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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR HA-CONVEX FUNCTIONS

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Abstract. Some new inequalities of Hermite-Hadamard type for HA-convex functions defined on positive intervals are given.

1. Introduction

Following [4] (see also [40]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t) f(x) + t f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be *HA-concave* or *harmonically concave*.

In order to avoid any confusion with the class of *AH-convex* functions, namely the functions satisfying the condition

$$f((1-t)x+ty) \leq \frac{f(x) f(y)}{(1-t) f(y) + t f(x)}, \tag{1.2}$$

we call the class of functions satisfying (1.1) as *HA-convex functions*.

If $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

If $[a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then f is HA-convex on $[a,b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$. Therefore, as examples of HA-convex functions we can take $f(t) = g\left(\frac{1}{t}\right)$, where g is any convex function on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

For a convex function $h : [c,d] \to \mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d h(t)\,dt \leq \frac{h(c)+h(d)}{2}. \tag{1.3}$$

For related results, see [1]-[20], [23]-[26], [27]-[36] and [37]-[48].

If we write the Hermite-Hadamard inequality for the convex function $g(t) = f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b}, \frac{1}{a}\right]$, then we have

$$f\left(\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}\right) \leq \frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right) dt \leq \frac{f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)}{2}$$

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that is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt\leq\frac{f(b)+f(a)}{2}. \tag{1.4}$$

Using the change of variable $s=\frac{1}{t}$, then

$$\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{t}\right)dt=\int_a^b\frac{f(s)}{s^2}ds$$

and by (1.4) we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(s)}{s^2}ds\leq\frac{f(b)+f(a)}{2}. \tag{1.5}$$

The inequality (1.5) has been obtained in a different manner in [40] by I. Işcan.

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for HA-convex functions.

2. A Refinement

We have the following representation result, see [25]. For the sake of completeness we give here a simple proof.

Lemma 1. *Let $g:[x,y]\subset\mathbb{R}\to\mathbb{C}$ be a Lebesgue integrable function on $[x,y]$. Then for any $\lambda\in[0,1]$ we have the representation*

$$\int_0^1 g[(1-t)x+ty]\,dt=(1-\lambda)\int_0^1 g[(1-t)((1-\lambda)x+\lambda y)+ty]\,dt \tag{2.1}$$
$$+\lambda\int_0^1 g[(1-t)x+t((1-\lambda)x+\lambda y)]\,dt.$$

Proof. For $\lambda=0$ and $\lambda=1$ the equality (2.1) is obvious.

Let $\lambda\in(0,1)$. Observe that

$$\int_0^1 g[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt$$
$$=\int_0^1 g[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt$$

and

$$\int_0^1 g[t(\lambda y+(1-\lambda)x)+(1-t)x]\,dt=\int_0^1 g[t\lambda y+(1-\lambda t)x]\,dt.$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)du$. Then

$$\int_0^1 g[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt=\frac{1}{1-\lambda}\int_\lambda^1 g[uy+(1-u)x]\,du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 g[t\lambda y+(1-\lambda t)x]\,dt=\frac{1}{\lambda}\int_0^\lambda g[uy+(1-u)x]\,du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 g\left[(1-t)(\lambda y+(1-\lambda)x)+ty\right]dt\\
&+\lambda\int_0^1 g\left[t(\lambda y+(1-\lambda)x)+(1-t)x\right]dt\\
&=\int_\lambda^1 g\left[uy+(1-u)x\right]du+\int_0^\lambda g\left[uy+(1-u)x\right]du\\
&=\int_0^1 g\left[uy+(1-u)x\right]du
\end{aligned}$$

and the identity (2.1) is proved. □

Corollary 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda\in[0,1]$, then we have the representation*

$$\begin{aligned}
\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt &= (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)((1-\lambda)a+\lambda b)+tb}\right)dt\\
&\quad+\lambda\int_0^1 f\left(\frac{ab}{(1-t)a+t((1-\lambda)a+\lambda b)}\right)dt.
\end{aligned}\tag{2.2}$$

Proof. Consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, $s\in\left[\frac{1}{b},\frac{1}{a}\right]$.

We have by (2.1) for g and $x=\frac{1}{b}$, $y=\frac{1}{a}$ that

$$\begin{aligned}
&\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt\\
&=\int_0^1 f\left(\frac{1}{(1-t)\frac{1}{b}+t\frac{1}{a}}\right)dt\\
&=\int_0^1 g\left((1-t)\frac{1}{b}+t\frac{1}{a}\right)dt\\
&=(1-\lambda)\int_0^1 g\left[(1-t)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}\right]dt\\
&+\lambda\int_0^1 g\left[(1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)\right]dt\\
&=(1-\lambda)\int_0^1 f\left(\frac{1}{(1-t)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}}\right)dt\\
&+\lambda\int_0^1 f\left(\frac{1}{(1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)}\right)dt\\
&=(1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)((1-\lambda)a+\lambda b)+tb}\right)dt\\
&+\lambda\int_0^1 f\left(\frac{ab}{(1-t)a+t((1-\lambda)a+\lambda b)}\right)dt.
\end{aligned}\tag{2.3}$$

□

The following result holds.

Theorem 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then for any $\lambda\in[0,1]$ we have the inequalities*

$$f\left(\frac{2ab}{a+b}\right)\leq(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{2.4}$$
$$\leq\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt$$
$$\leq\frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right]$$
$$\leq\frac{f(a)+f(b)}{2}.$$

Proof. Consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, $s\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Since g is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$, then by Hermite-Hadamard inequality for convex functions we have for $\lambda\in[0,1]$

$$g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}\right)=g\left(\frac{(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2}\right) \tag{2.5}$$
$$\leq\int_0^1 g\left((1-t)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}\right)dt$$
$$\leq\frac{g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+g\left(\frac{1}{a}\right)}{2}$$
$$=\frac{g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)+g\left(\frac{1}{a}\right)}{2}$$

and

$$g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right)=g\left(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2}\right) \tag{2.6}$$
$$\leq\int_0^1 g\left((1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)\right)dt$$
$$\leq\frac{g\left(\frac{1}{b}\right)+g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)}{2}$$
$$=\frac{g\left(\frac{1}{b}\right)+g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)}{2}.$$

If we multiply (2.5) by $(1-\lambda)$ and 2.6 by λ, add the obtained inequalities and use the first part of the equality (2.3) we get

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{2.7}$$
$$\leq\int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt$$
$$\leq(1-\lambda)\frac{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+f(a)}{2}+\lambda\frac{f(b)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{2}$$
$$=\frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right].$$

By the convexity of g we have

$$\begin{aligned}
&(1-\lambda) f\left(\frac{2ab}{(1-\lambda) a+(\lambda+1) b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda) a+\lambda b}\right) \\
&=(1-\lambda) g\left(\frac{(1-\lambda) a+(\lambda+1) b}{2ab}\right)+\lambda g\left(\frac{(2-\lambda) a+\lambda b}{2ab}\right) \\
&\geq g\left(\frac{(1-\lambda)[(1-\lambda) a+(\lambda+1) b]}{2ab}+\frac{\lambda[(2-\lambda) a+\lambda b]}{2ab}\right) \\
&=g\left(\frac{a+b}{2ab}\right)=f\left(\frac{2ab}{a+b}\right)
\end{aligned}$$

and

$$\begin{aligned}
&f\left(\frac{ab}{(1-\lambda) a+\lambda b}\right)+(1-\lambda) f(a)+\lambda f(b) \\
&=g\left((1-\lambda) \frac{1}{b}+\lambda \frac{1}{a}\right)+(1-\lambda) f(a)+\lambda f(b) \\
&\leq(1-\lambda) f(b)+\lambda f(a)+(1-\lambda) f(a)+\lambda f(b) \\
&=f(a)+f(b)
\end{aligned}$$

and the desired inequality (2.4) is proved. □

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
f\left(\frac{2ab}{a+b}\right) &\leq \frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right] \leq \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \\
&\leq \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.
\end{aligned} \tag{2.8}$$

3. New Results

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f: I \rightarrow \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x<y$, then $f_{-}^{\prime}(x) \leq f_{+}^{\prime}(x) \leq f_{-}^{\prime}(y) \leq f_{+}^{\prime}(y)$ which shows that both $f_{-}^{\prime}$ and $f_{+}^{\prime}$ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f: I \rightarrow \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi: I \rightarrow[-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a)+(x-a) \varphi(a) \text{ for any } x, a \in I. \tag{3.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $f_{-}^{\prime}$, $f_{+}^{\prime} \in \partial f$ and if $\varphi \in \partial f$, then

$$f_{-}^{\prime}(x) \leq \varphi(x) \leq f_{+}^{\prime}(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f=\left\{f^{\prime}\right\}$.

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

Theorem 2. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$f(L(a,b)) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{(L(a,b)-a)\,bf(b) + (b-L(a,b))\,af(a)}{(b-a)\,L(a,b)}. \tag{3.2}$$

Proof. Since $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g : \left[\frac{1}{b},\frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$. Therefore f has partial derivatives in each point of (a,b) and by the gradient inequality for g we have for any $x,y \in (a,b)$ that

$$\begin{aligned} f(x) - f(y) &= g\left(\frac{1}{x}\right) - g\left(\frac{1}{y}\right) \ge g'_+\left(\frac{1}{y}\right)\left(\frac{1}{x} - \frac{1}{y}\right) \\ &= g'_+\left(\frac{1}{y}\right)\frac{y-x}{xy}. \end{aligned} \tag{3.3}$$

Since

$$g'_+(s) = f'_-\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right), \quad s \in \left(\frac{1}{b},\frac{1}{a}\right)$$

then

$$g'_+\left(\frac{1}{y}\right) = f'_-(y)\left(-y^2\right)$$

and by (3.3) we have

$$f(x) - f(y) \ge f'_-(y)\frac{y-x}{xy}\left(-y^2\right) = f'_-(y)\,y\left(1 - \frac{y}{x}\right).$$

Therefore we have

$$f(x) - f(y) \ge f'_-(y)\,y\left(1 - \frac{y}{x}\right) \tag{3.4}$$

for any $x,y \in (a,b)$.

If we take the integral mean over x in (3.4), then we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx - f(y) &\ge \left(1 - y\frac{1}{b-a}\int_a^b \frac{1}{x}dx\right) f'_-(y)\,y \\ &= \left(1 - \frac{y}{L(a,b)}\right) f'_-(y)\,y \end{aligned} \tag{3.5}$$

for any $y \in (a,b)$.

Now, if we take $y = L(a,b)$ in (3.5), then we get the first inequality in (3.2).

Observe that for any $x \in [a,b]$ we have

$$\frac{1}{x} = \frac{\left(\frac{1}{a} - \frac{1}{x}\right)\frac{1}{b} + \left(\frac{1}{x} - \frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a} - \frac{1}{b}}.$$

By the convexity of g on $\left[\frac{1}{b},\frac{1}{a}\right]$ we then have

$$f(x)=g\left(\frac{1}{x}\right)=g\left(\frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}\right) \tag{3.6}$$
$$\leq\frac{\left(\frac{1}{a}-\frac{1}{x}\right)g\left(\frac{1}{b}\right)+\left(\frac{1}{x}-\frac{1}{b}\right)g\left(\frac{1}{a}\right)}{\frac{1}{a}-\frac{1}{b}}$$
$$=\frac{\left(\frac{1}{a}-\frac{1}{x}\right)f(b)+\left(\frac{1}{x}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}$$

for any $x\in[a,b]$.

Taking the integral mean in (3.6) we get

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{\left(\frac{1}{a}-\frac{1}{b-a}\int_a^b\frac{1}{x}dx\right)f(b)+\left(\frac{1}{b-a}\int_a^b\frac{1}{x}dx-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}$$
$$=\frac{\left(\frac{1}{a}-\frac{1}{L(a,b)}\right)f(b)+\left(\frac{1}{L(a,b)}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}$$
$$=\frac{\frac{L(a,b)-a}{aL(a,b)}f(b)+\frac{b-L(a,b)}{L(a,b)b}f(a)}{\frac{b-a}{ab}}$$

and the second inequality in (3.2) is also proved. □

Remark 1. *If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (3.5) we have the following inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f(y)\geq\left(1-\frac{y}{L(a,b)}\right)f'(y)\,y \tag{3.7}$$

for any $y\in(a,b)$.

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f(A(a,b))\geq\left(1-\frac{A(a,b)}{L(a,b)}\right)f'(A(a,b))\,A(a,b) \tag{3.8}$$

and if $f'(A(a,b))\leq 0$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx\geq f(A(a,b)). \tag{3.9}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f(I(a,b))\geq\left(1-\frac{I(a,b)}{L(a,b)}\right)f'(I(a,b))\,I(a,b) \tag{3.10}$$

and if $f'(I(a,b))\leq 0$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx\geq f(I(a,b)). \tag{3.11}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f(G(a,b))\geq\left(1-\frac{G(a,b)}{L(a,b)}\right)f'(G(a,b))\,G(a,b) \tag{3.12}$$

and if $f'(G(a,b)) \geq 0$, *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(G(a,b)). \tag{3.13}$$

We have:

Theorem 3. *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$f\left(\frac{a+b}{2}\right)\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{bf(b)+af(a)}{2}. \tag{3.14}$$

Proof. From the inequality (3.4), by multiplying with $x>0$ we have

$$xf(x)-xf(y) \geq f'_-(y)\,y\,(x-y) \tag{3.15}$$

for any $x,y\in(a,b)$.

Taking the integral mean over $x\in[a,b]$ we have

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{1}{b-a}\int_a^b xdx \geq \left(\frac{1}{b-a}\int_a^b xdx - y\right)f'_-(y)\,y,$$

that is equivalent to

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{a+b}{2} \geq \left(\frac{a+b}{2}-y\right)f'_-(y)\,y, \tag{3.16}$$

for any $y\in(a,b)$.

If we take in (3.16) $y=\frac{a+b}{2}$, then we get the first inequality in (3.14).

From the inequality (3.6) we also have

$$xf(x) \leq \frac{\left(\frac{x}{a}-1\right)f(b)+\left(1-\frac{x}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \tag{3.17}$$

for any $x\in[a,b]$.

Taking the integral mean on (3.17) we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b xf(x)\,dx &\leq \frac{\left(\frac{a+b}{2a}-1\right)f(b)+\left(1-\frac{a+b}{2b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \\ &= \frac{\frac{b-a}{2a}f(b)+\frac{b-a}{2b}f(a)}{\frac{b-a}{ab}} = \frac{bf(b)+af(a)}{2}\end{aligned} \tag{3.18}$$

and the second inequality in (3.14) is proved. □

Remark 2. *If* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *is a differentiable HA-convex function on the interval* (a,b), *then from (3.16) we have*

$$f(y)A(a,b) - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq (y-A(a,b))\,f'(y)\,y, \tag{3.19}$$

for any $y\in(a,b)$.

If we take in (3.19) $y=I(a,b)$, *then we get*

$$\begin{aligned}&f(I(a,b))A(a,b) - \frac{1}{b-a}\int_a^b xf(x)\,dx \\ &\leq (I(a,b)-A(a,b))\,f'(I(a,b))\,I(a,b).\end{aligned} \tag{3.20}$$

If $f'(I(a,b)) \geq 0,$ *then*

$$f(I(a,b))A(a,b) \leq \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{3.21}$$

If we take in (3.19) $y = L(a,b)$, *then we get*

$$f(L(a,b))A(a,b) - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq (L(a,b) - A(a,b))\,f'(L(a,b))\,L(a,b). \tag{3.22}$$

If $f'(L(a,b)) \geq 0,$ *then*

$$f(L(a,b))A(a,b) \leq \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{3.23}$$

If we take in (3.19) $y = G(a,b)$, *then we get*

$$f(G(a,b))A(a,b) - \frac{1}{b-a}\int_a^b xf(x)\,dx \leq (G(a,b) - A(a,b))\,f'(G(a,b))\,G(a,b). \tag{3.24}$$

If $f'(G(a,b)) \geq 0,$ *then*

$$f(G(a,b))A(a,b) \leq \frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{3.25}$$

We use the following results obtained by the author in [21] and [22]

Lemma 2. *Let* $h : [\alpha,\beta] \to \mathbb{R}$ *be a convex function on* $[\alpha,\beta]$. *Then we have the inequalities*

$$\begin{aligned} &\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right) - h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ &\leq \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt \\ &\leq \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta-\alpha) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right) - h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \\ &\leq \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right) \\ &\leq \frac{1}{8}\left[h'_-(\beta) - h'_+(\alpha)\right](\beta-\alpha). \end{aligned} \tag{3.27}$$

The constant $\frac{1}{8}$ *is best possible in (3.26) and (3.27).*

We have:

Theorem 4. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^2}(b-a)\tag{3.28}$$
$$\le\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt$$
$$\le\frac{1}{8}\left[\frac{f'_-(b)b^2-f'_+(a)a^2}{ab}\right](b-a)$$

and

$$\frac{1}{2}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^2}(b-a)\tag{3.29}$$
$$\le\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt-f\left(\frac{2ab}{a+b}\right)$$
$$\le\frac{1}{8}\left[\frac{f'_-(b)b^2-f'_+(a)a^2}{ab}\right](b-a).$$

Proof. Since $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

We know that

$$g'_\pm(s)=f'_\mp\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right),\ s\in\left(\frac{1}{b},\frac{1}{a}\right).$$

If we use the inequality (3.26) for the convex function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, then we have

$$\frac{1}{8}\left[f'_-\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(-\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)-f'_+\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(-\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)\right]$$
$$\times\left(\frac{1}{a}-\frac{1}{b}\right)$$
$$\le\frac{f(a)+f(b)}{2}-\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{s}\right)ds$$
$$\le\frac{1}{8}\left[f'_+\left(\frac{1}{\frac{1}{a}}\right)\left(-\frac{1}{\left(\frac{1}{a}\right)^2}\right)-f'_-\left(\frac{1}{\frac{1}{b}}\right)\left(-\frac{1}{\left(\frac{1}{b}\right)^2}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right)$$

that is equivalent to

$$\frac{1}{8}\left[f'_+\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)-f'_-\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)\right]$$
$$\times\left(\frac{1}{a}-\frac{1}{b}\right)$$
$$\leq\frac{f(a)+f(b)}{2}-\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{s}\right)ds$$
$$\leq\frac{1}{8}\left[f'_-\left(\frac{1}{\frac{1}{b}}\right)\left(\frac{1}{\left(\frac{1}{b}\right)^2}\right)-f'_+\left(\frac{1}{\frac{1}{a}}\right)\left(\frac{1}{\left(\frac{1}{a}\right)^2}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right),$$

namely

$$\frac{1}{8}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{4a^2b^2}{(a+b)^2}\left(\frac{b-a}{ab}\right)$$
$$\leq\frac{f(a)+f(b)}{2}-\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{s}\right)ds$$
$$\leq\frac{1}{8}\left[f'_-(b)\,b^2-f'_+(a)\,a^2\right]\left(\frac{b-a}{ab}\right).$$

Observe that

$$\int_{\frac{1}{b}}^{\frac{1}{a}}f\left(\frac{1}{s}\right)ds=\int_a^b\frac{f(t)}{t^2}dt$$

and the inequality (3.28) is proved.

The inequality (3.29) follows by (3.27). □

Corollary 3. *If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then*

$$0\leq\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt \tag{3.30}$$
$$\leq\frac{1}{8}\left[\frac{f'_-(b)\,b^2-f'_+(a)\,a^2}{ab}\right](b-a)$$

and

$$0\leq\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt-f\left(\frac{2ab}{a+b}\right) \tag{3.31}$$
$$\leq\frac{1}{8}\left[\frac{f'_-(b)\,b^2-f'_+(a)\,a^2}{ab}\right](b-a).$$

4. Related Results

We have the following result:

Theorem 5. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2}\left[xf(x)+\frac{(b-x)\,bf(b)+(x-a)\,af(a)}{b-a}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy \tag{4.1}$$

for any $x \in [a, b]$.

Proof. From (3.4) we have

$$f(x) - f(y) \geq f'_-(y)\, y \left(1 - \frac{y}{x}\right) \tag{4.2}$$

for any $x, y \in (a, b)$.

If we take the integral mean over y in (4.2), then we have

$$f(x) - \frac{1}{b-a}\int_a^b f(y)\, dy \geq \frac{1}{b-a}\int_a^b f'_-(y)\, y dy - \frac{1}{x}\frac{1}{b-a}\int_a^b f'_-(y)\, y^2 dy \tag{4.3}$$

for any $x \in (a, b)$.

Integrating by parts in the Lebesgue integral, we have

$$\int_a^b f'_-(y)\, y dy = bf(b) - af(a) - \int_a^b f(y)\, dy$$

and

$$\int_a^b f'_-(y)\, y^2 dy = b^2 f(b) - a^2 f(a) - 2\int_a^b y f(y)\, dy.$$

Utilising (4.3) we obtain

$$\begin{aligned} f(x) &- \frac{1}{b-a}\int_a^b f(y)\, dy \\ &\geq \frac{1}{b-a}\left(bf(b) - af(a) - \int_a^b f(y)\, dy\right) \\ &- \frac{1}{x}\frac{1}{b-a}\left(b^2 f(b) - a^2 f(a) - 2\int_a^b y f(y)\, dy\right) \\ &= \frac{bf(b) - af(a)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\, dy \\ &- \frac{1}{x}\frac{b^2 f(b) - a^2 f(a)}{b-a} + \frac{2}{x}\frac{1}{b-a}\int_a^b y f(y)\, dy \end{aligned} \tag{4.4}$$

that is equivalent to

$$f(x) + \frac{1}{x}\frac{b^2 f(b) - a^2 f(a)}{b-a} - \frac{bf(b) - af(a)}{b-a} \geq \frac{2}{x}\frac{1}{b-a}\int_a^b y f(y)\, dy.$$

If we multiply this inequality by $\frac{x}{2}$, then we get

$$\frac{1}{2}\left[x f(x) + \frac{b^2 f(b) - a^2 f(a) - xbf(b) + xaf(a)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b y f(y)\, dy,$$

and the inequality (4.1) is proved. □

Remark 3. *If we take in (4.1)* $x = \frac{a+b}{2}$*, then we get*

$$\frac{1}{2}\left[\frac{a+b}{2} f\left(\frac{a+b}{2}\right) + \frac{bf(b) + af(a)}{2}\right] \geq \frac{1}{b-a}\int_a^b y f(y)\, dy. \tag{4.5}$$

If we take in (4.1) $x = \frac{2ab}{a+b}$*, then we get*

$$\frac{1}{2}\left[\frac{2ab}{a+b} f\left(\frac{2ab}{a+b}\right) + \frac{b^2 f(b) + a^2 f(a)}{a+b}\right] \geq \frac{1}{b-a}\int_a^b y f(y)\, dy. \tag{4.6}$$

We have:

Theorem 6. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{x}\left[\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy\right] \geq \frac{1}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{f(y)}{y}dy-f(x)\right] \tag{4.7}$$

for any $x\in[a,b]$.

Proof. By dividing with $y>0$ in (3.4) we have

$$\frac{1}{y}f(x)-\frac{f(y)}{y}\geq f'_-(y)\left(1-\frac{y}{x}\right) \tag{4.8}$$

for any $x,y\in(a,b)$.

By taking the integral mean over y in (4.8) we obtain

$$\begin{aligned}&\frac{\ln b-\ln a}{b-a}f(x)-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy\\ &\geq\frac{1}{b-a}\int_a^b f'_-(y)\,dy-\frac{1}{x}\frac{1}{b-a}\int_a^b yf'_-(y)\,dy\\ &=\frac{f(b)-f(a)}{b-a}-\frac{1}{x}\frac{bf(b)-af(a)-\int_a^b f(y)\,dy}{b-a}\\ &=\frac{f(b)-f(a)}{b-a}-\frac{1}{x}\frac{bf(b)-af(a)\,y}{b-a}+\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{1}{x}\frac{bf(b)-af(a)\,y}{b-a}-\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy\\ &\geq\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{\ln b-\ln a}{b-a}f(x)\end{aligned}$$

or, to

$$\frac{1}{b-a}\left[\frac{b-x}{x}f(b)+\frac{x-a}{x}f(a)\right]-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \geq\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{\ln b-\ln a}{b-a}f(x) \tag{4.9}$$

for any $x\in(a,b)$.

Rearranging the terms in (4.9) produces the desired result (4.7). □

Remark 4. *If we take $x=L(a,b)$ in (4.7), then we get*

$$\frac{(b-L(a,b))f(b)+(L(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq\frac{1}{\ln b-\ln a}\int_a^b\frac{f(y)}{y}dy-f(L(a,b)). \tag{4.10}$$

If we take $x = A(a,b)$ *in (4.7), then we get*

$$\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{A(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(A(a,b))\right]. \tag{4.11}$$

If we take $x = H(a,b) := \frac{2ab}{a+b}$ *in (4.7), then we get*

$$\frac{bf(b)+af(a)}{b+a}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{H(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(H(a,b))\right]. \tag{4.12}$$

If we take $x = G(a,b)$ *in (4.7)), then we get*

$$\frac{(b-G(a,b))f(b)+(G(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{G(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(G(a,b))\right]. \tag{4.13}$$

If the function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is convex, then by Jensen's inequality we have

$$\begin{aligned}\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy &= \frac{1}{\int_a^b \frac{dy}{y}}\int_a^b \frac{f(y)}{y}dy \geq f\left(\frac{\int_a^b y\frac{dy}{y}}{\int_a^b \frac{dy}{y}}\right)\\ &= f\left(\frac{b-a}{\ln b-\ln a}\right) = f(L(a,b)).\end{aligned}$$

Therefore, for any function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ that is *convex and HA-convex*, by (4.10) we have

$$\frac{(b-L(a,b))f(b)+(L(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \geq \frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy - f(L(a,b)) \geq 0. \tag{4.14}$$

It is known that, if a function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is *GA-convex*, namely

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq (\geq)\,(1-\lambda)f(x)+\lambda f(y)$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$, then [25]

$$\frac{1}{\ln b-\ln a}\int_a^b \frac{f(y)}{y}dy \geq f(G(a,b)). \tag{4.15}$$

Therefore, for any function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ that is *GA-convex and HA-convex*, by (4.13) we have

$$\frac{(b-G(a,b))f(b)+(G(a,b)-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{4.16}$$
$$\geq\frac{G(a,b)}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{f(y)}{y}dy-f(G(a,b))\right]\geq 0.$$

Theorem 7. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2x}\left(\frac{f(b)\,a(b-x)+f(a)\,b(x-a)}{b-a}+xf(x)\right)\geq\frac{ab}{b-a}\int_a^b\frac{f(y)}{y^2}dy \tag{4.17}$$

for any $x\in[a,b]$.

Proof. From (3.4) we have, by division with $y^2>0$, that

$$\frac{1}{y^2}f(x)-\frac{1}{y^2}f(y)\geq\frac{f'_-(y)}{y}\left(1-\frac{y}{x}\right)$$

for any $x,y\in(a,b)$.

Taking the integral mean over y we have

$$f(x)\frac{1}{b-a}\int_a^b\frac{1}{y^2}dy-\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy$$
$$\geq\frac{1}{b-a}\int_a^b\frac{f'_-(y)}{y}dy-\frac{1}{x}\frac{1}{b-a}\int_a^b f'_-(y)\,dy$$

that is equivalent to

$$\frac{f(x)}{ab}-\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy$$
$$\geq\frac{1}{b-a}\left[\frac{f(b)}{b}-\frac{f(a)}{a}+\int_a^b\frac{f(y)}{y^2}dy\right]-\frac{1}{x}\frac{f(b)-f(a)}{b-a}$$
$$=\frac{1}{b-a}\left(\frac{f(b)}{b}-\frac{f(a)}{a}\right)+\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy-\frac{1}{x}\frac{f(b)-f(a)}{b-a},$$

for any $x\in(a,b)$. This can be written as

$$\frac{1}{x}\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\left(\frac{f(b)}{b}-\frac{f(a)}{a}\right)\geq\frac{2}{b-a}\int_a^b\frac{f(y)}{y^2}dy-\frac{f(x)}{ab}$$

or as

$$\frac{1}{2}\left(\frac{1}{b-a}\left[f(b)\frac{b-x}{xb}+f(a)\frac{x-a}{ax}\right]+\frac{f(x)}{ab}\right)\geq\frac{1}{b-a}\int_a^b\frac{f(y)}{y^2}dy.$$

This is equivalent to the desired result (4.17). □

Remark 5. *If we take in (4.17) $x=\frac{a+b}{2}$, then we get*

$$\frac{1}{2}\left(\frac{f(b)\,a+f(a)\,b}{a+b}+f\left(\frac{a+b}{2}\right)\right)\geq\frac{ab}{b-a}\int_a^b\frac{f(y)}{y^2}dy. \tag{4.18}$$

If we take in (3.19) $x = \frac{2ab}{a+b}$, *then we get*

$$\frac{1}{2}\left[\frac{f(b)+f(a)}{2}+f\left(\frac{2ab}{a+b}\right)\right] \geq \frac{ab}{b-a}\int_a^b \frac{f(y)}{y^2}dy. \tag{4.19}$$

5. Applications

We consider the *arithmetic mean* $A(a,b) = \frac{a+b}{2}$, the *geometric mean* $G(a,b) = \sqrt{ab}$ and *harmonic mean* $H(a,b) = \frac{2ab}{a+b}$ for the positive numbers $a, b > 0$.

The following well know order between these means, including logarithmic and identric means defined above, holds

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b). \tag{5.1}$$

If we consider the HA-convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t$ and we use the inequalities (2.4), then we have

$$\begin{aligned} H(a,b) &\leq (1-\lambda)\frac{2ab}{(1-\lambda)a+(\lambda+1)b}+\lambda\frac{2ab}{(2-\lambda)a+\lambda b} \\ &\leq \frac{G^2(a,b)}{L(a,b)} \leq \frac{1}{2}\left[\frac{ab}{(1-\lambda)a+\lambda b}+(1-\lambda)a+\lambda b\right] \leq A(a,b), \end{aligned} \tag{5.2}$$

for any $\lambda \in [0,1]$.

If we use the inequalities (3.30) and (3.31) we get

$$0 \leq A(a,b) - \frac{G^2(a,b)}{L(a,b)} \leq \frac{1}{4}\frac{A(a,b)}{G^2(a,b)}(b-a)^2 \tag{5.3}$$

and

$$0 \leq \frac{G^2(a,b)}{L(a,b)} - H(a,b) \leq \frac{1}{4}\frac{A(a,b)}{G^2(a,b)}(b-a)^2. \tag{5.4}$$

The first inequality in (5.3) also follows by (5.1).

Consider the function $f : (0,\infty) \to \mathbb{R}$, $f(t) = \frac{\ln t}{t}$. Observe that

$$g(t) = f\left(\frac{1}{t}\right) = -t\ln t,$$

which shows that f is HA-concave on $(0,\infty)$.

If we use the inequality (3.2) for HA-concave functions we have

$$\frac{\ln(L(a,b))}{L(a,b)} \geq \frac{1}{b-a}\int_a^b \frac{\ln t}{t}dt \geq \frac{(L(a,b)-a)\ln b + (b-L(a,b))\ln a}{(b-a)L(a,b)},$$

which is equivalent to

$$\frac{\ln(L(a,b))}{L(a,b)} \geq \frac{\ln G(a,b)}{L(a,b)} \geq \frac{(L(a,b)-a)\ln b + (b-L(a,b))\ln a}{(b-a)L(a,b)}. \tag{5.5}$$

The first inequality in (5.5) also follows (5.1).

From the second inequality we have

$$G(a,b) \geq b^{\frac{L(a,b)-a}{b-a}} a^{\frac{b-L(a,b)}{b-a}}. \tag{5.6}$$

If we write the inequality (4.19) for the HA-convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t$, then we have

$$\frac{A(a,b)+H(a,b)}{2} \geq \frac{G^2(a,b)}{L(a,b)}. \tag{5.7}$$

If we write the inequalities (4.5) and (4.6) for the HA-concave function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$, then we get

$$\sqrt{A(a,b)\,G(a,b)}\le I(a,b) \tag{5.8}$$

and

$$\frac{1}{2}\left[\ln\left(\frac{2ab}{a+b}\right)+\frac{b\ln b+a\ln a}{a+b}\right]\le \ln I(a,b)\,. \tag{5.9}$$

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