Received 24/10/15

TWO VARIABLES VERSION OF HERMITE-HADAMARD TYPE INEQUALITIES WITH APPLICATIONS TO ESTIMATION OF PRODUCT OF MOMENT OF TWO CONTINUOUS RANDOM VARIABLES.

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Abstract. In this paper, new Hermite-Hadamard type inequalities for co-ordinated convex and co-ordinated quasi convex functions are proved in a unique way. These results generalize many results proved in earlier works for these classes of functions. Finally, applications of our results are given to estimate the product of moments of two independent continuous random variables.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

An important inequality for convex functions is the Hermite-Hadamard's inequality (see for instance [7]), which is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function and $a, b \in I$ with $a < b$. The inequalities in (1.1) hold in reversed if f is a concave function.

It has been an important task to to provide sharp bounds for the quantities

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \text{ and } \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|.$$

Many mathematicians have tried to provide sharp bounds for the above two quantities over the past few years by proving different identities. Moreover, a lot of variants of the (1.1) have been proved by using different forms and generalizations of convexity, see for instance the works in [2, 4, 5, 6, 9, 23, 24] and the references cited in there.

The following generalization of convexity for functions of two variables, known as convexity on co-ordinates, was initiated by Dragomir [4].

Date: Today.

2000 *Mathematics Subject Classification.* 26D15, 26D20, 26D07.

Key words and phrases. Hermite-Hadamard's inequality, co-ordinated convex function, co-ordinated quasi-convex function, Hölder's integral inequality, Moment of random variable.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Let $[a,b]\times[c,d]$ be a bidimensional interval in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on $[a,b]\times[c,d]$ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$.

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b], y\in[c,d]$.

A different way of describing convexity of f on co-ordinates on $[a,b]\times[c,d]$ is given in the definition below.

Definition 1. [13] *A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be convex on co-ordinates on $[a,b]\times[c,d]$ if the inequality*

$$\begin{aligned}&f(tx+(1-t)y,sz+(1-s)w)\\&\leq tsf(x,z)+t(1-s)f(x,w)+s(1-t)f(y,z)+(1-t)(1-s)f(y,w)\end{aligned}$$

holds for all $(t,s)\in[0,1]\times[0,1]$ and $(x,z),(y,w)\in[a,b]\times[c,d]$.

It has been proved in [4] that every convex mapping $f:[a,b]\times[c,d]\to\mathbb{R}$ is convex on co-ordinates. Furthermore, there exist co-ordinated convex functions which are not convex, (see for example [4]).

The following Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [4]:

Theorem 1. [4] *Suppose that $f:[a,b]\times[c,d]\to\mathbb{R}$ is co-ordinated convex on $[a,b]\times[c,d]$. Then one has the inequalities:*

$$\begin{aligned}(1.2)\quad &f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\&\leq\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}$$

The above inequalities are sharp.

Latif et al. [17], proved the following Hermite-Hadamard type inequalities which provide a weighted generalization for the left side of (1.2).

Theorem 2. [17] *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for*

$[a,b]\times[c,d]\subset\Delta^{\circ}$ with $a<b$, $c<d$. If $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|$ is convex on the co-ordinates on $[a,b]\times[c,d]$, then

$$\begin{aligned}(1.3)\quad &\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\&\left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\&\leq\frac{(b-a)(d-c)}{4}\left[|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|\right]\\&\qquad\times\int_0^1\int_0^1\left(\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right)dtds,\end{aligned}$$

where $L_1(t)=\frac{1-t}{2}a+\frac{1+t}{2}b$ and $L_2(t)=\frac{1-s}{2}c+\frac{1+s}{2}d$.

Theorem 3. [17] *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on Δ° and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^{\circ}$ with $a<b$, $c<d$. If $f_{ts}\in L([a,b]\times[c,d])$ and $|f_{ts}|^q$ is convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq1$, then*

$$\begin{aligned}(1.4)\quad &\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_c^d\int_a^b p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right.\\&\left.-\int_c^d\int_a^b f\left(x,\frac{c+d}{2}\right)p(x,y)\,dxdy-\int_c^d\int_a^b f\left(\frac{a+b}{2},y\right)p(x,y)\,dxdy\right|\\&\leq(b-a)(d-c)\left[\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4}\right]^{\frac{1}{q}}\\&\qquad\times\int_0^1\int_0^1\left(\int_c^{L_2(s)}\int_a^{L_1(t)}p(x,y)\,dxdy\right)dtds,\end{aligned}$$

where $L_1(t)$ and $L_2(t)$ are defined in Theorem 2.

In a recent paper [22], Özdemir et al. gave the notion of co-ordinated quasi-convex functions which generalizes the notion of co-ordinated convex functions.

Definition 2. [22] *A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be quasi-convex on $[a,b]\times[c,d]$ if the inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\max\{f(x,y),f(z,w)\}$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$.

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are quasi-convex where defined for all $x\in[a,b]$, $y\in[c,d]$.

Another way of expressing the concept of co-ordinated quasi-convex functions is stated below.

Definition 3. [16] *A function $f : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be quasi-convex on co-ordinates on $[a,b] \times [c,d]$ if*

$$f(tx + (1-t)z, sy + (1-s)w) \leq \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}$$

holds for all $(x,y),(z,w) \in [a,b] \times [c,d]$ and $(s,t) \in [0,1] \times [0,1]$.

The class of co-ordinated quasi-convex functions on $[a,b] \times [c,d]$ is denoted by $QC([a,b] \times [c,d])$. It has also been proved in [22] that every quasi-convex function on $[a,b] \times [c,d]$ is quasi-convex on co-ordinates on $[a,b] \times [c,d]$ but the converse does not hold true.

The following inequalities related to the right side of (1.2) was obtained by Latif et al. in [16] for quasi-convex on co-ordinates on $[a,b] \times [c,d]$.

Theorem 4. [16] *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on Δ° and let $[a,b] \times [c,d] \subseteq \Delta^\circ$ with $a < b$, $c < d$. If $f_{ts} \in L([a,b] \times [c,d])$ and $|f_{ts}|$ is quasi-convex on the co-ordinates on $[a,b] \times [c,d]$, then the following inequality holds:*

$$(1.5)\quad \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_a^d f(x,y)\,dydx + \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c}\int_a^d [f(a,y)+f(b,y)]\,dy\right]\right|$$
$$\leq K\left[\sup\left\{|f_{ts}(a,c)|, \left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},c\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\right.$$
$$\sup\left\{|f_{ts}(a,d)|, \left|f_{ts}\left(a,\frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},d\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\sup\left\{|f_{ts}(b,c)|, \left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},c\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$\left.+\sup\left\{|f_{ts}(b,d)|, \left|f_{ts}\left(b,\frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},d\right)\right|, \left|f_{ts}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\right],$$

where $K = \frac{(b-a)(d-c)}{64}$.

For more recent results on this field, we refer the readers to [1, 5, 8], [10]-[22] and the references therein.

In the present paper, we establish new weighted integral inequalities of Hermite-Hadamard type for the classes of convex and quasi-convex functions on co-ordinates on $[a,b] \times [c,d]$ which generalize the results given in Theorem 2, Theorem 3 and Theorem 4. Applications of our results to estimate the moment of product of independent random variables are provided as well.

2. Some Auxiliary Results

In what follows we use the following notations

Let $w(x,y) : [a,b] \times [c,d] \to [0,\infty)$ be a continuous function such that

$$\int_a^b\int_c^d w(x,y)\,dydx = 1.$$

We denote the integral $\int_a^b \int_c^d xw(x,y)\,dydx$ by a_1, the integral $\int_a^b \int_c^d yw(x,y)\,dydx$ by c_1 and the integral $\int_a^b \int_c^d xyw(x,y)\,dydx$ by α_1, i.e.

$$
\begin{aligned}
a_1 &= \int_a^b \int_c^d xw(x,y)\,dydx\ ,\alpha_1 = \int_a^b \int_c^d xyw(x,y)\,dydx \text{ and} \\
c_1 &= \int_a^b \int_c^d yw(x,y)\,dx.
\end{aligned}
$$

Now we present a result in which the function $w(x,y)$ is symmetric on co-ordinates with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates.

Lemma 1. *If $w(x,y) : [a,b] \times [c,d] \to [0,\infty)$ is symmetric on co-ordinates with respect to the midpoints $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Then*

$$
a_1 = \frac{a+b}{2},\ c_1 = \frac{c+d}{2} \text{ and } \alpha_1 = \left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right).
$$

Proof. Since w is symmetric on co-ordinates with respect to the midpoint $\frac{a+b}{2}$, we have

$$
\begin{aligned}
a_1 &= \int_a^b \int_c^d xw(x,y)\,dydx = \int_a^b \int_c^d xw(a+b-x,y)\,dydx \\
&= \int_a^b \int_c^d (a+b-x)\,w(x,y)\,dydx
\end{aligned}
$$

which gives the desired result since

$$
\int_a^b \int_c^d w(x,y)\,dydx = 1.
$$

Similarly, one can prove that

$$
c_1 = \frac{c+d}{2}
$$

and

$$
\alpha_1 = \int_a^b \int_c^d xyw(x,y)\,dydx = \left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right).
$$

□

Lemma 2. *Let $f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially differentiable mapping on Ω° and $f_{ts} \in L([a,b] \times [c,d])$, where $[a,b] \times [c,d] \subseteq \Omega^\circ$ with $a < b$, $c < d$. Let*

$w:[a,b]\times[c,d]\to[0,\infty)$ *be a continuous mapping. Then*

$$\frac{1}{(b-a)(d-c)}\left[f(a,c)\int_a^b\int_c^d(b-x)(d-y)w(x,y)\,dydx\right.\tag{2.1}$$

$$+f(a,d)\int_a^b\int_c^d(b-x)(y-c)w(x,y)\,dydx+f(b,c)\int_a^b\int_c^d(x-a)(d-y)w(x,y)\,dydx$$

$$\left.+f(b,d)\int_a^b\int_c^d(x-a)(y-c)w(x,y)\,dydx\right]+\int_a^b\int_c^d f(x,y)w(x,y)\,dydx$$

$$-\frac{1}{d-c}\int_a^b\int_c^d(b-x)f(a,y)w(x,y)\,dydx-\frac{1}{d-c}\int_a^b\int_c^d(x-a)f(b,y)w(x,y)\,dydx$$

$$-\frac{1}{b-a}\int_a^b\int_c^d(d-y)f(x,c)w(x,y)\,dydx-\frac{1}{b-a}\int_a^b\int_c^d(y-c)f(x,d)w(x,y)\,dydx$$

$$=\frac{(a_1-a)(c_1-c)}{(b-a)(d-c)}\int_0^1\int_0^1 H(w,a,c,a_1,c_1;r,z)\,f_{rz}((1-r)a+a_1r,(1-z)c+c_1z)\,dzdr$$

$$+\frac{(b-a_1)(c_1-c)}{(b-a)(d-c)}\int_0^1\int_0^1 H(w,a_1,c,b,c_1;r,z)\,f_{rz}((1-r)a_1+br,(1-z)c+c_1z)\,dzdr$$

$$+\frac{(a_1-a)(d-c_1)}{(b-a)(d-c)}\int_0^1\int_0^1 H(w,a,c_1,a_1,d;r,z)\,f_{rz}((1-r)a+a_1r,(1-z)c_1+dz)\,dzdr$$

$$+\frac{(b-a_1)(d-c_1)}{(b-a)(d-c)}\int_0^1\int_0^1 H(w,a_1,c_1,b,d;r,z)\,f_{rz}((1-r)a_1+br,(1-z)c_1+dz)\,dzdr,$$

where

$$\begin{aligned}H(w,\alpha,\gamma,\beta,\delta;r,z)&=\int_{(1-r)\alpha+\beta r}^b\int_{(1-z)\gamma+\delta z}^d(b-x)(d-y)w(x,y)\,dydx\\&-\int_{(1-r)\alpha+\beta r}^b\int_c^{(1-z)\gamma+\delta z}(b-x)(y-c)w(x,y)\,dydx\\&-\int_a^{(1-r)\alpha+\beta r}\int_{(1-z)\gamma+\delta z}^d(x-a)(d-y)w(x,y)\,dydx\\&+\int_a^{(1-r)\alpha+\beta r}\int_c^{(1-z)\gamma+\delta z}(x-a)(y-c)w(x,y)\,dydx,\end{aligned}$$

$(\alpha,\gamma),(\beta,\delta)\in[a,b]\times[c,d]$.

Proof. The following identities hold

$$f(x,y)-f(a,y)-f(x,c)+f(a,c)=\int_a^b\int_c^d\sigma(x-t)\sigma(y-s)f_{ts}(t,s)\,dsdt,\tag{2.2}$$

$$f(x,y)-f(a,y)-f(x,d)+f(a,d)=-\int_a^b\int_c^d\sigma(x-t)\sigma(s-y)f_{ts}(t,s)\,dsdt,\tag{2.3}$$

$$f(x,y)-f(b,y)-f(x,c)+f(b,c)=-\int_a^b\int_c^d\sigma(t-x)\sigma(y-s)f_{ts}(t,s)\,dsdt\tag{2.4}$$

and

$$f(x,y)-f(b,y)-f(x,d)+f(b,d)=\int_a^b\int_c^d \sigma(t-x)\,\sigma(s-y)\,f_{ts}(t,s)\,dsdt, \tag{2.5}$$

where $\sigma(\cdot)$ is the Heavyside function defined by

$$\sigma(u)=\begin{cases}0, & u<0\\ 1 & u>0.\end{cases}$$

From (2.2), we have

$$\begin{aligned}
&\int_a^b\int_c^d (b-x)(d-y)\,f(x,y)\,w(x,y)\,dydx \\
&\quad-\int_a^b\int_c^d (b-x)(d-y)\,f(a,y)\,w(x,y)\,dydx \\
&-\int_a^b\int_c^d (b-x)(d-y)\,f(x,c)\,w(x,y)\,dydx+f(a,c)\int_a^b\int_c^d (b-x)(d-y)\,w(x,y)\,dydx \\
&=\int_a^b\int_c^d (b-x)(d-y)\left(\int_a^b\int_c^d \sigma(x-t)\,\sigma(y-s)\,f_{ts}(t,s)\,dsdt\right)w(x,y)\,dydx \\
&=\int_a^b\int_c^d\left(\int_t^b\int_s^d (b-x)(d-y)\,w(x,y)\,dydx\right)f_{ts}(t,s)\,dsdt.
\end{aligned} \tag{2.6}$$

Similarly, we also have

$$\begin{aligned}
&\int_a^b\int_c^d (b-x)(y-c)\,f(x,y)\,w(x,y)\,dydx \\
&-\int_a^b\int_c^d (b-x)(y-c)\,f(a,y)\,w(x,y)\,dydx-\int_a^b\int_c^d (b-x)(y-c)\,f(x,d)\,w(x,y)\,dydx \\
&\quad+f(a,d)\int_a^b\int_c^d (b-x)(y-c)\,w(x,y)\,dydx \\
&=-\int_a^b\int_c^d\left(\int_t^b\int_c^s (b-x)(y-c)\,w(x,y)\,dydx\right)f_{ts}(t,s)\,dsdt,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\int_a^b\int_c^d (x-a)(d-y)\,f(x,y)\,w(x,y)\,dydx \\
&-\int_a^b\int_c^d (x-a)(d-y)\,f(b,y)\,w(x,y)\,dydx-\int_a^b\int_c^d (x-a)(d-y)\,f(x,c)\,w(x,y)\,dydx \\
&\quad+f(b,c)\int_a^b\int_c^d (x-a)(d-y)\,w(x,y)\,dydx \\
&=-\int_a^b\int_c^d\left(\int_a^t\int_s^d (x-a)(d-y)\,w(x,y)\,dydx\right)f_{ts}(t,s)\,dsdt
\end{aligned} \tag{2.8}$$

and

$$
\begin{aligned}
(2.9)\quad & \int_a^b \int_c^d (x-a)(y-c) f(x,y) w(x,y)\, dydx \\
& - \int_a^b \int_c^d (x-a)(y-c) f(b,y) w(x,y)\, dydx \\
& - \int_a^b \int_c^d (x-a)(y-c) f(x,d) w(x,y)\, dydx \\
& + f(b,d) \int_a^b \int_c^d (x-a)(y-c) w(x,y)\, dydx \\
& = \int_a^b \int_c^d \left(\int_a^t \int_c^s (x-a)(y-c) w(x,y)\, dydx \right) f_{ts}(t,s)\, dsdt.
\end{aligned}
$$

From (2.6)-(2.9), we get

$$
\begin{aligned}
(2.10)\quad & \frac{1}{(b-a)(d-c)} \left[f(a,c) \int_a^b \int_c^d (b-x)(d-y) w(x,y)\, dydx \right. \\
& + f(a,d) \int_a^b \int_c^d (b-x)(y-c) w(x,y)\, dydx \\
& + f(b,c) \int_a^b \int_c^d (x-a)(d-y) w(x,y)\, dydx \\
& \left. + f(b,d) \int_a^b \int_c^d (x-a)(y-c) w(x,y)\, dydx \right] \\
& - \frac{1}{d-c} \int_a^b \int_c^d (d-y) f(x,c) w(x,y)\, dydx - \frac{1}{d-c} \int_a^b \int_c^d (y-c) f(x,d) w(x,y)\, dydx \\
& - \frac{1}{b-a} \int_a^b \int_c^d (b-x) f(a,y) w(x,y)\, dydx - \frac{1}{b-a} \int_a^b \int_c^d (x-a) f(b,y) w(x,y)\, dydx \\
& + \int_a^b \int_c^d f(x,y) w(x,y)\, dydx \\
& = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left[\int_t^b \int_s^d (b-x)(d-y) w(x,y)\, dydx \right. \\
& - \int_t^b \int_c^s (b-x)(y-c) w(x,y)\, dydx \\
& - \int_a^t \int_s^d (x-a)(d-y) w(x,y)\, dydx \\
& \left. + \int_a^t \int_c^s (x-a)(y-c) w(x,y)\, dydx \right] f_{ts}(t,s)\, dsdt.
\end{aligned}
$$

Now

$$\begin{aligned}(2.11)\quad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left[\int_t^b\int_s^d (b-x)(d-y)\,w(x,y)\,dydx\right.\\
&-\int_t^b\int_c^s (b-x)(y-c)\,w(x,y)\,dydx\\
&-\int_a^t\int_s^d (x-a)(d-y)\,w(x,y)\,dydx\\
&\left.+\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\right] f_{ts}(t,s)\,dsdt\\
&=\frac{1}{(b-a)(d-c)}\int_a^{a_1}\int_c^{c_1}\left[\int_t^b\int_s^d (b-x)(d-y)\,w(x,y)\,dydx\right.\\
&-\int_a^t\int_s^d (x-a)(d-y)\,w(x,y)\,dydx-\int_t^b\int_c^s (b-x)(y-c)\,w(x,y)\,dydx\\
&\left.+\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\right] f_{ts}(t,s)\,dsdt\\
&+\frac{1}{(b-a)(d-c)}\int_{a_1}^{b}\int_c^{c_1}\left[\int_t^b\int_s^d (b-x)(d-y)\,w(x,y)\,dydx\right.\\
&-\int_a^t\int_s^d (x-a)(d-y)\,w(x,y)\,dydx-\int_t^b\int_c^s (b-x)(y-c)\,w(x,y)\,dydx\\
&\left.+\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\right] f_{ts}(t,s)\,dsdt\\
&+\frac{1}{(b-a)(d-c)}\int_a^{a_1}\int_{c_1}^{d}\left[\int_t^b\int_s^d (b-x)(d-y)\,w(x,y)\,dydx\right.\\
&-\int_a^t\int_s^d (x-a)(d-y)\,w(x,y)\,dydx-\int_t^b\int_c^s (b-x)(y-c)\,w(x,y)\,dydx\\
&\left.+\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\right] f_{ts}(t,s)\,dsdt\\
&+\frac{1}{(b-a)(d-c)}\int_{a_1}^{b}\int_{c_1}^{d}\left[\int_t^b\int_s^d (b-x)(d-y)\,w(x,y)\,dydx\right.\\
&-\int_a^t\int_s^d (x-a)(d-y)\,w(x,y)\,dydx-\int_t^b\int_c^s (b-x)(y-c)\,w(x,y)\,dydx\\
&\left.+\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\right] f_{ts}(t,s)\,dsdt.\end{aligned}$$

After making suitable substitutions to the integrals in (2.11) together with (2.10), we get the required identity. □

Remark 1. *If we take* $w(x,y)=\frac{1}{(b-a)(d-c)}$*, for all* $(x,y)\in[a,b]\times[c,d]$*, the (2.1) reduces to*

$$\begin{aligned}(2.12)\quad &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\ &-\frac{1}{2(b-a)}\int_a^b[f(x,c)+f(x,d)]\,dx-\frac{1}{2(d-c)}\int_a^b[f(a,y)+f(b,y)]\,dx\\ &=\frac{(b-a)(d-c)}{16}\Bigg[\int_0^1\int_0^1 tsf_{st}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\ &+\int_0^1\int_0^1(-t)\,sf_{st}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\ &+\int_0^1\int_0^1 t(-s)\,f_{st}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\ &+\int_0^1\int_0^1(-s)(-t)\,f_{st}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\Bigg].\end{aligned}$$

The identity (2.12) was established in [16].

Corollary 1. *If the function* $w(x,y)$ *is symmetric with respect to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *on co-ordinates on* $[a,b]\times[c,d]$*, then*

$$\begin{aligned}(2.13)\quad &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\int_a^b\int_c^d f(x,y)\,w(x,y)\,dydx\\ &-\frac{1}{d-c}\int_a^b\int_c^d(d-y)\,f(x,c)\,w(x,y)\,dydx-\frac{1}{d-c}\int_a^b\int_c^d(y-c)\,f(x,d)\,w(x,y)\,dydx\\ &-\frac{1}{b-a}\int_a^b\int_c^d(b-x)\,f(b,y)\,w(x,y)\,dydx-\frac{1}{b-a}\int_a^b\int_c^d(x-a)\,f(b,y)\,w(x,y)\,dydx\\ &=\int_0^1\int_0^1 H(w,a,c,b,d;t,s)\,f_{ts}(at+(1-t)b,cs+(1-s)d)\,dsdt.\end{aligned}$$

Lemma 3. *Let* $A:C([a,b]\times[c,d])\to\mathbb{R}$ *be a positive linear functional on* $C([a,b]\times[c,d])$ *and let* e_i *and* k_j *be monomials* $e_i(x)=x^i$*,* $k_j(y)=y^j$*,* $x\in[a,b]$*,* $y\in[c,d]$*,* i*,* $j\in\mathbb{N}$*. Let* g *be a convex function on co-ordinates on* $[a,b]\times[c,d]$*, then the following inequality holds*

$$\begin{aligned}(2.14)\quad &A(g(e_1,k_1))\\ &\le\frac{1}{(b-a)(d-c)}\left[A((b-e_1)(d-k_1))\,g(a,c)+A((b-e_1)(k_1-c))\,g(a,d)\right.\\ &\qquad\left.+A((e_1-a)(d-k_1))\,g(b,c)+A((e_1-a)(k_1-c))\,g(b,d)\right].\end{aligned}$$

Proof. By using the convexity of g on co-ordinates on $[a,b]\times[c,d]$, we obtain

$$\begin{aligned}(2.15)\quad g(e_1,k_1)&\le\frac{(b-e_1)\,g(a,k_1)+(e_1-a)\,g(b,k_1)}{b-a}\\ &\le\frac{1}{(b-a)(d-c)}\left[(b-e_1)(d-k_1)\,g(a,c)+(b-e_1)(k_1-c)\,g(a,d)\right.\\ &\qquad\left.(e_1-a)(d-k_1)\,g(b,c)+(e_1-a)(k_1-c)\,g(b,d)\right].\end{aligned}$$

Since A is a positive linear functional, we get the inequality (2.14) by applying A on (2.15). □

3. Main Results

To make the presentation compact, we will use the following notations for the next theorem.

$$B_1(a,c,b,d)=\int_a^{a_1}\int_c^{c_1}(a_1-x)^2(c_1-y)^2w(x,y)\,dydx \tag{3.1}$$
$$-\int_a^{a_1}\int_{c_1}^{d}(a_1-x)^2(c_1-y)^2w(x,y)\,dydx-\int_{a_1}^{b}\int_c^{c_1}(a_1-x)^2(c_1-y)^2w(x,y)\,dydx$$
$$+\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)^2(c_1-y)^2w(x,y)\,dydx,$$

$$B_2(a,c,b,d)=\int_a^{a_1}\int_c^{c_1}(a_1-x)(c_1-y)^2w(x,y)\,dydx \tag{3.2}$$
$$-\int_a^{a_1}\int_{c_1}^{d}(a_1-x)(c_1-y)^2w(x,y)\,dydx$$
$$-\int_{a_1}^{b}\int_c^{c_1}(a_1-x)(c_1-y)^2w(x,y)\,dydx+\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)(c_1-y)^2w(x,y)\,dydx,$$

$$B_3(a,c,b,d)=\int_a^{a_1}\int_c^{c_1}(a_1-x)^2(c_1-y)w(x,y)\,dydx \tag{3.3}$$
$$-\int_a^{a_1}\int_{c_1}^{d}(a_1-x)^2(c_1-y)w(x,y)\,dydx-\int_{a_1}^{b}\int_c^{c_1}(a_1-x)^2(c_1-y)w(x,y)\,dydx$$
$$+\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)^2(c_1-y)w(x,y)\,dydx$$

and

$$B_4(a,c,b,d)=\int_a^{a_1}\int_c^{c_1}(a_1-x)(c_1-y)w(x,y)\,dydx \tag{3.4}$$
$$-\int_a^{a_1}\int_{c_1}^{d}(a_1-x)(c_1-y)w(x,y)\,dydx-\int_{a_1}^{b}\int_c^{c_1}(a_1-x)(c_1-y)w(x,y)\,dydx$$
$$+\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)(c_1-y)w(x,y)\,dydx.$$

Theorem 5. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially differentiable mapping on Ω° and $f_{ts}\in L([a,b]\times[c,d])$, where $[a,b]\times[c,d]\subseteq\Omega^\circ$ with $a<b$, $c<d$.*

If $w : [a,b] \times [c,d] \to [0,\infty)$ be a continuous mapping and $|f_{ts}|$ is convex on co-ordinates on $[a,b] \times [c,d]$, then the following inequality holds

$$\begin{aligned}
&\left| \int_a^b \int_c^d f(x,y)\, w(x,y)\, dydx - \int_a^b \int_c^d f(a_1,y)\, w(x,y)\, dydx \right.\\
&\qquad \left. - \int_a^b \int_c^d f(x,c_1)\, w(x,y)\, dydx + f(a_1,c_1) \right| \\
&\leq \frac{1}{(b-a)(d-c)} \left[|f_{ts}(a,c)|\, A_1(a,c,b,d) + |f_{ts}(a,d)|\, A_2(a,c,b,d) \right.\\
&\qquad \left. + |f_{ts}(b,c)|\, A_3(a,c,b,d) + |f_{ts}(b,d)|\, A_4(a,c,b,d) \right],
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
A_1(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(b-a_1)}{2} B_2(a,c,b,d) \\
&\quad + \frac{(d-c_1)}{2} B_3(a,c,b,d) + (b-a_1)(d-c_1)\, B_4(a,c,b,d),
\end{aligned}$$

$$\begin{aligned}
A_2(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(b-a_1)}{2} B_2(a,c,b,d) \\
&\quad + \frac{(c_1-c)}{2} B_3(a,c,b,d) + (b-a_1)(c_1-c)\, B_4(a,c,b,d),
\end{aligned}$$

$$\begin{aligned}
A_3(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(a_1-a)}{2} B_2(a,c,b,d) \\
&\quad + \frac{(d-c_1)}{2} B_3(a,c,b,d) + (a_1-a)(d-c_1)\, B_4(a,c,b,d)
\end{aligned}$$

and

$$\begin{aligned}
A_4(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(a_1-a)}{2} B_2(a,c,b,d) \\
&\quad + \frac{(c_1-c)}{2} B_3(a,c,b,d) + (a_1-a)(c_1-c)\, B_4(a,c,b,d).
\end{aligned}$$

Proof. We can write

$$\begin{aligned}
&f(x,y) - f(a_1,y) - f(x,c_1) + f(a_1,c_1) \\
&\qquad = \int_a^b \int_c^d \left[\sigma(x-t)\, \sigma(y-s) - \sigma(a_1-t)\, \sigma(y-s) \right.\\
&\qquad\qquad \left. - \sigma(x-t)\, \sigma(c_1-s) + \sigma(a_1-t)\, \sigma(c_1-s) \right] f_{ts}(t,s)\, dsdt
\end{aligned} \tag{3.6}$$

From (3.6), we obtain

$$\begin{aligned}(3.7)\quad &\int_a^b\int_c^d f(x,y)\,w(x,y)\,dydx-\int_a^b\int_c^d f(a_1,y)\,w(x,y)\,dydx\\&-\int_a^b\int_c^d f(x,c_1)\,w(x,y)\,dydx+f(a_1,c_1)\\&=\int_a^b\int_c^d\Bigg(\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\\&-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\Bigg)f_{ts}(t,s)\,dsdt\end{aligned}$$

Taking absolute value on both sides of (3.7) and applying Lemma 3,we have

$$\begin{aligned}(3.8)\quad &\left|\int_a^b\int_c^d f(x,y)\,w(x,y)\,dydx-\int_a^b\int_c^d f(a_1,y)\,w(x,y)\,dydx\right.\\&\left.-\int_a^b\int_c^d f(x,c_1)\,w(x,y)\,dydx+f(a_1,c_1)\right|\\&\le\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right||f_{ts}(t,s)|\,dsdt\\&\le\frac{1}{(b-a)(d-c)}\left[|f_{ts}(a,c)|\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(b-t)(d-s)\,dsdt\\&+|f_{ts}(a,d)|\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(b-t)(s-c)\,dsdt\\&+|f_{ts}(b,c)|\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(t-a)(d-s)\,dsdt\\&+|f_{ts}(b,d)|\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(t-a)(s-c)\,dsdt\right].\end{aligned}$$

Now

$$\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.$$
$$\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(b-t)(d-s)\,dsdt \tag{3.9}$$
$$=\int_a^{a_1}\int_c^{c_1}\left(\int_a^t\int_c^s w(x,y)\,dydx\right)(b-t)(d-s)\,dsdt$$
$$+\int_{a_1}^b\int_c^{c_1}\left(\int_t^b\int_c^s w(x,y)\,dydx\right)(b-t)(d-s)\,dsdt$$
$$+\int_a^{a_1}\int_{c_1}^d\left(\int_a^t\int_s^d w(x,y)\,dydx\right)(b-t)(d-s)\,dsdt$$
$$+\int_{a_1}^b\int_{c_1}^d\left(\int_t^b\int_s^d w(x,y)\,dydx\right)(b-t)(d-s)\,dsdt$$
$$=\frac{(d-c_1)^2(b-a_1)^2}{4}\int_a^{a_1}\int_c^{c_1} w(x,y)\,dydx-\frac{(d-c_1)^2(b-a_1)^2}{4}\int_{a_1}^b\int_c^{c_1} w(x,y)\,dydx$$
$$-\frac{(d-c_1)^2(b-a_1)^2}{4}\int_a^{a_1}\int_{c_1}^d w(x,y)\,dydx+\frac{(d-c_1)^2(b-a_1)^2}{4}\int_{a_1}^b\int_{c_1}^d w(x,y)\,dydx$$
$$-\frac{(d-c_1)^2}{2}\int_a^{a_1}\int_c^{c_1}\frac{(b-x)^2}{2}w(x,y)\,dydx+\frac{(d-c_1)^2}{2}\int_{a_1}^b\int_c^{c_1}\frac{(b-x)^2}{2}w(x,y)\,dydx$$
$$+\frac{(d-c_1)^2}{2}\int_a^{a_1}\int_{c_1}^d\frac{(b-x)^2}{2}w(x,y)\,dydx-\frac{(d-c_1)^2}{2}\int_{a_1}^b\int_{c_1}^d\frac{(b-x)^2}{2}w(x,y)\,dydx$$
$$-\frac{(b-a_1)^2}{2}\int_a^{a_1}\int_c^{c_1}\frac{(d-y)^2}{2}w(x,y)\,dydx+\frac{(b-a_1)^2}{2}\int_{a_1}^b\int_c^{c_1}\frac{(d-y)^2}{2}w(x,y)\,dydx$$
$$+\frac{(b-a_1)^2}{2}\int_a^{a_1}\int_{c_1}^d\frac{(d-y)^2}{2}w(x,y)\,dydx-\frac{(b-a_1)^2}{2}\int_{a_1}^b\int_{c_1}^d\frac{(d-y)^2}{2}w(x,y)\,dydx$$
$$+\int_a^{a_1}\int_c^{c_1}\frac{(d-y)^2(b-x)^2}{4}w(x,y)\,dydx-\int_{a_1}^b\int_c^{c_1}\frac{(d-y)^2(b-x)^2}{4}w(x,y)\,dydx$$
$$-\int_a^{a_1}\int_{c_1}^d\frac{(d-y)^2(b-x)^2}{4}w(x,y)\,dydx+\int_{a_1}^b\int_{c_1}^d\frac{(d-y)^2(b-x)^2}{4}w(x,y)\,dydx.$$

After simplification, we get

$$\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.$$
$$\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(b-t)(d-s)\,dsdt \tag{3.10}$$
$$=\frac{1}{4}B_1(a,c,b,d)+\frac{(b-a_1)}{2}B_2(a,c,b,d)+\frac{(d-c_1)}{2}B_3(a,c,b,d)$$
$$+(b-a_1)(d-c_1)\,B_4(a,c,b,d)=A_1(a,c,b,d).$$

Similarly, one can get that

$$\begin{aligned}(3.11)\quad &\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(b-t)(s-c)\,dsdt\\&=\frac{1}{4}B_1(a,c,b,d)+\frac{(b-a_1)}{2}B_2(a,c,b,d)+\frac{(c_1-c)}{2}B_3(a,c,b,d)\\&\qquad+(b-a_1)(c_1-c)\,B_4(a,c,b,d)=A_2(a,c,b,d),\end{aligned}$$

$$\begin{aligned}(3.12)\quad &\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(a-t)(d-s)\,dsdt\\&=\frac{1}{4}B_1(a,c,b,d)+\frac{(a_1-a)}{2}B_2(a,c,b,d)+\frac{(d-c_1)}{2}B_3(a,c,b,d)\\&\qquad+(a_1-a)(d-c_1)\,B_4(a,c,b,d)=A_3(a,c,b,d)\end{aligned}$$

and

$$\begin{aligned}(3.13)\quad &\int_a^b\int_c^d\left|\int_t^b\int_s^d w(x,y)\,dydx-\sigma(a_1-t)\int_a^b\int_s^d w(x,y)\,dydx\right.\\&\left.-\sigma(c_1-s)\int_t^b\int_c^d w(x,y)\,dydx+\sigma(a_1-t)\,\sigma(c_1-s)\right|(a-t)(s-c)\,dsdt\\&=\frac{1}{4}B_1(a,c,b,d)+\frac{(a_1-a)}{2}B_2(a,c,b,d)+\frac{(c_1-c)}{2}B_3(a,c,b,d)\\&\qquad+(a_1-a)(c_1-c)\,B_4(a,c,b,d)=A_4(a,c,b,d).\end{aligned}$$

Using (3.10)-(3.13) in (3.8), we get the required result. This completes the proof of the theorem. □

Corollary 2. *Suppose that the assumptions of Theorem 5 are satisfied and that $w(x,y)$ is symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates. The the following inequality holds:*

$$\begin{aligned}(3.14)\quad &\left|\int_a^b\int_c^d f(x,y)\,w(x,y)\,dydx-\int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)w(x,y)\,dydx\right.\\&\left.-\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)w(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\&\le\left[|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|\right]\\&\qquad\times\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)w(x,y)\,dydx.\end{aligned}$$

Proof. Since the function $w(x,y)$ is symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates. Hence the function $(a_1-x)^2(c_1-y)^2\,w(x,y)$ is symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates, the function $(a_1-x)(c_1-y)^2 w(x,y)$ symmetric with respect to $\frac{c+d}{2}$ and the function $(a_1-x)^2(c_1-y)w(x,y)$ is symmetric with respect to $\frac{a+b}{2}$. Therefore, we have

$$\int_a^{a_1}\int_c^{c_1}(a_1-x)^2(c_1-y)^2 w(x,y)\,dydx=\int_a^{a_1}\int_{c_1}^{d}(a_1-x)^2(c_1-y)^2 w(x,y)\,dydx$$
$$=\int_{a_1}^{b}\int_c^{c_1}(a_1-x)^2(c_1-y)^2 w(x,y)\,dydx=\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)^2(c_1-y)^2 w(x,y)\,dydx,$$
$$\int_a^{a_1}\int_c^{c_1}(a_1-x)(c_1-y)^2 w(x,y)\,dydx=\int_a^{a_1}\int_{c_1}^{d}(a_1-x)(c_1-y)^2 w(x,y)\,dydx,$$
$$\int_{a_1}^{b}\int_c^{c_1}(a_1-x)(c_1-y)^2 w(x,y)\,dydx=\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)(c_1-y)^2 w(x,y)\,dydx,$$
$$\int_a^{a_1}\int_c^{c_1}(a_1-x)^2(c_1-y)w(x,y)\,dydx=\int_a^{a_1}\int_{c_1}^{d}(a_1-x)^2(c_1-y)w(x,y)\,dydx$$

and

$$\int_{a_1}^{b}\int_c^{c_1}(a_1-x)^2(c_1-y)w(x,y)\,dydx=\int_{a_1}^{b}\int_{c_1}^{d}(a_1-x)^2(c_1-y)w(x,y)\,dydx.$$

From the above equations, we obtain

$$B_1(a,c,b,d)=B_2(a,c,b,d)=B_3(a,c,b,d)=0$$

and

$$B_4(a,c,b,d)=4\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)w(x,y)\,dydx.$$

Applying the above quantities in (3.10), we get (3.14). □

Remark 2. *If we take* $w(x,y)=\frac{g(x,y)}{\int_a^b\int_c^d g(x,y)dydx}$ *in (3.5) and* $g(x,y)$ *is symmetric with respect to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *on co-ordinates, then we get the inequality (1.3).*

Theorem 6. *Let* $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially differentiable mapping on* Ω° *and* $f_{ts}\in L([a,b]\times[c,d])$*, where* $[a,b]\times[c,d]\subseteq\Omega^\circ$ *with* $a<b$, $c<d$. *If* $w:[a,b]\times[c,d]\to[0,\infty)$ *be a continuous mapping and* $|f_{ts}|^q$ *is convex on co-ordinates on* $[a,b]\times[c,d]$ *for* $q\geq 1$*, then the following inequality holds*

$$\left|\int_a^b\int_c^d f(x,y)w(x,y)\,dydx-\int_a^b\int_c^d f(a_1,y)w(x,y)\,dydx\right.$$
$$\left.-\int_a^b\int_c^d f(x,c_1)w(x,y)\,dydx+f(a_1,c_1)\right|$$
$$\leq 4^{1-\frac{1}{q}}\left(\int_{a_1}^{b}\int_{c_1}^{d}(x-a_1)(y-c_1)w(x,y)\,dydx\right)^{1-\frac{1}{q}}\left(\frac{1}{(b-a)(d-c)}\right)^{\frac{1}{q}}$$
$$\times\left[|f_{ts}(a,c)|^q A_1(a,c,b,d)+|f_{ts}(a,d)|^q A_2(a,c,b,d)\right.$$
$$\left.+|f_{ts}(b,c)|^q A_3(a,c,b,d)+|f_{ts}(b,d)|^q A_4(a,c,b,d)\right]^{\frac{1}{q}},\tag{3.15}$$

where $A_1(a,c,b,d)$, $A_2(a,c,b,d)$, $A_3(a,c,b,d)$ *and* $A_4(a,c,b,d)$ *are defined in Theorem 5.*

Proof. An application of Hölder inequality in (3.8) yields that

$$\begin{aligned}
(3.16)\quad & \left| \int_a^b \int_c^d f(x,y)\, w(x,y)\, dydx - \int_a^b \int_c^d f(a_1,y)\, w(x,y)\, dydx \right. \\
& \left. - \int_a^b \int_c^d f(x,c_1)\, w(x,y)\, dydx + f(a_1,c_1) \right| \\
& \leq \left(\int_a^b \int_c^d \left| \int_t^b \int_s^d w(x,y)\, dydx - \sigma(a_1-t) \int_a^b \int_s^d w(x,y)\, dydx \right. \right. \\
& \left. \left. -\sigma(c_1-s) \int_t^b \int_c^d w(x,y)\, dydx + \sigma(a_1-t)\,\sigma(c_1-s) \right| dsdt \right)^{1-\frac{1}{q}} \\
& \times \left(\int_a^b \int_c^d \left| \int_t^b \int_s^d w(x,y)\, dydx - \sigma(a_1-t) \int_a^b \int_s^d w(x,y)\, dydx \right. \right. \\
& \left. \left. -\sigma(c_1-s) \int_t^b \int_c^d w(x,y)\, dydx + \sigma(a_1-t)\,\sigma(c_1-s) \right| |f_{ts}(t,s)|^q\, dsdt \right)^{\frac{1}{q}}.
\end{aligned}$$

Applying Lemma 3, we have

$$\begin{aligned}
(3.17)\quad & \int_a^b \int_c^d \left| \int_t^b \int_s^d w(x,y)\, dydx - \sigma(a_1-t) \int_a^b \int_s^d w(x,y)\, dydx \right. \\
& \left. -\sigma(c_1-s) \int_t^b \int_c^d w(x,y)\, dydx + \sigma(a_1-t)\,\sigma(c_1-s) \right| |f_{ts}(t,s)|^q\, dsdt \\
& \leq \frac{1}{(b-a)(d-c)} \left[|f_{ts}(a,c)|^q A_1(a,c,b,d) + |f_{ts}(a,d)|^q A_2(a,c,b,d) \right. \\
& \left. + |f_{ts}(b,c)|^q A_3(a,c,b,d) + |f_{ts}(b,d)|^q A_4(a,c,b,d) \right].
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
(3.18)\quad & \int_a^b \int_c^d \left| \int_t^b \int_s^d w(x,y)\, dydx - \sigma(a_1-t) \int_a^b \int_s^d w(x,y)\, dydx \right. \\
& \left. -\sigma(c_1-s) \int_t^b \int_c^d w(x,y)\, dydx + \sigma(a_1-t)\,\sigma(c_1-s) \right| dsdt \\
& = \int_a^{a_1} \int_c^{c_1} \left(\int_a^t \int_c^s w(x,y)\, dydx \right) dsdt + \int_{a_1}^b \int_c^{c_1} \left(\int_t^b \int_c^s w(x,y)\, dydx \right) dsdt \\
& + \int_a^{a_1} \int_{c_1}^d \left(\int_a^t \int_s^d w(x,y)\, dydx \right) dsdt + \int_{a_1}^b \int_{c_1}^d \left(\int_t^b \int_s^d w(x,y)\, dydx \right) dsdt \\
& = 4 \int_{a_1}^b \int_{c_1}^d (x-a_1)(y-c_1)\, w(x,y)\, dydx.
\end{aligned}$$

A combination of (3.16)-(3.18) gives (3.15). This completes the proof of the theorem. □

Remark 3. *If $w(x,y)$ is symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates, then from (3.15), we obtain the following inequality*

$$\left|\int_a^b\int_c^d f(x,y)\,w(x,y)\,dydx-\int_a^b\int_c^d f\left(\frac{a+b}{2},y\right)w(x,y)\,dydx\right.$$
$$\left.-\int_a^b\int_c^d f\left(x,\frac{c+d}{2}\right)w(x,y)\,dydx+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|$$
$$\leq 4\left[\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right]^{\frac{1}{q}}$$
$$\times\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)w(x,y)\,dydx. \tag{3.19}$$

Remark 4. *If $w(x,y)=\frac{g(x,y)}{\int_a^b\int_c^d g(x,y)dydx}$ and $g(x,y)$ is symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates, then the inequality (3.15) reduces to the result proved in* [18].

For our next results, we use the following notations.

$$\Psi(w,f):=\frac{1}{(b-a)(d-c)}\left[f(a,c)\int_a^b\int_c^d(b-x)(d-y)\,w(x,y)\,dydx\right.$$
$$+f(a,d)\int_a^b\int_c^d(b-x)(y-c)\,w(x,y)\,dydx+f(b,c)\int_a^b\int_c^d(x-a)(d-y)\,w(x,y)\,dydx$$
$$\left.+f(b,d)\int_a^b\int_c^d(x-a)(y-c)\,w(x,y)\,dydx\right]+\int_a^b\int_c^d f(x,y)\,w(x,y)\,dydx$$
$$-\frac{1}{b-a}\int_a^b\int_c^d(b-x)f(a,y)\,w(x,y)\,dydx-\frac{1}{b-a}\int_a^b\int_c^d(x-a)f(b,y)\,w(x,y)\,dydx$$
$$-\frac{1}{d-c}\int_a^b\int_c^d(d-y)f(x,c)\,w(x,y)\,dydx-\frac{1}{d-c}\int_a^b\int_c^d(y-c)f(x,d)\,w(x,y)\,dydx. \tag{3.20}$$

It is clear from (3.20) that

$$\Psi\left(\frac{1}{(b-a)(d-c)},f\right):=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$
$$-\frac{1}{2(b-a)}\int_a^b[f(x,d)+f(x,c)]\,dx-\frac{1}{2(d-c)}\int_c^d[f(a,y)+f(b,y)]\,dy$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx. \tag{3.21}$$

We also use the following notations

$$\begin{aligned}
\sup\left\{f_{rz}(a,c), f_{rz}(a_1,c), f_{rz}(a,c_1), f_{rz}(a_1,c_1)\right\} &= \eta_1(a,c,b,d),\\
\sup\left\{f_{rz}(a_1,c), f_{rz}(a_1,c_1), f_{rz}(b,c), f_{rz}(b,c_1)\right\} &= \eta_2(a,c,b,d),\\
\sup\left\{f_{rz}(a,c_1), f_{rz}(a,d), f_{rz}(a_1,c_1), f_{rz}(a_1,d)\right\} &= \eta_3(a,c,b,d),\\
\sup\left\{f_{rz}(a_1,c_1), f_{rz}(a_1,d), f_{rz}(b,c_1), f_{rz}(b,d)\right\} &= \eta_4(a,c,b,d).
\end{aligned}$$

The next result gives upper bound of $|\Psi(w,f)|$ when the function $f(x,y)$ is quasi-convex on co-ordinates.

Theorem 7. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially differentiable mapping on Ω° and $f_{ts}\in L([a,b]\times[c,d])$, where $[a,b]\times[c,d]\subseteq\Omega^\circ$ with $a<b$, $c<d$. If $w:[a,b]\times[c,d]\to[0,\infty)$ be a continuous mapping and $|f_{ts}|$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds*

(3.22)

$$\begin{aligned}
|\Psi(w,f)| &\le \frac{(a_1-a)(c_1-c)}{(b-a)(d-c)}\eta_1(a,c,b,d)\int_0^1\int_0^1 |H(w,a,c,a_1,c_1;r,z)|\,dzdr\\
&+\frac{(b-a_1)(c_1-c)}{(b-a)(d-c)}\eta_2(a,c,b,d)\int_0^1\int_0^1 |H(w,a_1,c,b,c_1;r,z)|\,dzdr\\
&+\frac{(a_1-a)(d-c_1)}{(b-a)(d-c)}\eta_3(a,c,b,d)\int_0^1\int_0^1 |H(w,a,c_1,a_1,d;r,z)|\,dzdr\\
&+\frac{(b-a_1)(d-c_1)}{(b-a)(d-c)}\eta_4(a,c,b,d)\int_0^1\int_0^1 |H(w,a_1,c_1,b,d;r,z)|\,dzdr,
\end{aligned}$$

Proof. Since $|f_{ts}|$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$, we have

$$\begin{aligned}
&f_{rz}((1-r)a+a_1r,(1-z)c+c_1z)\\
&\le\sup\left\{f_{rz}(a,c), f_{rz}(a_1,c), f_{rz}(a,c_1), f_{rz}(a_1,c_1)\right\}=\eta_1(a,c,b,d),
\end{aligned}$$

$$\begin{aligned}
&f_{rz}((1-r)a_1+br,(1-z)c+c_1z)\\
&\le\sup\left\{f_{rz}(a_1,c), f_{rz}(a_1,c_1), f_{rz}(b,c), f_{rz}(b,c_1)\right\}=\eta_2(a,c,b,d),
\end{aligned}$$

$$\begin{aligned}
&f_{rz}((1-r)a+a_1r,(1-z)c_1+dz)\\
&\le\sup\left\{f_{rz}(a,c_1), f_{rz}(a,d), f_{rz}(a_1,c_1), f_{rz}(a_1,d)\right\}=\eta_3(a,c,b,d)
\end{aligned}$$

and

$$\begin{aligned}
&f_{rz}((1-r)a_1+br,(1-z)c_1+dz)\\
&\le\sup\left\{f_{rz}(a_1,c_1), f_{rz}(a_1,d), f_{rz}(b,c_1), f_{rz}(b,d)\right\}=\eta_4(a,c,b,d)
\end{aligned}$$

for all $(r,z)\in[0,1]\times[0,1]$. Hence the inequality (3.22) follows from (2.1). □

Theorem 8. *Let $f:\Omega\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially differentiable mapping on Ω° and $f_{ts}\in L([a,b]\times[c,d])$, where $[a,b]\times[c,d]\subseteq\Omega^\circ$ with $a<b$, $c<d$. If $w:[a,b]\times[c,d]\to[0,\infty)$ is a continuous mapping symmetric with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates and $|f_{ts}|$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$,*

then the following inequality holds

$$
\begin{aligned}
(3.23)\quad & \Bigg| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \int_a^b \int_c^d f(x,y)\, w(x,y)\, dydx \\
& - \frac{1}{b-a}\int_a^b \int_c^d (b-x) f(a,y) w(x,y)\, dydx - \frac{1}{b-a}\int_a^b \int_c^d (x-a) f(b,y) w(x,y)\, dydx \\
& - \frac{1}{d-c}\int_a^b \int_c^d (d-y) f(x,c) w(x,y)\, dydx - \frac{1}{d-c}\int_a^b \int_c^d (y-c) f(x,d) w(x,y)\, dydx \Bigg| \\
& \leq \left(\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (x-a)(y-c)\, w(x,y)\, dydx \right) \\
& \times \Bigg[\sup \left\{ f_{rz}(a,c), f_{rz}\left(\frac{a+b}{2}, c\right), f_{rz}\left(a, \frac{c+d}{2}\right), f_{rz}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right\} \\
& + \sup \left\{ f_{rz}\left(\frac{a+b}{2}, c\right), f_{rz}\left(\frac{a+b}{2}, \frac{c+d}{2}\right), f_{rz}(b,c), f_{rz}\left(b, \frac{c+d}{2}\right) \right\} \\
& + \sup \left\{ f_{rz}\left(a, \frac{c+d}{2}\right), f_{rz}(a,d), f_{rz}\left(\frac{a+b}{2}, \frac{c+d}{2}\right), f_{rz}\left(\frac{a+b}{2}, d\right) \right\} \\
& + \sup \left\{ f_{rz}\left(\frac{a+b}{2}, \frac{c+d}{2}\right), f_{rz}\left(\frac{a+b}{2}, d\right), f_{rz}\left(b, \frac{c+d}{2}\right), f_{rz}(b,d) \right\} \Bigg].
\end{aligned}
$$

Proof. The symmetry of $w(x,y)$ with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ on co-ordinates gives

$$
\begin{aligned}
(3.24)\quad \Psi(w,f) &= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \\
& + \int_a^b \int_c^d f(x,y)\, w(x,y)\, dydx + \int_a^b \int_c^d f(x,y)\, w(x,y)\, dydx \\
& - \frac{1}{b-a}\int_a^b \int_c^d (b-x) f(a,y) w(x,y)\, dydx - \frac{1}{b-a}\int_a^b \int_c^d (x-a) f(b,y) w(x,y)\, dydx \\
& - \frac{1}{d-c}\int_a^b \int_c^d (d-y) f(x,c) w(x,y)\, dydx - \frac{1}{d-c}\int_a^b \int_c^d (y-c) f(x,d) w(x,y)\, dydx.
\end{aligned}
$$

We also observe that

$$
\begin{aligned}
& \frac{(a_1-a)(c_1-c)}{(b-a)(d-c)} \int_0^1 \int_0^1 |H(w,a,c,a_1,c_1;r,z)|\, dzdr \\
& = \frac{1}{(b-a)(d-c)} \int_a^{a_1} \int_c^{c_1} \Bigg| \int_a^t \int_c^s (x-a)(y-c)\, w(x,y)\, dydx \\
& - \int_a^t \int_s^d (x-a)(y-c)\, w(x,y)\, dydx - \int_t^b \int_c^s (x-a)(y-c)\, w(x,y)\, dydx \\
& + \int_t^d \int_s^c (x-a)(y-c)\, w(x,y)\, dydx \Bigg| dsdt,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(b-a_1)(c_1-c)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a_1,c,b,c_1;r,z)|\,dzdr\\
&=\frac{1}{(b-a)(d-c)}\int_{a_1}^{b}\int_{c}^{c_1}\bigg|\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&-\int_a^t\int_s^d (x-a)(y-c)\,w(x,y)\,dydx-\int_t^b\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&+\int_t^d\int_s^c (x-a)(y-c)\,w(x,y)\,dydx\bigg|\,dsdt,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(a_1-a)(d-c_1)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a_1,c,b,c_1;r,z)|\,dzdr\\
&=\frac{1}{(b-a)(d-c)}\int_{a}^{a_1}\int_{c_1}^{d}\bigg|\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&-\int_a^t\int_s^d (x-a)(y-c)\,w(x,y)\,dydx-\int_t^b\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&+\int_t^d\int_s^c (x-a)(y-c)\,w(x,y)\,dydx\bigg|\,dsdt
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(b-a_1)(d-c_1)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a_1,c_1,b,d;r,z)|\,dzdr\\
&=\frac{1}{(b-a)(d-c)}\int_{a_1}^{b}\int_{c_1}^{d}\bigg|\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&-\int_a^t\int_s^d (x-a)(y-c)\,w(x,y)\,dydx-\int_t^b\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&+\int_t^d\int_s^c (x-a)(y-c)\,w(x,y)\,dydx\bigg|\,dsdt.
\end{aligned}
$$

Consider the function $p:[a,b]\times[c,d]\to\mathbb{R}$ defined by

$$
\begin{aligned}
p(t,s)=&\int_a^t\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&-\int_a^t\int_s^d (x-a)(y-c)\,w(x,y)\,dydx-\int_t^b\int_c^s (x-a)(y-c)\,w(x,y)\,dydx\\
&+\int_t^d\int_s^c (x-a)(y-c)\,w(x,y)\,dydx.
\end{aligned}
$$

Then

$$
p_{ts}(t,s)=(b-a)(d-c)\,w(t,s)>0,(t,s)\in[a,b]\times[c,d]
$$

This shows that $p(t,s)$ is an increasing function on $[a,b]\times[c,d]$ and

$$
p(a_1,c_1)=0.
$$

Now it is easy to see that

$$\begin{aligned}
&\frac{(a_1-a)(c_1-c)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a,c,a_1,c_1;r,z)|\,dzdr \\
&=\frac{(b-a_1)(c_1-c)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a_1,c,b,c_1;r,z)|\,dzdr \\
&=\frac{(a_1-a)(d-c_1)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a_1,c,b,c_1;r,z)|\,dzdr \\
&=\frac{(b-a_1)(d-c_1)}{(b-a)(d-c)}\int_0^1\int_0^1 |H(w,a_1,c_1,b,d;r,z)|\,dzdr \\
&=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} (x-a)(y-c)\,w(x,y)\,dydx.
\end{aligned}$$

Hence the inequality (3.23) follows from the inequality (3.22). □

4. Applications to Random Variables

Let $0<a<b$, $0<c<d$, $\alpha,\beta\in\mathbb{R}$ and let X and Y be two independent continuous random variables having the bi-variate continuous probability density function $w:[a,b]\times[c,d]\to[0,\infty)$. The α-moment of X and the β-moment of Y about the origin are respectively defined as follows

$$E_\alpha(X)=\int_a^b t^\alpha w_1(t)\,dt,\; E_\beta(Y)=\int_c^d s^\beta w_2(s)\,ds$$

which are assumed to be finite, here $w_1:[a,b]\to[0,\infty)$ and $w_2:[c,d]\to[0,\infty)$ are the marginal probability density functions of X and Y. Since X and Y are independent random variables, we have

$$w(t,s)=w_1(t)\,w_2(s)$$

for all $(t,s)\in[a,b]\times[c,d]$. Hence

$$\begin{aligned}
E_{\alpha,\beta}(XY)&=\int_a^b\int_c^d t^\alpha s^\beta w(t,s)\,dsdt=\left(\int_a^b t^\alpha w_1(t)\,dt\right)\left(\int_c^d s^\beta w_2(s)\,ds\right)\\
&=E_\alpha(X)\,E_\beta(Y).
\end{aligned}$$

With the above notations, it is obvious that

$$\begin{aligned}
B_1(a,c,b,d)&=\int_a^{E(X)}\int_c^{E(Y)}(E(X)-x)^2(E(Y)-y)^2\,w(x,y)\,dydx\\
&-\int_a^{E(X)}\int_{E(Y)}^{d}(E(X)-x)^2(E(Y)-y)^2\,w(x,y)\,dydx\\
&-\int_{E(X)}^{b}\int_c^{E(Y)}(E(X)-x)^2(E(Y)-y)^2\,w(x,y)\,dydx\\
&\qquad+\int_{E(X)}^{b}\int_{E(Y)}^{d}(E(X)-x)^2(E(Y)-y)^2\,w(x,y)\,dydx,
\end{aligned}$$

$$
\begin{aligned}
B_2(a,c,b,d) &= \int_a^{E(X)} \int_c^{E(Y)} (E(X)-x)(E(Y)-y)^2 w(x,y)\,dydx \\
&\quad - \int_a^{E(X)} \int_{E(Y)}^{d} (E(X)-x)(E(Y)-y)^2 w(x,y)\,dydx \\
&\quad - \int_{E(X)}^{b} \int_c^{E(Y)} (E(X)-x)(E(Y)-y)^2 w(x,y)\,dydx \\
&\qquad + \int_{E(X)}^{b} \int_{E(Y)}^{d} (E(X)-x)(E(Y)-y)^2 w(x,y)\,dydx,
\end{aligned}
$$

$$
\begin{aligned}
B_3(a,c,b,d) &= \int_a^{E(X)} \int_c^{E(Y)} (E(X)-x)^2 (E(Y)-y) w(x,y)\,dydx \\
&\quad - \int_a^{E(X)} \int_{E(Y)}^{d} (E(X)-x)^2 (E(Y)-y) w(x,y)\,dydx \\
&\quad - \int_{E(X)}^{b} \int_c^{E(Y)} (E(X)-x)^2 (E(Y)-y) w(x,y)\,dydx \\
&\qquad + \int_{E(X)}^{b} \int_{E(Y)}^{d} (E(X)-x)^2 (E(Y)-y) w(x,y)\,dydx
\end{aligned}
$$

and

$$
\begin{aligned}
B_4(a,c,b,d) &= \int_a^{E(X)} \int_c^{E(Y)} (E(X)-x)(E(Y)-y) w(x,y)\,dydx \\
&\quad - \int_a^{E(X)} \int_{E(Y)}^{d} (E(X)-x)(E(Y)-y) w(x,y)\,dydx \\
&\quad - \int_{E(X)}^{b} \int_c^{E(Y)} (E(X)-x)(c_1-y) w(x,y)\,dydx \\
&\qquad + \int_{E(X)}^{b} \int_{E(Y)}^{d} (E(X)-x)(E(Y)-y) w(x,y)\,dydx.
\end{aligned}
$$

Moreover, we get that

$$
\begin{aligned}
A_1(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(b-E(X))}{2} B_2(a,c,b,d) \\
&\quad + \frac{(d-E(Y))}{2} B_3(a,c,b,d) + (b-E(X))(d-E(Y)) B_4(a,c,b,d),
\end{aligned}
$$

$$
\begin{aligned}
A_2(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(b-E(X))}{2} B_2(a,c,b,d) \\
&\quad + \frac{(E(Y)-c)}{2} B_3(a,c,b,d) + (b-E(X))(E(Y)-c) B_4(a,c,b,d),
\end{aligned}
$$

$$
\begin{aligned}
A_3(a,c,b,d) &= \frac{1}{4} B_1(a,c,b,d) + \frac{(E(X)-a)}{2} B_2(a,c,b,d) \\
&\quad + \frac{(d-E(Y))}{2} B_3(a,c,b,d) + (E(X)-a)(d-E(Y)) B_4(a,c,b,d)
\end{aligned}
$$

and

$$A_4(a,c,b,d) = \frac{1}{4}B_1(a,c,b,d) + \frac{(E(X)-a)}{2}B_2(a,c,b,d) + \frac{(E(Y)-c)}{2}B_3(a,c,b,d) + (E(X)-a)(E(Y)-c)B_4(a,c,b,d).$$

Now we give some applications of our result to random variables.

Theorem 9. *The inequality*

$$\left|E_\alpha(X)E_\beta(Y) - [E(X)]^\alpha E_\beta(Y) - E_\alpha(X)[E(Y)]^\beta + [E(X)]^\alpha [E(Y)]^\beta\right| \tag{4.1}$$
$$\leq \frac{\alpha\beta}{(b-a)(d-c)}\left[a^{\alpha-1}c^{\beta-1}A_1(a,c,b,d) + a^{\alpha-1}d^{\beta-1}A_2(a,c,b,d) + b^{\alpha-1}c^{\beta-1}A_3(a,c,b,d) + b^{\alpha-1}b^{\beta-1}A_4(a,c,b,d)\right],$$

holds for $0 < a < b$, $0 < c < d$ *and* $\alpha, \beta \geq 2$, *where* $A_1(a,c,b,d)$, $A_2(a,c,b,d)$, $A_3(a,c,b,d)$ *and* $A_4(a,c,b,d)$ *are defined above.*

Proof. Let $f(t,s) = t^\alpha s^\beta$ on $[a,b]\times[c,d]$ for $\alpha,\beta \geq 2$, we observe that $|f_{ts}(t,s)| = \alpha\beta t^{\alpha-1}s^{\beta-1}$ is convex on co-ordinates on $[a,b]\times[c,d]$.
Now

$$\int_a^b\int_c^d f(t,s)w(t,s)\,dsdt = \int_a^b\int_c^d t^\alpha s^\beta w_1(t)w_2(s)\,dsdt = \left(\int_a^b t^\alpha w_1(t)\,dt\right)\left(\int_c^d s^\beta w_2(s)\,ds\right) = E_\alpha(X)E_\beta(Y)$$

$$\int_a^b\int_c^d f(a_1,t)w(t,s)\,dsdt = [E(X)]^\alpha E_\beta(Y),$$
$$\int_a^b\int_c^d f(t,c_1)w(t,s)\,dsdt = E_\alpha(X)[E(Y)]^\beta$$

and

$$f(a_1,c_1) = [E(X)]^\alpha [E(Y)]^\beta.$$

Hence the inequality (4.1) follows from the inequality (3.5). □

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