EXPLICIT SOLUTION TO MODULAR OPERATOR EQUATION \( TXS^* - SX^*T^* = A \)

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Abstract. In this paper, by using some block operator matrix techniques, we find explicit solution of the operator equation \( TXS^* - SX^*T^* = A \) in the general setting of the adjointable operators between Hilbert \( C^* \)-modules. Furthermore, we solve the operator equation \( TXS^* - SX^*T^* = A \), when \( \text{ran}(T) + \text{ran}(S) \) is closed.

1. Introduction and preliminaries

The equation \( TXS^* - SX^*T^* = A \) was studied by Yuan [8] for finite matrices and Xu et al. [7] generalized the results to Hilbert \( C^* \)-modules, under the condition that \( \text{ran}(S) \) is contained in \( \text{ran}(T) \). When \( T \) equals an identity matrix or identity operator, this equation reduces to \( XS^* - SX^* = A \), which was studied by Braden [1] for finite matrices, and Djordjevic [3] for the Hilbert space operators. In this paper, by using block operator matrix techniques and properties of the Moore-Penrose inverse, we provide a new approach to the study of the equation \( TXS^* - SX^*T^* = A \) for adjointable Hilbert module operators than those with closed ranges. Furthermore, we solve the operator equation \( TXS^* - SX^*T^* = A \), when \( \text{ran}(T) + \text{ran}(S) \) is closed.

Throughout this paper, \( A \) is a \( C^* \)-algebra. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Hilbert \( A \)-modules, and \( L(\mathcal{X}, \mathcal{Y}) \) be the set of the adjointable operators from \( \mathcal{X} \) to \( \mathcal{Y} \). For any \( T \in L(\mathcal{X}, \mathcal{Y}) \), the range and the null space of \( T \) are denoted by \( \text{ran}(T) \) and \( \text{ker}(T) \) respectively. In case \( \mathcal{X} = \mathcal{Y} \), \( L(\mathcal{X}, \mathcal{X}) \) which we abbreviate to \( L(\mathcal{X}) \), is a \( C^* \)-algebra. The identity operator on \( \mathcal{X} \) is denoted by \( 1_\mathcal{X} \) or \( 1 \) if there is no ambiguity.

Theorem 1.1. [5, Theorem 3.2] Suppose that \( T \in L(\mathcal{X}, \mathcal{Y}) \) has closed range. Then

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• \( \ker(T) \) is orthogonally complemented in \( \mathcal{X} \), with complement \( \operatorname{ran}(T^*) \).
• \( \operatorname{ran}(T) \) is orthogonally complemented in \( \mathcal{Y} \), with complement \( \ker(T^*) \).
• The map \( T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \) has closed range.

Xu and Sheng [6] showed that a bounded adjointable operator between two Hilbert \( \mathcal{A} \)-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse of \( T \), denoted by \( T^\dagger \), is the unique operator \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) satisfying the following conditions:

\[
TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.
\]

It is well-known that \( T^\dagger \) exists if and only if \( \operatorname{ran}(T) \) is closed, and in this case \( (T^\dagger)^* = (T^*)^\dagger \). Let \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) be a closed range, then \( TT^\dagger \) is the orthogonal projection from \( \mathcal{Y} \) onto \( \operatorname{ran}(T) \) and \( T^\dagger T \) is the orthogonal projection from \( \mathcal{X} \) onto \( \operatorname{ran}(T^*) \). Projection, in the sense that they are self adjoint idempotent operators.

A matrix form of a bounded adjointable operator \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) can be induced by some natural decompositions of Hilbert \( C^* \)-modules. Indeed, if \( \mathcal{M} \) and \( \mathcal{N} \) are closed orthogonally complemented submodules of \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, and \( \mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp \), then \( T \) can be written as the following \( 2 \times 2 \) matrix

\[
T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},
\]

where, \( T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \quad T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}), \quad T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp) \) and \( T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp) \). Note that \( P_\mathcal{M} \) denotes the projection corresponding to \( \mathcal{M} \).

In fact \( T_1 = P_\mathcal{N}TP_\mathcal{M}, \quad T_2 = P_\mathcal{N}T(1 - P_\mathcal{M}) \quad T_3 = (1 - P_\mathcal{N})TP_\mathcal{M} \) and \( T_4 = (1 - P_\mathcal{N})T(1 - P_\mathcal{M}) \).

**Lemma 1.2.** (see [4, Corollary 1.2]) Suppose that \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) has closed range. Then \( T \) has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules \( \mathcal{X} = \operatorname{ran}(T^*) \oplus \ker(T) \) and \( \mathcal{Y} = \operatorname{ran}(T) \oplus \ker(T^*) \):

\[
T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix} \rightarrow \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix},
\]
where $T_1$ is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix}. $$

2. Solutions to $TXS^* - SX^*T^* = A$

In this section, we will study the operator equation $TXS^* - SX^*T^* = A$ in the general context of the Hilbert $C^*$-modules. First, in the following theorem we solve to the operator equation $TXS^* - SX^*T^* = A$, in the case when $S$ and $T$ are invertible operators.

**Theorem 2.1.** Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $A$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* - SX^*T^* = A$.

(b) $A = -A^*$

If (a) or (b) is satisfied, then any solution to Eq. (2.1)

$$TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfy $Z^* = Z$.

**Proof.** (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (a): Note that, if $A = -A^*$ then $X = \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}Z(S^*)^{-1}$ is a solution to Eq. (2.1). The following sentences state this claim

$$T\left(\frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}Z(S^*)^{-1}\right)S^* - S\left(\frac{1}{2}S^{-1}A^*(T^*)^{-1} + S^{-1}Z^*(T^*)^{-1}\right)T^*$$

$$= \frac{1}{2}(TT^{-1}A(S^*)^{-1}S^* - SS^{-1}A^*(T^*)^{-1}T^*) + TT^{-1}Z(S^*)^{-1}S^* - SS^{-1}Z^*(T^*)^{-1}T^*$$

$$= A + Z - Z^* = A.$$
On the other hand, let $X$ be any solution to (2.1). Then $X = T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1}$. We have

$$X = T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1}$$

$$= \frac{1}{2}T^{-1}A(S^*)^{-1} + \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1}$$

$$= \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}(\frac{1}{2}A + SX^*T^*)S^*)^{-1}. $$

Taking $Z = \frac{1}{2}A + SX^*T^*$, we get $Z^* = Z$. \qed

**Corollary 2.2.** Suppose that $\mathcal{Y}, \mathcal{Z}$ are Hilbert $A$-modules and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator, $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ to the operator equation $TX - X^*T^* = A$.

(b) $A = -A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.

If (a) or (b) is satisfied, then any solution to Eq. (2.3)

$$TX - X^*T^* = A, \quad X \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$$

has the form

$$X = \frac{1}{2}T^{-1}A + T^{-1}Z,$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfy $Z^* = Z$.

In the following theorems we obtain explicit solutions to the operator equation

$$TXS^* - SX^*T^* = A,$$  

via matrix form and complemented submodules.

**Theorem 2.3.** Suppose $S \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ has closed range and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(\mathcal{Z})$ to Eq. (2.5).

(b) $A = -A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.

If (a) or (b) is satisfied, then any solution to Eq. (2.5) has the form

$$X = \frac{1}{2}TT^\dagger S^* - T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger A(1 - TT^\dagger)(S^*)^{-1}$$

$$+ (1 - TT^\dagger)Y(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^*(Z - Z^*)T = 0$, and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary.
**Proof.** \((a) \Rightarrow (b):\) Obviously, \(A = -A^*\). Also,

\[
(1 - TT^\dagger)A(1 - TT^\dagger) = (1 - TT^\dagger)(TXS^* - SX^*T^*)(1 - TT^\dagger)
\]

\[
= (T - TT^\dagger T)XS^*(1 - TT^\dagger) - (1 - TT^\dagger)SX^*(T^* - T^*TT^\dagger) = 0.
\]

\((b) \Rightarrow (a):\) Note that the condition \((1 - TT^\dagger)A(1 - TT^\dagger) = 0\) is equivalent to \(A = ATT^\dagger + TT^\dagger A + TT^\dagger ATT^\dagger\). On the other hand, since \(T^*(Z - Z^*)T = 0\), then \((Z - Z^*)T \in \ker(T^*) = \ker(T^\dagger)\). Therefore \(T^\dagger(Z - Z^*)T = 0\) or equivalently \(TT^\dagger ZTT^\dagger - TT^\dagger Z^*(T^\dagger)^*T^* = 0\). Hence we have

\[
\frac{1}{2}TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger + TT^\dagger A(1 - TT^\dagger) + T(1 - T^\dagger T)Y
\]

\[
- \frac{1}{2}TT^\dagger A^*(T^\dagger)^*T^* - TT^\dagger Z^*(T^\dagger)^*T^* - (1 - TT^\dagger)A^*(T^\dagger)^*T^* - Y^*(1 - T^\dagger T)T^*
\]

\[
= ATT^\dagger + TT^\dagger A + TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger - TT^\dagger Z^*(T^\dagger)^*T^* = A.
\]

That is, any operator \(X\) of the form (2.6) is a solution to Eq. (2.5).

Now, suppose that

\[
(2.7) \quad X = X_0(S^*)^{-1}, \quad X_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})
\]

is a solution to Eq. (2.5). We let (2.7) in Eq. (2.5). Hence (2.5) get into the following equation

\[
(2.8) \quad TX_0 - X^*_0T^* = A , \quad X_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})
\]

Since \(T\) has closed range, we have \(\mathcal{Z} = \text{ran}(T^*) \oplus \ker(T)\) and \(\mathcal{Y} = \text{ran}(T) \oplus \ker(T^*)\). Now, \(T\) has the matrix form

\[
T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix},
\]

where \(T_1\) is invertible. On the other hand, \(A = -A^*\) and \((1 - TT^\dagger)A(1 - TT^\dagger) = 0\) imply that \(A\) has the form

\[
A = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix},
\]

where \(A_1 = -A_1^*\). Let \(X_0\) have the form

\[
X_0 = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix}.
\]
Now by using matrix form for operators $T, X_0$ and $A$, we have
\[
\begin{bmatrix}
T_1 & 0 \\
0 & 0 \\

\end{bmatrix}
\begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4 \\

\end{bmatrix}
- \begin{bmatrix}
X_1^* & X_3^* \\
X_2^* & X_4^* \\

\end{bmatrix}
\begin{bmatrix}
T_1^* & 0 \\
0 & 0 \\

\end{bmatrix}
= \begin{bmatrix}
A_1 & A_2 \\
-A_2^* & 0 \\

\end{bmatrix},
\]
or equivalently
\[
\begin{bmatrix}
T_1X_1 - X_1^*T_1^* \\
-X_2^*T_1^* \\

\end{bmatrix}
= \begin{bmatrix}
A_1 & A_2 \\
-A_2^* & 0 \\

\end{bmatrix}.
\]

Therefore
\[
T_1X_1 - X_1^*T_1^* = A_1, \quad (2.9)
\]
\[
T_1X_2 = A_2. \quad (2.10)
\]

Now, we obtain $X_1$ and $X_2$. By Lemma 1.1, $T_1$ is invertible. Hence, Corollary 2.2 implies that
\[
X_1 = \frac{1}{2} T_1^{-1} A_1 + T_1^{-1} Z_1,
\]
where $Z_1 \in \mathcal{L}(\text{ran}(T))$ satisfy $Z_1^* = Z_1$. Now, multiplying $T_1^{-1}$ from the left to Eq. (2.10), we get
\[
X_2 = T_1^{-1} A_2. \quad (2.11)
\]

Hence
\[
\begin{bmatrix}
\frac{1}{2} T_1^{-1} A_1 + T_1^{-1} Z_1 & T_1^{-1} A_2 \\
X_3 & X_4 \\

\end{bmatrix}
\]
is a solution to Eq. (2.8), such that $X_3, X_4$ are arbitrary operators. Let
\[
Y = \begin{bmatrix}
Y_1 & Y_2 \\
X_3 & X_4 \\

\end{bmatrix}
: \begin{bmatrix}
\text{ran}(T) \\
\text{ker}(T) \\

\end{bmatrix}
\rightarrow \begin{bmatrix}
\text{ran}(T^*) \\
\text{ker}(T) \\

\end{bmatrix},
\]
and
\[
Z = \begin{bmatrix}
Z_1 & Z_2 \\
Z_3 & Z_4 \\

\end{bmatrix}
: \begin{bmatrix}
\text{ran}(T) \\
\text{ker}(T) \\

\end{bmatrix}
\rightarrow \begin{bmatrix}
\text{ran}(T^*) \\
\text{ker}(T^*) \\

\end{bmatrix}.
\]

Then
\[
\frac{1}{2} T^\dagger A T T^\dagger = \begin{bmatrix}
\frac{1}{2} T_1^{-1} A_1 & 0 \\
0 & 0 \\

\end{bmatrix}, \quad T^\dagger T Z T^\dagger = \begin{bmatrix}
T_1^{-1} Z_1 & 0 \\
0 & 0 \\

\end{bmatrix}
\]
and
\[
T^\dagger A (1 - T T^\dagger) = \begin{bmatrix}
0 & T_1^{-1} A_2 \\
0 & 0 \\

\end{bmatrix} , \quad (1 - T^\dagger T) Y = \begin{bmatrix}
0 & 0 \\
X_3 & X_4 \\

\end{bmatrix}.
\]

Consequently, $X_0 = \frac{1}{2} T^\dagger A T T^\dagger + T^\dagger T Z T^\dagger + T^\dagger A (1 - T T^\dagger) + (1 - T^\dagger T) Y$, where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^* (Z - Z^*) T = 0$, and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary. 

□
Remark 2.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules, we use the notation $\mathcal{X} \oplus \mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ x_1 \\ y_1 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle,$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, $i = 1, 2$. To simplify the notation, we use $x \oplus y$ to denote $\left( \begin{array}{c} x \\ y \end{array} \right) \in \mathcal{X} \oplus \mathcal{Y}$.

In the following theorem we obtain explicit solution to the operator equation $TXS^* - SX^*T^* = A$ when $(1 - P_{\text{ran}(S)})T$ and $S$ have closed ranges.

Theorem 2.5. Suppose that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ and $A \in \mathcal{L}(\mathcal{Y})$ and such that $(1 - SS^\dagger)T$ and $S$ have closed ranges. If the equation

$$(2.12) \quad TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

is solvable, then

$$(2.13) \quad \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} \dagger$$

Proof. Taking $H = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}: \mathcal{Z} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$. The operator $H$ has closed range, since let $\{z_n \oplus x_n\}$ be sequence chosen in $\mathcal{Z} \oplus \mathcal{X}, \{z_n\}, \{x_n\}$ be sequences chosen in $\mathcal{Z}$ and $\mathcal{X}$, respectively such that $T(z_n) + S(x_n) \to y$ for some $y \in \mathcal{Y}$. Then

$$(1 - SS^\dagger)T(z_n) = (1 - SS^\dagger)(T(z_n) + S(x_n)) \to (1 - SS^\dagger)(y).$$

Since $\text{ran}((1 - SS^\dagger)T)$ is assumed to be closed, $(1 - SS^\dagger)(y) = (1 - SS^\dagger)T(z_1)$ for some $z_1 \in \mathcal{Z}$. It follows that $y - T(x_1) \in \ker(1 - SS^\dagger) = \text{ran}(S)$, hence $y = T(z_1) + S(x)$ for some $x \in \mathcal{X}$. Therefore $H$ has closed range. Let $Y = \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix}: \mathcal{Z} \oplus \mathcal{X} \to \mathcal{Z} \oplus \mathcal{X}$ and $H^* = \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}: \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Z} \oplus \mathcal{X}$ and

$B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}: \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{Y}$, therefore Eq. (2.12) get into

$$(2.14) \quad HYH^* = B.$$
Since $H$ has closed range and equation (2.12) has solution, then equation (2.14) has solution, therefore with multiplying $HH^\dagger$ on the left and multiply $H^*(H^*)^\dagger$ on the right, we have

$$Y = H^\dagger B(H^*)^\dagger.$$

□

The proof of the following remark is the same as in the matrix case.

**Remark 2.6.** Let $T \in \mathcal{L}(Y, Z)$ and $S \in \mathcal{L}(X, Y)$ have closed ranges, and $A \in \mathcal{L}(X, Z)$. Then the equation

(2.15) $TXS = A, \quad X \in \mathcal{L}(Y)$

has a solution if and only if

(2.16) $TT^\dagger AS^\dagger S = A$.

In which case, any solution of Eq. (2.15) has the form

(2.17) $X = T^\dagger AS^\dagger$.

Now, we solve to the operator equation $TXS^* - SX^*T^* = A$ in the case when $\text{ran}(T) + \text{ran}(S)$ is closed.

**Theorem 2.7.** Suppose that $X$ is a Hilbert $\mathcal{A}$-module, $S, T, A \in \mathcal{L}(X)$ such that $\text{ran}(T) + \text{ran}(S)$ is closed. Then the following statements are equivalent:

(a) There exists a solution $X \in \mathcal{L}(X)$ to the operator equation $TXS^* - SX^*T^* = A$.

(b) $A = -A^*$ and $P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} = A$.

If (a) or (b) is satisfied, then any solution to Eq.

(2.18) $TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(X)$

has the form

(2.19) $X = T^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger S$.

Proof. (a) $\Rightarrow$ (b) Suppose that Eq. (2.18) has a solution $X \in \mathcal{L}(X)$. Then obviously $A = -A^*$. On the other hand, equation (2.18) get into

(2.20) \[
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & X \\
-X^* & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
= \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}.
\]
Since Eq. (2.18) is solvable, then Eq. (2.20) is solvable. Since $\text{ran}(T) + \text{ran}(S)$ is closed, then [2, Lemma 4] implies that $\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger$ exists. Hence, by Remark 2.15 we have

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$  

By applying [2, Lemma 4, Corollary 5] are shown that

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} T^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} T^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

That is, $P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} = A$.

$(b) \Rightarrow (a)$: If $P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} = A$, then we have

$$\begin{bmatrix} P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} (TT^* + SS^*)(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger (TT^* + SS^*) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (TT^* + SS^*)(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger (TT^* + SS^*) \\ 0 & 0 \end{bmatrix}.$$  

Therefore, Remark 2.15 implies that Eq. (2.20) is solvable and hence Eq. (2.18) is solvable.
Now, by applying Remark 2.15 and [2, Lemma 4, Corollary 5] imply that
\[
0 \begin{bmatrix} X & -X^* \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ 0 & S^* \end{bmatrix} = \begin{bmatrix} T^*(TT^* + SS^*)^\dagger & (TT^* + SS^*)^\dagger T \\ S^*(TT^* + SS^*)^\dagger & 0 \end{bmatrix} \begin{bmatrix} (TT^* + SS^*) \dagger T \\ 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ 0 & S^* \end{bmatrix} = \begin{bmatrix} T^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger T \\ S^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger T \end{bmatrix} \begin{bmatrix} (TT^* + SS^*) \dagger T \\ 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ 0 & S^* \end{bmatrix} = 0.
\]

Therefore
\[
T^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger T = S^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger S = 0.
\]

Consequently, X has the form (2.19). \(\square\)

Using exactly similar arguments, we obtain the following analogue of Theorem 2.1, in which to Eq. (2.1) is replaced by

(2.21) \[ TXS^* + SX^*T^* = A. \]

All results of this section can be rewritten for to Eq. (2.21), considering the following theorem.

**Theorem 2.8.** Let \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) be Hilbert \( \mathcal{A} \)-modules, \( S \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) and \( T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}) \) be invertible operators and \( A \in \mathcal{L}(\mathcal{Y}) \). Then the following statements are equivalent:

(a) There exists a solution \( X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \) to the operator equation \( TXS^* + SX^*T^* = A \).

(b) \( A = A^* \).

If (a) or (b) is satisfied, then any solution to Eq.

(2.22) \[ TXS^* + SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \]

has the form

\[
X = \frac{1}{2}T^{-1}A(S^*)^{-1} - T^{-1}Z(S^*)^{-1},
\]

where \( Z \in \mathcal{L}(\mathcal{Y}) \) satisfy \( Z^* = -Z \).
EXPLICIT SOLUTION TO MODULAR OPERATOR EQUATION $TXS^* - SX^*T^* = A$

REFERENCES


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