A Note on Numerical Radius and the Kreĭn-Lin Inequality

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Abstract

In this note we show that the Kreĭn-Lin triangle inequality can be naturally applied to obtain an elegant reverse for a classical numerical radius power inequality for bounded linear operators on complex Hilbert space due to C. Pearcy.

1 Introduction

Let \((H, \langle \cdot, \cdot \rangle)\) be a complex inner product space and \(x, y \in H\) two nonzero vectors. One can define the 
angle between the vectors \(x, y\) either by the standard formula
\[
\cos \Phi_{x,y} = \frac{\text{Re} \langle x, y \rangle}{\|x\| \|y\|}
\]
or by
\[
\cos \Psi_{x,y} = \frac{|\langle x, y \rangle|}{\|x\| \|y\|}.
\]
The function \(\Psi_{x,y}\) is a natural metric on complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors
\[
\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{x,z}
\]
for any \(x, y, z \in H \setminus \{0\}\).
By using the representation
\[ \Psi_{x,y} = \inf_{\alpha, \beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x, \beta y} \]
and Krein’s inequality (1), M. Lin [6] has shown recently that the following triangle inequality is also valid
\[ \Psi_{x,y} \leq \Psi_{x,z} + \Psi_{y,z} \]
for any \( x, y, z \in H \setminus \{0\} \).

In this note we show that the Krein-Lin triangle inequality (3) can be naturally applied to obtain an elegant reverse for a classical numerical radius power inequality for bounded linear operators on complex Hilbert space due to C. Pearcy [7].

2 A Reverse Inequality

Let \((H; \langle \cdot, \cdot \rangle)\) be a complex Hilbert space. The numerical range of an operator \(T\) is the subset of the complex numbers \(\mathbb{C}\) given by [3, p. 1]:
\[ W(T) = \{ \langle Tx, x \rangle, \ x \in H, \ |x| = 1 \}. \]

The numerical radius \(w(T)\) of an operator \(T\) on \(H\) is given by [3, p. 8]:
\[ w(T) = \sup \{|\lambda|, \lambda \in W(T)\} = \sup \{|\langle Tx, x \rangle|, \ |x| = 1 \}. \]

It is well known that \(w(\cdot)\) is a norm on the Banach algebra \(B(H)\) of all bounded linear operators \(T: H \to H\). This norm is equivalent to the operator norm. In fact, the following more precise result holds [3, p. 9]:
\[ w(T) \leq ||T|| \leq 2w(T), \]
for any \(T \in B(H)\).

For other results on numerical radii, see [4, Chapter 11], [3] and the recent monograph [2].

The following result is well known in the literature [7]:
\[ w(T^n) \leq w^n(T) \]
for each positive integer \(n\) and any operator \(T \in B(H)\).

The following elegant reverse inequalities for \(n = 2\) can be derived from the Krein-Lin triangle inequality (3).

**Theorem 1** For any \(T \in B(H)\) we have
\[ w^2(T) \leq w(T^2) + \inf_{\lambda \in \mathbb{C}} ||T - \lambda I||^2. \]
Proof. The inequality (3) is equivalent to
\[
\cos \Psi_{x,y} \geq \cos (\Psi_{x,z} + \Psi_{y,z}) = \cos \Psi_{x,z} \cos \Psi_{y,z} - \sin \Psi_{x,z} \sin \Psi_{y,z},
\]
or to
\[
\frac{|\langle x, y \rangle|}{\|x\| \|y\|} + \sqrt{1 - \frac{|\langle x, z \rangle|^2}{\|x\|^2 \|z\|^2}} \sqrt{1 - \frac{|\langle y, z \rangle|^2}{\|y\|^2 \|z\|^2}} \geq \frac{|\langle x, z \rangle|}{\|x\| \|z\|} \frac{|\langle y, z \rangle|}{\|y\| \|z\|}
\]
for any \( x, y, z \in H \setminus \{0\} \).

If we multiply (10) by \( \|x\| \|z\|^2 \|y\| > 0 \) then we get
\[
|\langle x, y \rangle| \|z\|^2 + \sqrt{\|x\|^2 \|z\|^2 - |\langle x, z \rangle|^2} \sqrt{\|y\|^2 \|z\|^2 - |\langle y, z \rangle|^2} \\
\geq |\langle x, z \rangle| |\langle y, z \rangle|.
\]
We notice that the inequality (10) remains true, becoming equality, if either \( x = 0 \) or \( y = 0 \) or \( z = 0 \).

We know that for any \( u, e \in H \) with \( \|e\| = 1 \) we have the representation (see for instance [1, Lemma 2.4])
\[
\|u\|^2 - |\langle u, e \rangle|^2 = \|u - \langle u, e \rangle e\|^2 = \inf_{\lambda \in \mathbb{C}} \|u - \lambda e\|^2.
\]
Then by (10) we have for any \( x, y, z \in H \) with \( \|z\| = 1 \) that
\[
|\langle x, y \rangle| + \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| \inf_{\mu \in \mathbb{C}} \|y - \mu z\| \geq |\langle x, z \rangle| |\langle y, z \rangle|.
\]
By taking \( x = Tz \) and \( y = T^*z \) in (11) we get
\[
|\langle Tz, z \rangle| |\langle T^*z, z \rangle| \leq |\langle Tz, T^*z \rangle| + \inf_{\lambda \in \mathbb{C}} \|Tz - \lambda z\| \inf_{\mu \in \mathbb{C}} \|T^*z - \mu z\| \\
\leq |\langle Tz, T^*z \rangle| + \|Tz - \lambda z\| \|T^*z - \mu z\|
\]
for any \( z \in H \) with \( \|z\| = 1 \) and \( \lambda, \mu \in \mathbb{C} \).

Therefore
\[
|\langle Tz, z \rangle|^2 \leq |\langle T^2z, z \rangle| + \|Tz - \lambda z\| \|T^*z - \mu z\|
\]
for any \( z \in H \) with \( \|z\| = 1 \) and \( \lambda, \mu \in \mathbb{C} \).

By taking the supremum over \( z \in H \) with \( \|z\| = 1 \), we deduce
\[
w^2(T) \leq w(T^2) + \|T - \lambda I\| \|T^* - \mu I\|
\]
for any \( \lambda, \mu \in \mathbb{C} \).

Finally, by taking the infimum in (12) over \( \lambda, \mu \in \mathbb{C} \) and since
\[
\inf_{\mu \in \mathbb{C}} \|T^* - \mu I\| = \inf_{\mu \in \mathbb{C}} \|T - \bar{\mu} I\| = \inf_{\lambda \in \mathbb{C}} \|T - \lambda I\|
\]
we deduce the desired result (7).

Corollary 2 Let \( T \in B(H) \). If there exist \( \omega \in \mathbb{C} \) and \( r > 0 \) such that \( \|T - \omega I\| \leq r \), then
\[
w^2(T) \leq w(T^2) + r^2.
\]
References

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