SOME ADDITIVE INEQUALITIES FOR HEINZ OPERATOR MEAN

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ABSTRACT. In this paper we obtain some new additive inequalities for Heinz operator mean.

1. Introduction

Throughout this paper $A$, $B$ are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0, 1]$

$$ A \nabla \nu B := (1 - \nu) A + \nu B, $$

the weighted operator arithmetic mean, and

$$ A^\# \nu B := A^{1/2} \left( A^{-1/2} BA^{-1/2} \right)^{\nu} A^{1/2}, $$

the weighted operator geometric mean [14]. When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A^\# B$ for brevity, respectively.

Define the Heinz operator mean by

$$ H_\nu (A, B) := \frac{1}{2} (A^\# \nu B + A^\#_{1-\nu} B). $$

The following interpolatory inequality is obvious

$$ A^\# B \leq H_\nu (A, B) \leq A \nabla B \tag{1.1} $$

for any $\nu \in [0, 1]$.

We recall that Specht’s ratio is defined by [16]

$$ S(h) := \begin{cases} 
\frac{h^{1+\frac{1}{h-1}}}{\exp \left( \frac{h}{h^{1-1}} \right)} & \text{if } h \in (0, 1) \cup (1, \infty), \\
1 & \text{if } h = 1.
\end{cases} \tag{1.2} $$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S \left( \frac{1}{h} \right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

The following result provides an upper and lower bound for the Heinz mean in terms of the operator geometric mean $A^\# B$:

**Theorem 1** (Dragomir, 2015 [6]). Assume that $A$, $B$ are positive invertible operators and the constants $M > m > 0$ are such that

$$ mA \leq B \leq MA. \tag{1.3} $$
Then we have
\begin{equation}
\omega_{\nu} (m, M) A^* B \leq H_{\nu} (A, B) \leq \Omega_{\nu} (m, M) A^* B,
\end{equation}
where
\begin{equation}
\Omega_{\nu} (m, M) := \begin{cases} 
S (m^{2\nu-1}) & \text{if } M < 1, \\
\max \{ S (m^{2\nu-1}), S (M^{2\nu-1}) \} & \text{if } m \leq 1 \leq M, \\
S (M^{2\nu-1}) & \text{if } 1 < m
\end{cases}
\end{equation}
and
\begin{equation}
\omega_{\nu} (m, M) := \begin{cases} 
S (M^{1-\frac{1}{2\nu}}) & \text{if } M < 1, \\
1 & \text{if } m \leq 1 \leq M, \\
S (m^{1-\frac{1}{2\nu}}) & \text{if } 1 < m
\end{cases}
\end{equation}
where \( \nu \in [0, 1] \).

We consider the Kantorovich’s constant defined by
\begin{equation}
K (h) := \frac{(h + 1)^2}{4h}, \quad h > 0.
\end{equation}
The function \( K \) is decreasing on \((0, 1)\) and increasing on \([1, \infty)\), \( K (h) \geq 1 \) for any \( h > 0 \) and \( K (h) = K \left( \frac{1}{h} \right) \) for any \( h > 0 \).

We have:

\textbf{Theorem 2} (Dragomir, 2015 [7]). Assume that \( A, B \) are positive invertible operators and the constants \( M > m > 0 \) are such that the condition (1.3) is valid. Then for any \( \nu \in [0, 1] \) we have
\begin{equation}
( A^* B \leq ) H_{\nu} (A, B) \leq \exp [ \Theta_{\nu} (m, M) - 1] A^* B
\end{equation}
where
\begin{equation}
\Theta_{\nu} (m, M) := \begin{cases} 
K (m^{2\nu-1}) & \text{if } M < 1, \\
\max \{ K (m^{2\nu-1}), K (M^{2\nu-1}) \} & \text{if } m \leq 1 \leq M, \\
K (M^{2\nu-1}) & \text{if } 1 < m
\end{cases}
\end{equation}
and
\begin{equation}
(0 \leq ) H_{\nu} (A, B) - A^* B \leq \frac{1}{4m^{1-\nu}} \max_{x \in [m, M]} D (x^{2\nu-1}) A,
\end{equation}
where the function \( D : (0, \infty) \rightarrow [0, \infty) \) is defined by \( D (x) = (x - 1) \ln x \).

The following bounds for the Heinz mean \( H_{\nu} (A, B) \) in terms of \( A \triangledown B \) are also valid:

\textbf{Theorem 3} (Dragomir, 2015 [7]). With the assumptions of Theorem 4 we have
\begin{equation}
(0 \leq ) A \triangledown B - H_{\nu} (A, B) \leq \nu (1 - \nu) \Upsilon (m, M) A,
\end{equation}
where

\[ Y(m, M) := \begin{cases} (m - 1) \ln m & \text{if } M < 1, \\ \max \{(m - 1) \ln m, (M - 1) \ln M\} & \text{if } m \leq 1 \leq M, \\ (M - 1) \ln M & \text{if } 1 < m \end{cases} \]

and

\[ A \nabla B \exp \left[-4\nu (1 - \nu) (F(m, M) - 1)\right] \leq H_{\nu}(A, B) \leq A \nabla B \]

where

\[ F(m, M) := \begin{cases} K(m) & \text{if } M < 1, \\ \max \{K(m), K(M)\} & \text{if } m \leq 1 \leq M, \\ K(M) & \text{if } 1 < m. \end{cases} \]

For other recent results on geometric operator mean inequalities, see [2]-[13], [15] and [17]-[18].

Motivated by the above results, we establish in this paper some inequalities for the quantities

\[ H_{\nu}(A, B) - A \nabla B \quad \text{and} \quad A \nabla B - H_{\nu}(A, B) \]

under various assumptions for positive invertible operators \( A, B \).

2. Bounds for \( H_{\nu}(A, B) - A \nabla B \)

First we notice the following simple result:

**Theorem 4.** Assume that \( A, B \) are positive invertible operators and the constants \( M > m > 0 \) are such that the condition (1.3) holds. If we consider the function \( f_{\nu} : [0, \infty) \to \mathbb{R} \) for \( \nu \in [0, 1] \) defined by

\[ f_{\nu}(x) = \frac{1}{2} (x^\nu + x^{1-\nu}) \],

then we have

\[ f_{\nu}(m) A \leq H_{\nu}(A, B) \leq f_{\nu}(M) A. \]

**Proof.** We observe that

\[ f_{\nu}'(x) = \frac{1}{2} (\nu x^{\nu-1} + (1 - \nu) x^{-\nu}) , \]

which is positive for \( x \in (0, \infty) \).

Therefore \( f_{\nu} \) is increasing on \((0, \infty)\) and

\[ f_{\nu}(m) = \min_{x \in [m, M]} f_{\nu}(x) \leq f_{\nu}(x) \leq \max_{x \in [m, M]} f_{\nu}(x) = f_{\nu}(M) \]

for any \( x \in [m, M] \).

Using the continuous functional calculus, we have for any operator \( X \) with \( mI \leq X \leq MI \) that

\[ f_{\nu}(m) I \leq \frac{1}{2} (X^\nu + X^{1-\nu}) \leq f_{\nu}(M) I. \]
From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI \leq A^{-1/2}BA^{-1/2} \leq MI.$$

Now, writing the inequality (2.2) for $X = A^{-1/2}BA^{-1/2}$, we get

$$f_\nu(m) I \leq \frac{1}{2} \left[ \left( A^{-1/2}BA^{-1/2} \right)^\nu + \left( A^{-1/2}BA^{-1/2} \right)^{1-\nu} \right] \leq f_\nu(M) I.$$

Finally, if we multiply both sides of (2.3) by $A^{-1/2}$ we get the desired result (2.1).

**Corollary 1.** Let $A, B$ be two positive operators. For positive real numbers $m, m', M, M'$, put $h := \frac{M}{m}, h' := \frac{M'}{m'}$, and let $\nu \in [0, 1]$.

(i) If $0 < mI \leq \frac{M}{m} I < M'I \leq B \leq MI$, then

$$f_\nu(h') A \leq H_\nu(A, B) \leq f_\nu(h) A.$$

(ii) If $0 < mI \leq B \leq \frac{M}{m} I < A \leq MI$, then

$$f_\nu(h) \frac{h}{h'} A \leq H_\nu(A, B) \leq f_\nu(h') \frac{h'}{h} A.$$

**Proof.** If the condition (i) is valid, then we have for $X = A^{-1/2}BA^{-1/2}$

$$I < \frac{M'}{m'} I = h'I \leq X \leq hI = \frac{M}{m} I,$$

which, by (2.2) gives the desired result (2.4).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h} I \leq X \leq \frac{1}{h'} I < I,$$

which, by (2.2) gives

$$f_\nu\left(\frac{1}{h}\right) A \leq H_\nu(A, B) \leq f_\nu\left(\frac{1}{h'}\right) A$$

that is equivalent to (2.5), since

$$f_\nu\left(\frac{1}{h}\right) = f_\nu\left(\frac{h}{h'}\right).$$

We need the following lemma:

**Lemma 1.** Consider the function $g_\nu : [0, \infty) \rightarrow \mathbb{R}$ for $\nu \in (0, 1)$ defined by

$$g_\nu(x) = \frac{1}{2} (x^\nu + x^{1-\nu}) - \sqrt{x} \geq 0.$$

Then $g_\nu(0) = g_\nu(1) = 0$, $g_\nu$ is increasing on $(0, x_\nu)$ with a local maximum in

$$x_\nu := \left( \frac{\nu}{1-\nu} \right)^{\frac{1}{2\nu}} \in (0, 1),$$

is decreasing on $(x_\nu, 1)$ with a local minimum in $x = 1$ and increasing on $(1, \infty)$ with $\lim_{x \rightarrow \infty} g_\nu(x) = \infty.$
Proof. (i) If \( \nu \in \left( 0, \frac{1}{2} \right) \), then

\[
g'_\nu(x) = \frac{1}{2} \left( \frac{\nu}{x^{1-\nu}} + \frac{1-\nu}{x^\nu} - \frac{1}{x^{1/2}} \right)
= \frac{1}{2} \nu + \frac{1}{x^{1-\nu}} - \frac{1}{x^{1/2}}.
\]

If we denote \( u = x^{1-2\nu} \), then we have

\[
\nu + (1 - \nu) x^{1-2\nu} - x^{-\frac{1-2\nu}{2}} = (1 - \nu) u^2 - u + \nu.
= (1 - \nu) \left( u - \frac{\nu}{1 - \nu} \right) (u - 1)
= (1 - \nu) \left( x^{1-2\nu} - \frac{\nu}{1 - \nu} \right) \left( x^{\frac{1-2\nu}{2}} - 1 \right).
\]

We observe that \( g'_\nu(x) = 0 \) only for \( x = 1 \) and \( x_{\nu} = \left( \frac{\nu}{1 - \nu} \right)^{\frac{2}{1-2\nu}} \in (0, 1) \). Also \( g'_\nu(x) > 0 \) for \( x \in (0, x_{\nu}) \cup (1, \infty) \) and \( g'_\nu(x) < 0 \) for \( x \in (x_{\nu}, 1) \). These imply the desired conclusion.

(ii) If \( \nu \in \left( \frac{1}{2}, 1 \right) \), then

\[
g'_\nu(x) = \frac{1}{2} \left( 1 - \nu + \nu x^{2\nu-1} - x^{\frac{2\nu-1}{2}} \right)
\]

If we denote \( z = x^{\frac{2\nu-1}{2}} \), then we have

\[
1 - \nu + \nu x^{2\nu-1} - x^{\frac{2\nu-1}{2}} = \nu z^2 - z + 1 - \nu
= \nu \left( z - \frac{1 - \nu}{\nu} \right) (z - 1)
= \nu \left( x^{\frac{2\nu-1}{2}} - \frac{1 - \nu}{\nu} \right) (x^{\frac{2\nu-1}{2}} - 1).
\]

We observe that \( g'_\nu(x) = 0 \) only for \( x = 1 \) and \( x_{\nu} = \left( \frac{1 - \nu}{\nu} \right)^{\frac{2}{2\nu-1}} \in (0, 1) \). Also \( g'_\nu(x) > 0 \) for \( x \in (0, x_{\nu}) \cup (1, \infty) \) and \( g'_\nu(x) < 0 \) for \( x \in (x_{\nu}, 1) \). These imply the desired conclusion. \( \square \)

The above lemma allows us to obtain various bounds for the nonnegative quantity

\[
H_\nu(A, B) - A_B^\sharp
\]

when some conditions for the involved operators \( A \) and \( B \) are known.

**Theorem 5.** Assume that \( A, B \) are positive invertible operators with \( B \preceq A \). Then for \( \nu \in (0, 1) \) we have

\[
(0 \leq H_\nu(A, B) - A_B^\sharp \leq g_\nu(x_{\nu}) A,
\]

where \( g_\nu \) is defined by (2.6) and \( x_{\nu} \) by (2.7).

**Proof.** From Lemma 1 we have for \( \nu \in (0, 1) \) that

\[
0 \leq \frac{1}{2} (x^\nu + x^{1-\nu}) - \sqrt{x} \leq g_\nu(x_{\nu})
\]

for any \( x \in [0, 1] \).
Using the continuous functional calculus, we have for any operator $X$ with $0 \leq X \leq I$ that
\begin{equation}
0 \leq \frac{1}{2} (X^\nu + X^{1-\nu}) - X^{1/2} \leq g_\nu (x_\nu)
\end{equation}
for $\nu \in (0, 1)$.

By multiplying both sides of the inequality $0 \leq B \leq A$ with $A^{-1/2}$ we get
\[0 \leq A^{-1/2}BA^{-1/2} \leq I.
\]

If we use the inequality (2.9) for $X = A^{-1/2}BA^{-1/2}$, then we get
\begin{equation}
0 \leq \frac{1}{2} \left[ \left( A^{-1/2}BA^{-1/2} \right)^\nu + \left( A^{-1/2}BA^{-1/2} \right)^{1-\nu} \right] - \left( A^{-1/2}BA^{-1/2} \right)^{1/2} \leq g_\nu (x_\nu) I
\end{equation}
for $\nu \in (0, 1)$.

Finally, if we multiply both sides of (2.10) with $A^{1/2}$, then we get the desired result (2.8).

**Theorem 6.** Assume that $A$, $B$ are positive invertible operators and the constants $M > m \geq 0$ are such that the condition (1.3) holds. Let $\nu \in (0, 1)$.

(i) If $0 \leq m < M \leq 1$, then
\begin{equation}
\gamma_\nu (m, M) A \leq H_\nu (A, B) - A^*_\nu B \leq \Gamma_\nu (m, M) A,
\end{equation}
where
\begin{equation}
\gamma_\nu (m, M) := \begin{cases} 
g_\nu (m) & \text{if } 0 \leq m < M \leq x_\nu, \\
\min \{ g_\nu (m), g_\nu (M) \} & \text{if } 0 \leq m \leq x_\nu \leq M \leq 1, \\
g_\nu (M) & \text{if } x_\nu \leq m < M
\end{cases}
\end{equation}
\begin{equation}
\Gamma_\nu (m, M) := \begin{cases} 
g_\nu (M) & \text{if } 0 \leq m < M \leq x_\nu, \\
g_\nu (x_\nu) & \text{if } 0 \leq m \leq x_\nu \leq M \leq 1, \\
g_\nu (m) & \text{if } x_\nu \leq m \leq M \leq 1
\end{cases}
\end{equation}
where $g_\nu$ is defined by (2.6) and $x_\nu$ by (2.7).

(ii) If $1 \leq m < M < \infty$, then
\begin{equation}
g_\nu (m) A \leq H_\nu (A, B) - A^*_\nu B \leq g_\nu (M) A.
\end{equation}
Proof. (i) If $0 \leq m < M \leq 1$ then by Lemma 1 we have for $\nu \in (0,1)$ that

\[
\begin{cases}
g_\nu(m) & \text{if } 0 \leq m < M \leq x_\nu \\
\min \{g_\nu(m), g_\nu(M)\} & \text{if } 0 \leq m \leq x_\nu \leq M \leq 1 \\
g_\nu(M) & \text{if } x_\nu \leq m < M \\
g_\nu(x) & \text{if } 0 \leq m < M \leq x_\nu \\
g_\nu(x_\nu) & \text{if } 0 \leq m \leq x_\nu \leq M \leq 1 \\
g_\nu(M) & \text{if } x_\nu \leq m < M \leq 1
\end{cases}
\]

for any $x \in [m,M]$.

Now, on making use of a similar argument to the one in the proof of Theorem 5, we obtain the desired result (2.13).

(ii) Obvious by the properties of function $g_\nu$. \qed

The interested reader may obtain similar bounds for other locations of $0 \leq m < M < \infty$. The details are omitted.

The following particular case holds:

**Corollary 2.** Let $A, B$ be two positive operators. For positive real numbers $m, m', M, M'$, put $h := \frac{M}{m}, h' := \frac{M'}{m'}$ and let $\nu \in (0,1)$.

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

\[
(2.15) \quad g_\nu(h') A \leq H_\nu(A,B) - A_2^2B \leq g_\nu(h) A.
\]

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

\[
(2.16) \quad \tilde{\gamma}_\nu(h,h') A \leq H_\nu(A,B) - A_2^2B \leq \tilde{\Gamma}_\nu(h,h') A,
\]

where

\[
(2.17) \quad \tilde{\gamma}_\nu(h,h') := \begin{cases}
g_\nu(h) & \text{if } 0 \leq \frac{1}{h} < \frac{1}{h'} \leq x_\nu, \\
\min \left\{ \frac{g_\nu(h)}{h}, \frac{g_\nu(h')}{h'} \right\} & \text{if } 0 \leq \frac{1}{h} \leq x_\nu \leq \frac{1}{h'} \leq 1, \\
\frac{g_\nu(h')}{h'} & \text{if } x_\nu \leq \frac{1}{h} < \frac{1}{h'}
\end{cases}
\]

and

\[
(2.18) \quad \tilde{\Gamma}_\nu(h,h') := \begin{cases}
g_\nu(h') & \text{if } 0 \leq \frac{1}{h} < \frac{1}{h'} \leq x_\nu, \\
g_\nu(x_\nu) & \text{if } 0 \leq \frac{1}{h} \leq x_\nu \leq \frac{1}{h'} \leq 1, \\
\frac{g_\nu(h)}{h} & \text{if } x_\nu \leq \frac{1}{h} < \frac{1}{h'} \leq 1.
\end{cases}
\]
3. Bounds for $A\nabla B - H_\nu (A, B)$

In order to provide some upper and lower bounds for the quantity

$$A \nabla B - H_\nu (A, B)$$

where $A$, $B$ are positive invertible operators, we need the following lemma.

**Lemma 2.** Consider the function $h_\nu : [0, \infty) \to \mathbb{R}$ for $\nu \in (0, 1)$ defined by

$$(3.1) \quad h_\nu (x) = \frac{x + 1}{2} - \frac{1}{2} (x^\nu + x^{1-\nu}) \geq 0.$$  

Then $h_\nu$ is decreasing on $[0, 1)$ and increasing on $(1, \infty)$ with $x = 1$ its global minimum. We have $h_\nu (0) = \frac{1}{2}$, $\lim_{x \to \infty} h_\nu (x) = \infty$ and $h_\nu$ is convex on $(0, \infty)$.

**Proof.** We have

$$h_\nu' (x) = \frac{1}{2} \left( 1 - \frac{\nu}{x^{1-\nu}} - \frac{1-\nu}{x^\nu} \right)$$

and

$$h_\nu'' (x) = \frac{1}{2} \nu (1 - \nu) \left( x^{\nu-2} + x^{-\nu-1} \right)$$

for any $x \in (0, \infty)$ and $\nu \in (0, 1)$.

We observe that $h_\nu' (1) = 0$ and $h_\nu'' (x) > 0$ for any $x \in (0, \infty)$ and $\nu \in (0, 1)$. These imply that the equation $h_\nu' (x) = 0$ has only one solution on $(0, \infty)$, namely $x = 1$. Since $h_\nu' (x) < 0$ for $x \in (0, 1)$ and $h_\nu' (x) > 0$ for $x \in (1, \infty)$, then we deduce the desired conclusion. \hfill $\Box$

**Theorem 7.** Assume that $A$, $B$ are positive invertible operators, the constants $M > m \geq 0$ are such that the condition $(1.3)$ holds and $\nu \in (0, 1)$. Then we have

$$(3.2) \quad \delta_\nu (m, M) A \leq A \nabla B - H_\nu (A, B) \leq \Delta_\nu (m, M) A,$$

where

$$(3.3) \quad \delta_\nu (m, M) := \begin{cases} h_\nu (M) & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ h_\nu (m) & \text{if } 1 < m \end{cases}$$

and

$$(3.4) \quad \Delta_\nu (m, M) := \begin{cases} h_\nu (m) & \text{if } M < 1, \\ \max \{h_\nu (m), h_\nu (M)\} & \text{if } m \leq 1 \leq M, \\ h_\nu (M) & \text{if } 1 < m, \end{cases}$$

where $h_\nu$ is defined by $(3.1)$.
Proof. Using Lemma 2 we have
\[
\begin{cases}
h_\nu(M) \text{ if } M < 1, \\
0 \text{ if } m \leq 1 \leq M, & \leq h_\nu(x) \\
h_\nu(m) \text{ if } 1 < m,
\end{cases}
\]
for any \( x \in [m, M] \) and \( \nu \in (0, 1) \).

Using the continuous functional calculus, we have for any operator \( X \) with \( mI \leq X \leq MI \) that
\[
\delta_\nu(m, M) I \leq \frac{X + I}{2} - \frac{1}{2} (X^\nu + X^{1-\nu}) \leq \Delta_\nu(m, M) I.
\]

From (1.3) we have, by multiplying both sides with \( A^{-1/2} \) that
\[
mI \leq A^{-1/2}BA^{-1/2} \leq MI.
\]
Now, writing the inequality (3.5) for \( X = A^{-1/2}BA^{-1/2} \), we get
\[
\delta_\nu(m, M) I \\
\leq \frac{A^{-1/2}BA^{-1/2} + I}{2} - \frac{1}{2} \left( (A^{-1/2}BA^{-1/2})^\nu + (A^{-1/2}BA^{-1/2})^{1-\nu} \right) \\
\leq \Delta_\nu(m, M) I.
\]
Finally, if we multiply both sides of (3.6) by \( A^{1/2} \) we get the desired result (3.2).

Corollary 3. Let \( A, B \) be two positive operators. For positive real numbers \( m, m', \)
\( M, M' \), put \( h := \frac{M}{m} \), \( h' := \frac{M'}{m} \) and let \( \nu \in (0, 1) \).

(i) If \( 0 < mI \leq A \leq m'I < M'I \leq B \leq MI \), then
\[
h_\nu(h') A \leq A \nabla B - H_\nu(A, B) \leq h_\nu(h) A.
\]
(ii) If \( 0 < mI \leq B \leq m'I < M'I \leq A \leq MI \), then
\[
\frac{h_\nu(h')}{h'} A \leq A \nabla B - H_\nu(A, B) \leq \frac{h_\nu(h)}{h} A.
\]

References

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