)b)}{(1-2\lambda)^2(b-a)^2} \\
&\quad + \frac{2f(\lambda a+(1-\lambda)b)}{(1-2\lambda)^3(b-a)^3} - \frac{2f(\lambda b+(1-\lambda)a)}{(1-2\lambda)^3(b-a)^3}
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_0^1 t f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= \frac{f''(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)} - \frac{f'(\lambda a+(1-\lambda)b)}{(1-2\lambda)^2(b-a)^2} + \frac{f'(\lambda b+(1-\lambda)a)}{(1-2\lambda)^2(b-a)^2}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
I &= \int_0^1 t(1-t)(2t-1) f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= \frac{f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)}{(1-2\lambda)^2(b-a)^2} \\
&\quad - \frac{6\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^3(b-a)^3} + \frac{12}{(1-2\lambda)^4(b-a)^4}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx.
\end{aligned}$$

This completes the proof. □

Remark 1. *If we take $\lambda = 1$ or $\lambda = 0$ in Lemma 2, then the identity (2.1) reduces the identity (1.2) which is proved in* [4].

Now, we state the main results:

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a three times differentiable mapping on (a,b) with $a < b$. If $|f'''|^q$ is s-convex on $[a,b]$ for same fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right] \right. \\
&\left. -\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right] + \frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx \right| \\
&\leq \frac{(1-2\lambda)^4(b-a)^3}{192}\left(\frac{2^{2-s}\left(6+s+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\right)^{\frac{1}{q}} \\
&\times \left(|f'''(\lambda a+(1-\lambda)b)|^q + |f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}} \qquad (2.2)
\end{aligned}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$.

Proof. Using Lemma 2, s-convexity of $|f'''|^q$ and well-known Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right]\right. \\
&\left.-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} f(x)dx\right| \\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\int_0^1 t(1-t)|2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt \\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\int_0^1 t(1-t)|2t-1|\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 t(1-t)|2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(|f'''(\lambda a+(1-\lambda)b)|^q\int_0^1 t^{s+1}(1-t)|2t-1|\,dt\right. \\
&\left.+|f'''(\lambda b+(1-\lambda)a)|^q\int_0^1 t(1-t)^{s+1}|2t-1|\,dt\right)^{\frac{1}{q}}. \\
=\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{6+s+2^{s+2}s}{(s+2)(s+3)(s+4)}\left[|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right]\right)^{\frac{1}{q}} \\
=\ &\frac{(1-2\lambda)^4(b-a)^3}{192}\left(\frac{2^{2-s}\left(6+s+2^{s+2}s\right)}{(s+2)(s+3)(s+4)}\right)^{\frac{1}{q}} \\
&\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

The proof of Theorem 4 is comleted. □

Remark 2. *If we take* $\lambda=1$ *or* $\lambda=0$ *in Theorem 4, then the inequality (2.2) reduces the inequality (1.3) which is proved in* [4]

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be a three times differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'''|^q$ *is* s*-convex on* $[a,b]$ *for same fixed* $s\in(0,1]$ *and* $q>1$, *then* *the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right] \\
&-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\Bigg| \\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2^{1-s}(s2^s+1)}{(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}} \qquad (2.3)
\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2, s-convexity of $|f'''|^q$ and well-known Hölder's inequality, we obtain

$$\begin{aligned}
&\Bigg|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right] \\
&-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\Bigg| \\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\int_0^1 t(1-t)|2t-1|\left|f'''[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]\right|dt \\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\int_0^1 t^p(1-t)^p|2t-1|\,dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1|2t-1|\left|f'''[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)]\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(1-2\lambda)^4 (b-a)^3}{12} \left(\frac{1}{2^{2p+1}(p+1)} \right)^{\frac{1}{p}} \\
&\times \left(|f'''(\lambda a + (1-\lambda) b)|^q \int_0^1 |2t-1| t^s dt + |f'''(\lambda b + (1-\lambda) a)|^q \int_0^1 |2t-1| (1-t)^s dt \right)^{\frac{1}{q}} \\
&\leq \frac{(1-2\lambda)^4 (b-a)^3}{12} \left(\frac{1}{2^{2p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{s2^s+1}{2^s(s+1)(s+2)} \right)^{\frac{1}{q}} \\
&\times \left(|f'''(\lambda a + (1-\lambda) b)|^q + |f'''(\lambda b + (1-\lambda) a)|^q \right)^{\frac{1}{q}} \\
&\leq \frac{(1-2\lambda)^4 (b-a)^3}{96} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2^{1-s}(s2^s+1)}{(s+1)(s+2)} \right)^{\frac{1}{q}} \\
&\times \left(|f'''(\lambda a + (1-\lambda) b)|^q + |f'''(\lambda b + (1-\lambda) a)|^q \right)^{\frac{1}{q}}
\end{aligned}
$$

which is the inequality (2.3). □

Remark 3. *If we choose $\lambda = 0$ or $\lambda = 1$ in Theorem 5, then the inequality (2.3) reduces the inequality (1.4).*

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be a three times differentiable mapping on (a, b) with $a < b$. If $|f'''|^q$ is s-convex on $[a, b]$ for same fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:*

$$
\begin{aligned}
&\Bigg| \frac{(1-2\lambda)^2 (b-a)}{12} \left[f'(\lambda a + (1-\lambda) b) - f'(\lambda b + (1-\lambda) a) \right] \\
&- \frac{(1-2\lambda)}{2} \left[f(\lambda a + (1-\lambda) b) + f(\lambda b + (1-\lambda) a) \right] + \frac{1}{b-a} \int_{\lambda b + (1-\lambda) a}^{\lambda a + (1-\lambda) b} f(x) dx \Bigg| \\
&\leq \frac{(1-2\lambda)^4 (b-a)^3}{24} \left(\frac{1}{(p+1)(p+3)} \right)^{\frac{1}{p}} \left(\frac{2}{(s+2)(s+3)} \right)^{\frac{1}{q}} \\
&\times \left(|f'''(\lambda a + (1-\lambda) b)|^q + |f'''(\lambda b + (1-\lambda) a)|^q \right)^{\frac{1}{q}}. \qquad (2.4)
\end{aligned}
$$

where $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2, s-convexity of $|f'''|^q$ and well-known Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{(1-2\lambda)^2(b-a)}{12}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right]\right.\\
&\left.-\frac{(1-2\lambda)}{2}\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]+\frac{1}{b-a}\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b}f(x)dx\right|\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\int_0^1 t(1-t)|2t-1|\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\int_0^1 t(1-t)|2t-1|^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 t(1-t)\left|f'''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{2(p+1)(p+3)}\right)^{\frac{1}{p}}\\
&\times\left(|f'''(\lambda a+(1-\lambda)b)|^q\int_0^1 t^{s+1}(1-t)dt+|f'''(\lambda b+(1-\lambda)a)|^q\int_0^1 t(1-t)^{s+1}dt\right)^{\frac{1}{q}}\\
=\ &\frac{(1-2\lambda)^4(b-a)^3}{12}\left(\frac{1}{2(p+1)(p+3)}\right)^{\frac{1}{p}}\left(\frac{1}{(s+2)(s+3)}\right)^{\frac{1}{q}}\\
&\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}\\
=\ &\frac{(1-2\lambda)^4(b-a)^3}{24}\left(\frac{1}{(p+1)(p+3)}\right)^{\frac{1}{p}}\left(\frac{2}{(s+2)(s+3)}\right)^{\frac{1}{q}}\\
&\times\left(|f'''(\lambda a+(1-\lambda)b)|^q+|f'''(\lambda b+(1-\lambda)a)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 4. *If we take $\lambda = 0$ or $\lambda = 1$ in Theorem 6, then the inequality (2.4) becomes the inequality (1.5).*

References

[1] A.G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev. Colombiana Math., 28 (1994), 7-12.

[2] M. K. Bakula and J. Pečarić, *Note on some Hadamard-type inequalities,* Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.

[3] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen,* Pupl. Inst. Math. 23(1978), 13-20.

[4] L. Chun and F. Qi, *Integral Inequalities of Hermite-Hadamard Type for Functions Whose 3rd Derivatives Are s-Convex,* Applied Mathematics, 2012, 3, 1680-1685.http://dx.doi.org/10.4236/am.2012.311232.

[5] L. Chun and F. Qi, *Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex,* Journal of Inequalities and Applications 2013, 2013:451 doi:10.1186/1029-242X-2013-451.

[6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[7] S. S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., 11(5) (1998), 91-95.

[8] H. Hudzik and L. Maligranda, *Some remarks on* $s-$convex functions, Aequationes Math. 48 (1994), 100-111.

[9] J Park, *Hermite-Hadamard Type Inequalities for Functions whose Third Derivatives are Convex and s-Convex,* Applied Mathematical Sciences, Vol. 8, 2014, no. 1, 13 - 31

[10] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[11] S. Qaisar, S. Hussain and C. He, *On new inequalities of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex with applications*, Journal of the Egyptian Mathematical Society (2014) 22, 19–22.

[12] B-Y Xi, S-H Wang, and F Qi, *Some Inequalities of Hermite-Hadamard Type for Functions Whose 3rd Derivatives Are P-Convex,* Applied Mathematics, 2012, 3, 1898-1902.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: sarikayamz@gmail.com

E-mail address: hsyn.budak@gmail.com