A Note on the Wang-Zhang and Schwarz Inequalities

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Abstract
In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

1 Introduction
Let \((H, \langle \cdot, \cdot \rangle)\) be a complex inner product space and \(x, y \in H\) two nonzero vectors. One can define the \textit{angle} between the vectors \(x, y\) either by
\[
\cos \Phi_{x,y} = \frac{\text{Re} \langle x, y \rangle}{\|x\| \|y\|}
\] or by
\[
\cos \Psi_{x,y} = \frac{|\langle x, y \rangle|}{\|x\| \|y\|}.
\]
The function \(\Psi_{x,y}\) is a natural metric on complex projective space \([6]\).

In 1969 M. K. Kreğin \([5]\) obtained the following inequality for angles between two vectors
\[
\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{z,y}
\]
for any \(x, y, z \in H \setminus \{0\}\).

By using the representation
\[
\Psi_{x,y} = \inf_{\alpha, \beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x, \beta y}
\]

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and Kreǐn’s inequality (1), M. Lin [6] has shown recently that the following triangle inequality is also valid

\[ \Psi_{x,y} \leq \Psi_{x,z} + \Psi_{y,z} \]  

(3)

for any \( x, y, z \in H \setminus \{0\} \).

The following inequality has been obtained by Wang and Zhang in [9] (see also [11, p. 195])

\[ \sqrt{1 - \frac{|\langle x, y \rangle|^2}{\|x\|^2 \|y\|^2}} \leq \sqrt{1 - \frac{|\langle x, z \rangle|^2}{\|x\|^2 \|z\|^2}} + \sqrt{1 - \frac{|\langle y, z \rangle|^2}{\|y\|^2 \|z\|^2}} \]  

(4)

for any \( x, y, z \in H \setminus \{0\} \). Using the above notations it can be written as [6]

\[ \sin \Psi_{x,y} \leq \sin \Psi_{x,z} + \sin \Psi_{y,z} \]  

(5)

for any \( x, y, z \in H \setminus \{0\} \). It also provides another triangle type inequality complementing the Kreǐn and Lin inequalities above.

In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

2 Reverse of Schwarz Inequality

In the sequel we assume that \((H, \langle \cdot, \cdot \rangle)\) is a complex inner product space. The inequality

\[ |\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2 \]  

for \( x, y \in H \) is well know in the literature as the **Schwarz inequality**. The equality holds in (6) if \( x \) and \( y \) are linearly dependent.

**Theorem 1** Let \( x, y, z \in H \) with \( \|z\| = 1 \) and \( \alpha, \beta \in \mathbb{C}, r, s > 0 \) such that

\[ \|x - \alpha z\| \leq r \text{ and } \|y - \beta z\| \leq s. \]  

(7)

Then

\[ (0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq (r \|y\| + s \|x\|)^2. \]  

(8)

**Proof.** If we multiply (4) by \( \|x\| \|y\| \|z\| > 0 \), then we get

\[ \|z\| \sqrt{\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2} \leq \|y\| \sqrt{\|x\|^2 \|z\|^2 - |\langle x, z \rangle|^2} + \|x\| \sqrt{\|y\|^2 \|z\|^2 - |\langle y, z \rangle|^2} \]  

(9)

for any \( x, y, z \in H \setminus \{0\} \).

We observe that, if either \( x = 0 \) or \( y = 0 \), then the inequality (9) reduces to an equality.
Let \( z \in H \) with \( \|z\| = 1 \), and since (see for instance [2, Lemma 2.4])

\[
\|x\|^2 - |\langle x, z \rangle|^2 = \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\|^2 \quad \text{and} \quad \|y\|^2 - |\langle y, z \rangle|^2 = \inf_{\mu \in \mathbb{C}} \|y - \mu z\|^2
\]

then by (9) we have

\[
\sqrt{\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2} \leq \|y\| \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| + \|x\| \inf_{\mu \in \mathbb{C}} \|y - \mu z\|, \quad (10)
\]

for any \( x, y, z \in H \) with \( \|z\| = 1 \).

Since, by (7)

\[
\inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| \leq \|x - \alpha z\| \leq r \quad \text{and} \quad \inf_{\mu \in \mathbb{C}} \|y - \mu z\| \leq \|y - \beta z\| \leq s,
\]

then by (10) we obtain the desired result (8). \( \blacksquare \)

**Corollary 2.** Let \( x, y, z \in H \) with \( \|z\| = 1 \) and \( \lambda, \Lambda, \gamma, \Gamma \in \mathbb{C} \) with \( \lambda \neq \Lambda, \gamma \neq \Gamma \) and such that either

\[
\text{Re} \left( \langle \Lambda z - x, x - \lambda z \rangle \right) \geq 0 \quad \text{and} \quad \text{Re} \left( \langle \Gamma z - y, y - \gamma z \rangle \right) \geq 0 \quad (11)
\]

or, equivalently

\[
\left\| x - \frac{\lambda + \Lambda}{2} z \right\| \leq \frac{1}{2} |\Lambda - \lambda| \quad \text{and} \quad \left\| y - \frac{\gamma + \Gamma}{2} z \right\| \leq \frac{1}{2} |\Gamma - \gamma|
\]

are valid. Then

\[
(0 \leq) \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} (|\Lambda - \lambda| \|y\| + |\Gamma - \gamma| \|x\|)^2. \quad (12)
\]

**Proof.** Follows by Theorem 1 on observing that

\[
\text{Re} \left( \Delta e - u, u - \delta e \right) = \frac{1}{4} |\Delta - \delta|^2 - \left\| u - \frac{\delta + \Delta}{2} e \right\|^2
\]

for any \( \delta, \Delta \in \mathbb{C} \) with \( \delta \neq \Delta \) and \( u, e \in H \) with \( \|e\| = 1 \). \( \blacksquare \)

We give an example for \( n \)-tuples of complex numbers.

Let \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) and \( z = (z_1, \ldots, z_n) \) be \( n \)-tuples of complex numbers, \( p = (p_1, \ldots, p_n) \) a probability distribution, i.e. \( p_i > 0 \) \( i \in \{1, \ldots, n\} \) and \( \sum_{i=1}^n p_i = 1 \), with \( \sum_{i=1}^n p_i |z_i|^2 = 1 \) and \( \lambda, \Lambda, \gamma, \Gamma \in \mathbb{C} \) with \( \lambda \neq \Lambda, \gamma \neq \Gamma \) and such that

\[
\text{Re} \left[ \langle \Lambda z_i - x_i, (x_i - \lambda z_i) \rangle \right] \geq 0 \quad \text{and} \quad \text{Re} \left[ \langle \Gamma z_i - y_i, (y_i - \gamma z_i) \rangle \right] \geq 0
\]

or, equivalently

\[
\left| x_i - \frac{\lambda + \Lambda}{2} z_i \right| \leq \frac{1}{2} |\Lambda - \lambda| \quad \text{and} \quad \left| y_i - \frac{\gamma + \Gamma}{2} z_i \right| \leq \frac{1}{2} |\Gamma - \gamma|
\]

are valid. Then

\[
(0 \leq) \sum_{i=1}^n p_i \|x_i\|^2 \|y_i\|^2 - \sum_{i=1}^n p_i |\langle x_i, y_i \rangle|^2 \leq \frac{1}{4} \sum_{i=1}^n p_i (|\Lambda - \lambda| \|y_i\| + |\Gamma - \gamma| \|x_i\|)^2.
\]
for any $i \in \{1, ..., n\}$. Then
\[
\sum_{i=1}^{n} p_i \Re \left( (Az_i - x_i) \left( x_i - \overline{x_i} \right) \right) \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i \Re \left( (\Gamma z_i - \overline{y_i}) \left( y_i - \overline{y_i} \right) \right) \geq 0
\]
and by applying Corollary 2 for the inner product $\langle \cdot, \cdot \rangle_p : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ with $\langle x, y \rangle_p = \sum_{i=1}^{n} p_i x_i \overline{y_i}$, we have
\[
0 \leq \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \overline{y_i} \right|^2 \quad \text{(13)}
\]
\[
\leq \frac{1}{4} \left[ |\Lambda - \lambda| \left( \sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} + |\Gamma - \gamma| \left( \sum_{i=1}^{n} p_i |x_i|^2 \right)^{1/2} \right]^2.
\]
If $0 < a \leq a_i \leq A < \infty$ and $0 < b \leq b_i \leq B < \infty$ for any $i \in \{1, ..., n\}$ then by (13) we have for any $p = (p_1, ..., p_n)$ a probability distribution that
\[
0 \leq \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left( \sum_{i=1}^{n} p_i a_i b_i \right)^2 \quad \text{(14)}
\]
\[
\leq \frac{1}{4} \left[ (A-a) \left( \sum_{i=1}^{n} p_i b_i^2 \right)^{1/2} + (B-b) \left( \sum_{i=1}^{n} p_i a_i^2 \right)^{1/2} \right]^2.
\]
The interested reader may compare this new result with the classical reverses of Schwarz inequality obtained by Diaz and Metcalf [1], Ozeki [4], G. Pólya and G. Szegő [7], Shisha and Mond [8] and Cassels [10].

For other reverses of Schwarz inequality in complex inner product spaces see the monograph [3] and the references therein.

References


