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SOME INEQUALITIES FOR RELATIVE OPERATOR ENTROPY

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Abstract. In this paper, by the use of some recent refinements and reverses of Young's inequality, we obtain some inequalities for relative operator entropy.

1. Introduction

Kamei and Fujii [6], [7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [13].

In general, we can define

$$S(A|B) := s - \lim_{\varepsilon\to 0+} S(A+\varepsilon I|B)$$

if it exists, here I is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A = S(A|I) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [8, p. 149-p. 155] we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} = B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2};$$

(ii) We have the inequalities

$$S(A|B) \leq A(\ln\|B\| - \ln A) \text{ and } S(A|B) \leq B - A;$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

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(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \le S(T^* AT|T^* BT).$$

The relative operator entropy is jointly concave, namely for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \ge tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \le (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [14]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1)\cup(1,\infty) \\ \\ 1 & \text{if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \le (1-\nu)a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.4}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.4) is due to Tominaga [15] while the first one is due to Furuichi [3].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \le (1-\nu)a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [17] while the second by Liao et al. [12].

Kittaneh and Manasrah [9], [10] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu} b^{\nu} \le R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.7}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.7) to an identity.

In the recent paper [1] we obtained the following reverses of Young's inequality as well:

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{1.8}$$

and

$$1 \le \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{1.9}$$

where $a,\ b > 0$, $\nu \in [0,1]$.

In [2] we obtained the following inequalities that improve the corresponding results of Furuichi and Minculete from [5]

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\} &\le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\le \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right] \end{aligned} \tag{1.11}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$.

In this paper, by making use of the inequalities (1.4), (1.4), (1.9) and (1.11) we establish some new inequalities for the relative operator entropy $S(A|B)$, for positive invertible operators A and B that satisfy the condition

$$mA \le B \le MA \tag{1.12}$$

for some m, M with $0 < m < M$.

2. Trapezoid Error Estimates

As shown below, by making use of the geometric mean-arithmetic mean inequality, one can prove that

$$\frac{\ln m}{M-m}(MA - B) + \frac{\ln M}{M-m}(B - mA) \le S(A|B) \tag{2.1}$$

for positive invertible operators A and B that satisfy the condition (1.12).

Therefore, it is a natural question to ask how far the right term is from the left term in (2.1).

In the following, we provide some upper and positive lower bounds for the difference

$$S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA)$$

under the above assumptions.

Theorem 1. *Let A, B be two positive invertible operators such that the condition (1.12) is valid, then we have*

$$0\leq A^{\frac{1}{2}}\Upsilon_{m,M}\left(A^{-1/2}BA^{-1/2}\right)A^{\frac{1}{2}} \leq S\left(A|B\right)-\frac{\ln m}{M-m}\left(MA-B\right)-\frac{\ln M}{M-m}\left(B-mA\right)\leq\ln S\left(\frac{M}{m}\right)A \tag{2.2}$$

where

$$\Upsilon_{m,M}\left(x\right):=\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|x-\frac{m+M}{2}\right|}\right)\geq 0 \tag{2.3}$$

for $x\in\left[m,M\right]$.

Proof. From (1.4) we have

$$S\left(\left(\frac{M}{m}\right)^{\min\{\nu,1-\nu\}}\right)m^{1-\nu}M^{\nu}\leq\left(1-\nu\right)m+\nu M\leq S\left(\frac{M}{m}\right)m^{1-\nu}M^{\nu}, \tag{2.4}$$

for any $\nu\in\left[0,1\right]$.

If we take in (2.4) $\nu=\frac{x-m}{M-m}\in\left[0,1\right]$ with $x\in\left[m,M\right]$ then we get

$$S\left(\left(\frac{M}{m}\right)^{\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}}\right)m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}}\leq x\leq S\left(\frac{M}{m}\right)m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}},$$

and by taking the logarithm we obtain

$$\ln S\left(\left(\frac{M}{m}\right)^{\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}}\right) \leq\ln x-\frac{M-x}{M-m}\ln m-\frac{x-m}{M-m}\ln M\leq\ln S\left(\frac{M}{m}\right). \tag{2.5}$$

Since

$$\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}=\frac{1}{2}-\left|\frac{x-\frac{m+M}{2}}{M-m}\right|$$

for any $x\in\left[m,M\right]$, then by (2.5) we get

$$\Upsilon_{m,M}\left(x\right)\leq\ln x-\frac{M-x}{M-m}\ln m-\frac{x-m}{M-m}\ln M\leq\ln S\left(\frac{M}{m}\right) \tag{2.6}$$

for any $x\in\left[m,M\right]$, where $\Upsilon_{m,M}$ is (the continuous function) defined by (2.3).

Using the continuous functional calculus we have from (2.6) that

$$\Upsilon_{m,M}\left(X\right)\leq\ln X-\frac{\ln m}{M-m}\left(MI-X\right)-\frac{\ln M}{M-m}\left(X-mI\right)\leq\ln S\left(\frac{M}{m}\right)I \tag{2.7}$$

for any selfadjoint operator X with the property that $mI\leq X\leq MI$.

Multiplying both sides of (1.12) by $A^{-1/2}$ we get

$$mI\leq A^{-1/2}BA^{-1/2}\leq MI$$

and by replacing X by $A^{-1/2}BA^{-1/2}$ in (2.7) we get

$$\begin{aligned}(2.8)\qquad &\Upsilon_{m,M}\left(A^{-1/2}BA^{-1/2}\right)\\ &\leq \ln A^{-1/2}BA^{-1/2}\\ &-\frac{\ln m}{M-m}\left(MI-A^{-1/2}BA^{-1/2}\right)-\frac{\ln M}{M-m}\left(A^{-1/2}BA^{-1/2}-mI\right)\\ &\leq \ln S\left(\frac{M}{m}\right)I.\end{aligned}$$

Multiplying both sides of (2.8) by $A^{1/2}$ we get the desired result (2.2). □

Corollary 1. *Assume that $pI \leq C \leq PI$ for some p, P with $0 < p < P$. Then we have for operator entropy $\eta(C) = -C\ln C$ that*

$$\begin{aligned}(2.9)\qquad 0 &\leq C\Psi_{p,P}\left(C^{-1}\right) \leq \eta(C)+\frac{P\ln P}{P-p}(C-pI)+\frac{p\ln p}{P-p}(PI-C)\\ &\leq \ln S\left(\frac{P}{p}\right)C\end{aligned}$$

where

$$\Psi_{p,P}(x) = \ln S\left(\left(\frac{P}{p}\right)^{\frac{1}{2}-\frac{pP}{P-p}\left|x-\frac{p+P}{2pP}\right|}\right)$$

where $x \in \left[\frac{1}{P}, \frac{1}{p}\right]$.

Proof. We have

$$\frac{1}{P}C \leq I \leq \frac{1}{p}C.$$

If we take $B = I$, $A = C$, $m = \frac{1}{P}$ and $M = \frac{1}{p}$ in Theorem 1, then we get

$$\begin{aligned}&C^{\frac{1}{2}}\Upsilon_{\frac{1}{P},\frac{1}{p}}\left(C^{-1}\right)C^{\frac{1}{2}}\\ &\leq S(C|I)-\frac{\ln\frac{1}{P}}{\frac{1}{p}-\frac{1}{P}}\left(\frac{1}{p}C-I\right)-\frac{\ln\frac{1}{p}}{\frac{1}{p}-\frac{1}{P}}\left(I-\frac{1}{P}C\right)\leq \ln S\left(\frac{P}{p}\right)C,\end{aligned}$$

namely

$$\begin{aligned}&C^{\frac{1}{2}}\Psi_{p,P}\left(C^{-1}\right)C^{\frac{1}{2}}\\ &=C\Psi_{p,P}\left(C^{-1}\right)\\ &\leq S(C|I)+\frac{P\ln P}{P-p}(C-pI)+\frac{p\ln p}{P-p}(PI-C)\leq \ln S\left(\frac{P}{p}\right)C,\end{aligned}$$

where

$$\Upsilon_{\frac{1}{P},\frac{1}{p}}(x) = \Psi_{p,P}(x) = \ln S\left(\left(\frac{P}{p}\right)^{\frac{1}{2}-\frac{pP}{P-p}\left|x-\frac{p+P}{2pP}\right|}\right),$$

with $x \in \left[\frac{1}{P}, \frac{1}{p}\right]$. □

We also have:

Theorem 2. *With the assumptions of Theorem 1 we have*

$$
\begin{aligned}
(2.10)\quad 0 &\le \left(\frac{1}{2}A-\frac{1}{M-m}A^{1/2}\left|A^{-1/2}\left(B-\frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}\right)K\left(\frac{M}{m}\right)\\
&\le S(A|B)-\frac{\ln m}{M-m}(MA-B)-\frac{\ln M}{M-m}(B-mA)\\
&\le \left(\frac{1}{2}A+\frac{1}{M-m}A^{1/2}\left|A^{-1/2}\left(B-\frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}\right)K\left(\frac{M}{m}\right).
\end{aligned}
$$

Proof. Using the inequality (1.6) we have

$$
\begin{aligned}
(2.11)\quad K^{\min\{\nu,1-\nu\}}\left(\frac{M}{m}\right)m^{1-\nu}M^{\nu} &\le (1-\nu)m+\nu M\\
&\le K^{\max\{\nu,1-\nu\}}\left(\frac{M}{m}\right)m^{1-\nu}M^{\nu}
\end{aligned}
$$

for any $\nu\in[0,1]$.

If we take in (2.11) $\nu=\frac{x-m}{M-m}\in[0,1]$ with $x\in[m,M]$ then we get

$$
\begin{aligned}
&K^{\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}}\left(\frac{M}{m}\right)m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}}\\
&\le x\le K^{\max\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}}\left(\frac{M}{m}\right)m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&(0\le)\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}K\left(\frac{M}{m}\right)\\
&\le \ln x-\frac{M-x}{M-m}\ln m-\frac{x-m}{M-m}\ln M\\
&\le \max\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}K\left(\frac{M}{m}\right)
\end{aligned}
$$

or to

$$
\begin{aligned}
&(0\le)\left(\frac{1}{2}-\frac{1}{M-m}\left|x-\frac{m+M}{2}\right|\right)K\left(\frac{M}{m}\right)\\
&\le \ln x-\frac{M-x}{M-m}\ln m-\frac{x-m}{M-m}\ln M\\
&\le \left(\frac{1}{2}+\frac{1}{M-m}\left|x-\frac{m+M}{2}\right|\right)K\left(\frac{M}{m}\right).
\end{aligned}
$$

By making use of a similar argument to the one in the proof of Theorem 1 we get the desired result (2.10). □

Remark 1. *If A and B commute, then*

$$
A^{1/2}\left|A^{-1/2}\left(B-\frac{m+M}{2}A\right)A^{-1/2}\right|A^{1/2}=\left|B-\frac{m+M}{2}A\right|,
$$

$$
S(A|B)=A(\ln B-\ln A)
$$

and by (2.10) we have

$$\begin{aligned} 0 &\le \left(\frac{1}{2}A - \frac{1}{M-m}\left|B - \frac{m+M}{2}A\right|\right) K\left(\frac{M}{m}\right) \\ &\le A(\ln B - \ln A) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA) \\ &\le \left(\frac{1}{2}A + \frac{1}{M-m}\left|B - \frac{m+M}{2}A\right|\right) K\left(\frac{M}{m}\right). \end{aligned} \tag{2.12}$$

Corollary 2. *With the assumptions of Corollary 1 we have*

$$\begin{aligned} &\left(\frac{1}{2}C - \frac{pP}{P-p}\left|I - \frac{p+P}{2pP}C\right|\right) K\left(\frac{P}{p}\right) \\ &\le \eta(C) + \frac{P\ln P}{P-p}(C - pI) + \frac{p\ln p}{P-p}(PI - C) \\ &\le \left(\frac{1}{2}C + \frac{pP}{P-p}\left|I - \frac{p+P}{2pP}C\right|\right) K\left(\frac{P}{p}\right). \end{aligned} \tag{2.13}$$

Proof. Follows by Theorem 2 on choosing $B = I$, $A = C$, $m = \frac{1}{P}$ and $M = \frac{1}{p}$ and taking into account that, by the continuous functional calculus for C, we have

$$C^{1/2}\left|C^{-1/2}\left(I - \frac{p+P}{2pP}C\right)C^{-1/2}\right|C^{1/2} = \left|I - \frac{p+P}{2pP}C\right|.$$

□

Theorem 3. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} (0 \le)\, &S(A|B) - \frac{\ln m}{M-m}(MA - B) - \frac{\ln M}{M-m}(B - mA) \\ &\le \frac{4}{(M-m)^2}\left(K\left(\frac{M}{m}\right) - 1\right)(B - mA)A^{-1}(MA - B) \\ &\le \left(K\left(\frac{M}{m}\right) - 1\right)A. \end{aligned} \tag{2.14}$$

Proof. From the inequality (1.9) we have

$$(1 \le)\, \frac{(1-\nu)m + \nu M}{m^{1-\nu}M^{\nu}} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{M}{m}\right) - 1\right)\right], \tag{2.15}$$

for any $\nu \in [0,1]$.

If we take in (2.15) $\nu = \frac{x-m}{M-m} \in [0,1]$ with $x \in [m, M]$ then we get

$$(1 \le)\, \frac{x}{m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}}} \le \exp\left[\frac{4(x-m)(M-x)}{(M-m)^2}\left(K\left(\frac{M}{m}\right) - 1\right)\right]. \tag{2.16}$$

Taking the logarithm in (2.16) we get

$$(0 \le) \ln x - \frac{M-x}{M-m}\ln m - \frac{x-m}{M-m}\ln M \le \frac{4(x-m)(M-x)}{(M-m)^2}\left(K\left(\frac{M}{m}\right) - 1\right)$$

for any $x \in [m, M]$.

Making use of a similar argument to the one from the proof of Theorem 1 we get

$$\begin{aligned}
&S\left(A|B\right)-\frac{\ln m}{M-m}\left(MA-B\right)-\frac{\ln M}{M-m}\left(B-mA\right)\\
&\leq\frac{4}{\left(M-m\right)^{2}}\left(K\left(\frac{M}{m}\right)-1\right)\\
&\times A^{1/2}\left(A^{-1/2}BA^{-1/2}-m\right)\left(M-A^{-1/2}BA^{-1/2}\right)A^{1/2}
\end{aligned}$$

and since

$$\begin{aligned}
&A^{1/2}\left(A^{-1/2}BA^{-1/2}-m\right)\left(M-A^{-1/2}BA^{-1/2}\right)A^{1/2}\\
&=A^{1/2}\left(A^{-1/2}\left(B-mA\right)A^{-1/2}\right)\left(A^{-1/2}\left(MA-B\right)A^{-1/2}\right)A^{1/2}\\
&=\left(B-mA\right)A^{-1}\left(MA-B\right),
\end{aligned}$$

we obtain the first part of (2.14).

The second part follows by the inequality

$$\frac{4\left(x-m\right)\left(M-x\right)}{\left(M-m\right)^{2}}\leq 1$$

for any $x\in[m,M]$. □

Corollary 3. *With the assumptions of Corollary 1 we have*

$$\begin{aligned}
&(0\leq)\,\eta\left(C\right)+\frac{P\ln P}{P-p}\left(C-pI\right)+\frac{p\ln p}{P-p}\left(PI-C\right)\\
&\leq\frac{4pP}{\left(P-p\right)^{2}}\left(K\left(\frac{P}{p}\right)-1\right)\left(IP-C\right)C^{-1}\left(C-Ip\right)\\
&\leq\left(K\left(\frac{P}{p}\right)-1\right)C.
\end{aligned}\tag{2.17}$$

Finally, we have:

Theorem 4. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
&(0\leq)\,S\left(A|B\right)-\frac{\ln m}{M-m}\left(MA-B\right)-\frac{\ln M}{M-m}\left(B-mA\right)\\
&\leq\frac{1}{2m^{2}}\left(B-mA\right)A^{-1}\left(MA-B\right)
\end{aligned}\tag{2.18}$$

Proof. From the inequality (1.11) we have

$$\frac{\left(1-\nu\right)m+\nu M}{m^{1-\nu}M^{\nu}}\leq\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(\frac{M}{m}-1\right)^{2}\right]\tag{2.19}$$

for any $\nu\in[0,1]$.

If we take in (2.15) $\nu=\frac{x-m}{M-m}\in[0,1]$ with $x\in[m,M]$ then we get

$$\frac{x}{m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}}}\leq\exp\left[\frac{1}{2}\frac{\left(x-m\right)\left(M-x\right)}{\left(M-m\right)^{2}}\left(\frac{M}{m}-1\right)^{2}\right]$$

that is equivalent to

$$\frac{x}{m^{\frac{M-x}{M-m}}M^{\frac{x-m}{M-m}}}\leq\exp\left[\frac{1}{2}\frac{\left(x-m\right)\left(M-x\right)}{m^{2}}\right].$$

On taking the logarithm, we get

$$\ln x - \frac{M-x}{M-m}\ln m - \frac{x-m}{M-m}\ln M \le \frac{1}{2}\frac{(x-m)(M-x)}{m^2}, \tag{2.20}$$

for any $x \in [m, M]$.

Making use of a similar argument to the one from the proofs of Theorem 1 and Theorem 3 we get the desired result (2.18). □

Corollary 4. *With the assumptions of Corollary 1 we have*

$$\begin{aligned}(0 \le)\, \eta(C) + \frac{P\ln P}{P-p}(C-pI) + \frac{p\ln p}{P-p}(PI-C) \\ \le \frac{1}{2}\frac{P}{p}(IP-C)\,C^{-1}(C-Ip).\end{aligned} \tag{2.21}$$

3. Inequalities Via Uhlmann's Representation

In [16], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s - \lim_{t\to 0}\frac{A\sharp_t B - A}{t} \tag{3.1}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu} A^{1/2},\ \nu \in [0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

We have:

Theorem 5. *Let A, B be two positive invertible operators, then we have*

$$S(A|B) \le 2(A\sharp B - A) \le B - A. \tag{3.2}$$

Proof. From the inequality (1.7) for $\nu \in \left(0, \frac{1}{2}\right)$ and a, $b > 0$ we have

$$\nu\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)\,a + \nu b - a^{1-\nu}b^{\nu}$$

that is equivalent to

$$a - 2\sqrt{ab} + b \le b - a + \frac{1}{\nu}\left(a - a^{1-\nu}b^{\nu}\right)$$

and to

$$\frac{1}{\nu}\left(a^{1-\nu}b^{\nu} - a\right) \le 2\left(\sqrt{ab} - a\right). \tag{3.3}$$

If we take in (3.3) $a = 1$ then we get

$$\frac{1}{\nu}\left(b^{\nu} - 1\right) \le 2\left(b^{1/2} - 1\right), \tag{3.4}$$

for any $\nu \in \left(0, \frac{1}{2}\right)$ and a, $b > 0$.

If we use the continuous functional calculus, then we have for any positive operator X that

$$\frac{1}{\nu}\left(X^{\nu} - 1\right) \le 2\left(X^{1/2} - 1\right), \tag{3.5}$$

for any $\nu \in \left(0, \frac{1}{2}\right)$.

If we take in (3.5) $X = A^{-1/2}BA^{-1/2}$ then we get

$$\frac{1}{\nu}\left(\left(A^{-1/2}BA^{-1/2}\right)^{\nu}-1\right) \leq 2\left(\left(A^{-1/2}BA^{-1/2}\right)^{1/2}-1\right), \tag{3.6}$$

for any $\nu \in \left(0, \frac{1}{2}\right)$.

Multiplying both sides of (3.6) by $A^{1/2}$ we get

$$\begin{aligned}\frac{1}{\nu}&\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}-A\right) \\ &\leq 2\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}-A\right),\end{aligned} \tag{3.7}$$

for any $\nu \in \left(0, \frac{1}{2}\right)$.

By taking the strong limit over $\nu \to 0+$ in (3.7) and by using the representation (3.1) we obtain the first inequality in (3.2).

By the operator geometric mean - arithmetic mean inequality $A\sharp B \leq \frac{1}{2}(A+B)$ we deduce the second part of (3.2). $\square$

Remark 2. *The inequality (3.2) is an improvement of the result from (ii) in the introduction.*

Corollary 5. *For any positive invertible operator C we have*

$$\eta(C) \leq 2\left(C^{1/2}-C\right) \leq I - C.$$

Theorem 6. *Let A, B be two positive invertible operators, then we have*

$$\begin{aligned}&(0 \leq)\, \frac{1}{2}A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)^{2}\left(\frac{1}{2}I-\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)A^{1/2} \\ &\leq B - A - S(A|B) \\ &\leq \frac{1}{2}A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)^{2}\left(\frac{1}{2}I+\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)A^{1/2}.\end{aligned} \tag{3.8}$$

Proof. From (1.8) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\} &\leq (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \\ &\leq \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\}\end{aligned}$$

for any for $\nu \in (0,1)$ and a, $b > 0$.

This is equivalent to

$$\begin{aligned}\frac{1}{2}(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\} &\leq b-a+\frac{1}{\nu}\left(a-a^{1-\nu}b^{\nu}\right) \\ &\leq \frac{1}{2}(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\}\end{aligned} \tag{3.9}$$

for any for $\nu \in (0,1)$ and a, $b > 0$.

If we replace in (3.9) $a = 1$ and $b = x$, then we get

$$\begin{aligned}(0 \leq)\, \frac{1}{2}(1-\nu)(\ln x)^{2}\min\{1,x\} &\leq x-1+\frac{1}{\nu}\left(1-x^{\nu}\right) \\ &\leq \frac{1}{2}(1-\nu)(\ln x)^{2}\max\{1,x\}\end{aligned}$$

for any for $\nu \in (0,1)$ and $x > 0$.

If we use the continuous functional calculus, then we have for any positive operator X that

$$\begin{aligned}(3.10)\quad \frac{1}{2}(1-\nu)(\ln X)^2\left(\frac{1}{2}I-\left|X-\frac{1}{2}I\right|\right) &\le X-1+\frac{1}{\nu}(1-X^{\nu})\\ &\le \frac{1}{2}(1-\nu)(\ln X)^2\left(\frac{1}{2}I+\left|X-\frac{1}{2}I\right|\right)\end{aligned}$$

for any for $\nu\in(0,1)$.

If we take in (3.10) $X=A^{-1/2}BA^{-1/2}$ then we get

$$\begin{aligned}(3.11)\qquad & \frac{1}{2}(1-\nu)\left(\ln A^{-1/2}BA^{-1/2}\right)^2\left(\frac{1}{2}I-\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)\\ &\le A^{-1/2}BA^{-1/2}-1+\frac{1}{\nu}\left(1-\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\right)\\ &\le \frac{1}{2}(1-\nu)\left(\ln A^{-1/2}BA^{-1/2}\right)^2\left(\frac{1}{2}I+\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)\end{aligned}$$

for any for $\nu\in(0,1)$.

Multiplying both sides of (3.11) by $A^{1/2}$ we get

$$\begin{aligned}(3.12)\quad & \frac{1}{2}(1-\nu)A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)^2\left(\frac{1}{2}I-\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)A^{1/2}\\ &\le B-A+\frac{1}{\nu}A^{1/2}\left(I-\left(A^{-1/2}BA^{-1/2}\right)^{\nu}\right)A^{1/2}\\ &\le \frac{1}{2}(1-\nu)A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)^2\left(\frac{1}{2}I+\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)A^{1/2}\end{aligned}$$

for any for $\nu\in(0,1)$.

This is an inequality of interest in itself.

Now, if we let $\nu\to 0+$ in (3.12), then we get

$$\begin{aligned}&(0\le)\,\frac{1}{2}A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)^2\left(\frac{1}{2}I-\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)A^{1/2}\\ &\le B-A-S(A|B)\\ &\le \frac{1}{2}A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)^2\left(\frac{1}{2}I+\left|A^{-1/2}BA^{-1/2}-\frac{1}{2}I\right|\right)A^{1/2},\end{aligned}$$

which proves the desired result (3.8). □

Corollary 6. *For any positive invertible operator C we have*

$$\begin{aligned}(3.13)\qquad & (0\le)\,\frac{1}{2}(\ln C)^2\left(\frac{1}{2}I-\left|C-\frac{1}{2}I\right|\right)\\ &\le I-C-\eta(C)\le\frac{1}{2}(\ln C)^2\left(\frac{1}{2}I+\left|C-\frac{1}{2}I\right|\right).\end{aligned}$$

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