

**QUADRATIC WEIGHTED GEOMETRIC MEAN IN HERMITIAN  
UNITAL BANACH \*-ALGEBRAS**

S. S. DRAGOMIR<sup>1,2</sup>

ABSTRACT. In this paper we introduce the *quadratic weighted geometric mean*

$$x \mathbb{S}_\nu y := ||yx^{-1}|^\nu x|^2$$

for invertible elements  $x, y$  in a Hermitian unital Banach \*-algebra and real number  $\nu$ . We show that

$$x \mathbb{S}_\nu y = |x|^2 \sharp_\nu |y|^2,$$

where  $\sharp_\nu$  is the usual geometric mean and provide some inequalities for this mean under various assumptions for the elements involved.

1. INTRODUCTION

Let  $A$  be a unital Banach \*-algebra with unit 1. An element  $a \in A$  is called *selfadjoint* if  $a^* = a$ .  $A$  is called *Hermitian* if every selfadjoint element  $a$  in  $A$  has real *spectrum*  $\sigma(a)$ , namely  $\sigma(a) \subset \mathbb{R}$ .

In what follows we assume that  $A$  is a Hermitian unital Banach \*-algebra.

We say that an element  $a$  is *nonnegative* and write this as  $a \geq 0$  if  $a^* = a$  and  $\sigma(a) \subset [0, \infty)$ . We say that  $a$  is *positive* and write  $a > 0$  if  $a \geq 0$  and  $0 \notin \sigma(a)$ . Thus  $a > 0$  implies that its inverse  $a^{-1}$  exists. Denote the set of all invertible elements of  $A$  by  $\text{Inv}(A)$ . If  $a, b \in \text{Inv}(A)$ , then  $ab \in \text{Inv}(A)$  and  $(ab)^{-1} = b^{-1}a^{-1}$ . Also, saying that  $a \geq b$  means that  $a - b \geq 0$  and, similarly  $a > b$  means that  $a - b > 0$ .

The *Shirali-Ford theorem* asserts that [12] (see also [2, Theorem 41.5])

$$(SF) \quad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [11], Tanahashi and Uchiyama [13] proved the following fundamental properties (see also [5]):

- (i) If  $a, b \in A$ , then  $a \geq 0, b \geq 0$  imply  $a + b \geq 0$  and  $\alpha \geq 0$  implies  $\alpha a \geq 0$ ;
- (ii) If  $a, b \in A$ , then  $a > 0, b \geq 0$  imply  $a + b > 0$ ;
- (iii) If  $a, b \in A$ , then either  $a \geq b > 0$  or  $a > b \geq 0$  imply  $a > 0$ ;
- (iv) If  $a > 0$ , then  $a^{-1} > 0$ ;
- (v) If  $c > 0$ , then  $0 < b < a$  if and only if  $cbc < cac$ , also  $0 < b \leq a$  if and only if  $cbc \leq cac$ ;
- (vi) If  $0 < a < 1$ , then  $1 < a^{-1}$ ;
- (vii) If  $0 < b < a$ , then  $0 < a^{-1} < b^{-1}$ , also if  $0 < b \leq a$ , then  $0 < a^{-1} \leq b^{-1}$ .

---

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

*Key words and phrases.* Weighted geometric mean, Weighted harmonic mean, Young's inequality, Operator modulus, Arithmetic mean-geometric mean-harmonic mean inequality.

Okayasu [11] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach  $*$ -algebra with continuous involution, namely if  $a, b \in A$  and  $p \in [0, 1]$  then  $a > b$  ( $a \geq b$ ) implies that  $a^p > b^p$  ( $a^p \geq b^p$ ).

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let  $a \in A$  and  $a > 0$ , then  $0 \notin \sigma(a)$  and the fact that  $\sigma(a)$  is a compact subset of  $\mathbb{C}$  implies that  $\inf\{z : z \in \sigma(a)\} > 0$  and  $\sup\{z : z \in \sigma(a)\} < \infty$ . Choose  $\gamma$  to be close rectifiable curve in  $\{\operatorname{Re} z > 0\}$ , the right half open plane of the complex plane, such that  $\sigma(a) \subset \operatorname{ins}(\gamma)$ , the inside of  $\gamma$ . Let  $G$  be an open subset of  $\mathbb{C}$  with  $\sigma(a) \subset G$ . If  $f : G \rightarrow \mathbb{C}$  is analytic, we define an element  $f(a)$  in  $A$  by

$$f(a) := \frac{1}{2\pi i} \int_{\gamma} f(z) (z - a)^{-1} dz.$$

It is well known (see for instance [3, pp. 201-204]) that  $f(a)$  does not depend on the choice of  $\gamma$  and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any  $\alpha \in \mathbb{R}$  we define for  $a \in A$  and  $a > 0$ , the real power

$$a^\alpha := \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z - a)^{-1} dz,$$

where  $z^\alpha$  is the principal  $\alpha$ -power of  $z$ . Since  $A$  is a Banach  $*$ -algebra, then  $a^\alpha \in A$ . Moreover, since  $z^\alpha$  is analytic in  $\{\operatorname{Re} z > 0\}$ , then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [5], we list below some important properties of real powers:

- (viii) If  $0 < a \in A$  and  $\alpha \in \mathbb{R}$ , then  $a^\alpha \in A$  with  $a^\alpha > 0$  and  $(a^2)^{1/2} = a$ , [13, Lemma 6];
- (ix) If  $0 < a \in A$  and  $\alpha, \beta \in \mathbb{R}$ , then  $a^\alpha a^\beta = a^{\alpha+\beta}$ ;
- (x) If  $0 < a \in A$  and  $\alpha \in \mathbb{R}$ , then  $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$ ;
- (xi) If  $0 < a, b \in A$ ,  $\alpha, \beta \in \mathbb{R}$  and  $ab = ba$ , then  $a^\alpha b^\beta = b^\beta a^\alpha$ .

We define the following means for  $\nu \in [0, 1]$ , see also [5] for different notations:

$$(A) \quad a \nabla_\nu b := (1 - \nu)a + \nu b, \quad a, b \in A$$

the *weighted arithmetic mean* of  $(a, b)$ ,

$$(H) \quad a !_\nu b := ((1 - \nu)a^{-1} + \nu b^{-1})^{-1}, \quad a, b > 0$$

the *weighted harmonic mean* of positive elements  $(a, b)$  and

$$(G) \quad a \#_\nu b := a^{1/2} \left( a^{-1/2} b a^{-1/2} \right)^\nu a^{1/2}$$

the *weighted geometric mean* of positive elements  $(a, b)$ . Our notations above are motivated by the classical notations used in operator theory. For simplicity, if  $\nu = \frac{1}{2}$ , we use the simpler notations  $a \nabla b$ ,  $a ! b$  and  $a \# b$ . The definition of weighted geometric mean can be extended for any real  $\nu$ .

In [5], B. Q. Feng proved the following properties of these means in  $A$  a Hermitian unital Banach  $*$ -algebra:

- (xii) If  $0 < a, b \in A$ , then  $a ! b = b ! a$  and  $a \# b = b \# a$ ;

(xiii) If  $0 < a, b \in A$  and  $c \in \text{Inv}(A)$ , then

$$c^* (a!b) c = (c^*ac)!(c^*bc) \text{ and } c^* (a\sharp b) c = (c^*ac)\sharp(c^*bc);$$

(xiv) If  $0 < a, b \in A$  and  $\nu \in [0, 1]$ , then

$$(a!_\nu b)^{-1} = (a^{-1}) \nabla_\nu (b^{-1}) \text{ and } (a^{-1}) \sharp_\nu (b^{-1}) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [5] the following inequality between the weighted means introduced above:

$$(HGA) \quad a \nabla_\nu b \geq a\sharp_\nu b \geq a!_\nu b$$

for any  $0 < a, b \in A$  and  $\nu \in [0, 1]$ .

In [13], Tanahashi and Uchiyama obtained the following identity of interest:

**Lemma 1.** *If  $0 < c, d$  and  $\lambda$  is a real number, then*

$$(1.1) \quad (dcd)^\lambda = dc^{1/2} \left( c^{1/2} d^2 c^{1/2} \right)^{\lambda-1} c^{1/2} d.$$

We can prove the following fact:

**Proposition 1.** *For any  $0 < a, b \in A$  we have*

$$(1.2) \quad b\sharp_{1-\nu} a = a\sharp_\nu b$$

for any real number  $\nu$ .

*Proof.* We take in (1.1)  $d = b^{-1/2}$  and  $c = a$  to get

$$\left( b^{-1/2} a b^{-1/2} \right)^\lambda = b^{-1/2} a^{1/2} \left( a^{1/2} b^{-1} a^{1/2} \right)^{\lambda-1} a^{1/2} b^{-1/2}.$$

If we multiply both sides of this equality by  $b^{1/2}$  we get

$$(1.3) \quad b^{1/2} \left( b^{-1/2} a b^{-1/2} \right)^\lambda b^{1/2} = a^{1/2} \left( a^{1/2} b^{-1} a^{1/2} \right)^{\lambda-1} a^{1/2}.$$

Since

$$\left( a^{1/2} b^{-1} a^{1/2} \right)^{\lambda-1} = \left[ \left( a^{1/2} b^{-1} a^{1/2} \right)^{-1} \right]^{1-\lambda} = \left( a^{-1/2} b a^{-1/2} \right)^{1-\lambda}$$

then by (1.3) we get

$$a\sharp_{1-\nu} b = b\sharp_\nu a.$$

By swapping in this equality  $a$  with  $b$  we get the desired result (1.2).  $\square$

In this paper we introduce the *quadratic weighted geometric mean* for invertible elements  $x, y$  in a Hermitian unital Banach  $*$ -algebra and real number  $\nu$ . We show that it can be represented in terms of  $\sharp_\nu$ , which is the usual geometric mean and provide some inequalities for this mean under various assumptions for the elements involved.

## 2. QUADRATIC WEIGHTED GEOMETRIC MEAN

In what follows we assume that  $A$  is a Hermitian unital Banach  $*$ -algebra.

We observe that if  $x \in \text{Inv}(A)$ , then  $x^* \in \text{Inv}(A)$ , which implies that  $x^*x \in \text{Inv}(A)$ . Therefore by Shirali-Ford theorem we have  $x^*x > 0$ . If we define the modulus of the element  $c \in A$  by  $|c| := (c^*c)^{1/2}$  then for  $c \in \text{Inv}(A)$  we have  $|c|^2 > 0$  and by (viii),  $|c| > 0$ . If  $c > 0$ , then by (viii) we have  $|c| = c$ .

For  $x, y \in \text{Inv}(A)$  we consider the element

$$(2.1) \quad d := (x^*)^{-1} y^* y x^{-1} = (y x^{-1})^* y x^{-1} = |y x^{-1}|^2.$$

Since  $y x^{-1} \in \text{Inv}(A)$  then  $d > 0$ ,  $d \in \text{Inv}(A)$ ,  $d^{-1} = |y x^{-1}|^{-2}$ , and also

$$(2.2) \quad d^{-1} = \left( (x^*)^{-1} y^* y x^{-1} \right)^{-1} = x y^{-1} (y^{-1})^* x^* = \left| (y^{-1})^* x^* \right|^2.$$

For  $\nu \in \mathbb{R}$ , by using the property (viii) we get that  $d^\nu = |y x^{-1}|^{2\nu} > 0$  and  $d^{\nu/2} = |y x^{-1}|^\nu > 0$ . Since

$$x^* d^\nu x = x^* |y x^{-1}|^{2\nu} x = \left| |y x^{-1}|^\nu x \right|^2$$

and  $|y x^{-1}|^\nu x \in \text{Inv}(A)$ , it follows that  $x^* d^\nu x > 0$ .

We introduce the *quadratic weighted mean* of  $(x, y)$  with  $x, y \in \text{Inv}(A)$  and the real weight  $\nu \in \mathbb{R}$ , as the positive element denoted by  $x \mathbb{S}_\nu y$  and defined by

$$(S) \quad x \mathbb{S}_\nu y := x^* \left( (x^*)^{-1} y^* y x^{-1} \right)^\nu x = x^* |y x^{-1}|^{2\nu} x = \left| |y x^{-1}|^\nu x \right|^2.$$

When  $\nu = 1/2$ , we denote  $x \mathbb{S}_{1/2} y$  by  $x \mathbb{S} y$  and we have

$$x \mathbb{S} y = x^* \left( (x^*)^{-1} y^* y x^{-1} \right)^{1/2} x = x^* |y x^{-1}| x = \left| |y x^{-1}|^{1/2} x \right|^2.$$

We can also introduce the *1/2-quadratic weighted mean* of  $(x, y)$  with  $x, y \in \text{Inv}(A)$  and the real weight  $\nu \in \mathbb{R}$  by

$$(1/2-S) \quad x \mathbb{S}_\nu^{1/2} y := (x \mathbb{S}_\nu y)^{1/2} = \left| |y x^{-1}|^\nu x \right|.$$

Correspondingly, when  $\nu = 1/2$  we denote  $x \mathbb{S}^{1/2} y$  and we have

$$x \mathbb{S}^{1/2} y = \left| |y x^{-1}|^{1/2} x \right|.$$

The following equalities hold:

**Proposition 2.** For any  $x, y \in \text{Inv}(A)$  and  $\nu \in \mathbb{R}$  we have

$$(2.3) \quad (x \mathbb{S}_\nu y)^{-1} = (x^*)^{-1} \mathbb{S}_\nu (y^*)^{-1}$$

and

$$(2.4) \quad (x^{-1}) \mathbb{S}_\nu (y^{-1}) = (x^* \mathbb{S}_\nu y^*)^{-1}.$$

*Proof.* We observe that for any  $x, y \in \text{Inv}(A)$  and  $\nu \in \mathbb{R}$  we have

$$(x \mathbb{S}_\nu y)^{-1} = \left( x^* \left( (x^*)^{-1} y^* y x^{-1} \right)^\nu x \right)^{-1} = x^{-1} \left( x y^{-1} (y^*)^{-1} x^* \right)^\nu (x^*)^{-1}$$

and

$$\begin{aligned}
 & (x^*)^{-1} \mathbb{S}_\nu (y^*)^{-1} \\
 &= \left( (x^*)^{-1} \right)^* \left( \left( \left( (x^*)^{-1} \right)^* \right)^{-1} \left( (y^*)^{-1} \right)^* (y^*)^{-1} \left( (x^*)^{-1} \right)^{-1} \right)^\nu (x^*)^{-1} \\
 &= x^{-1} \left( xy^{-1} (y^*)^{-1} x^* \right)^\nu (x^*)^{-1},
 \end{aligned}$$

which proves (2.3).

If we replace in (2.3)  $x$  by  $x^{-1}$  and  $y$  by  $y^{-1}$  we get

$$\left( (x^{-1}) \mathbb{S}_\nu (y^{-1}) \right)^{-1} = x^* \mathbb{S}_\nu y^*$$

and by taking the inverse in this equality we get (2.4).  $\square$

If we take in (S)  $x = a^{1/2}$  and  $y = b^{1/2}$  with  $a, b > 0$  then we get

$$a^{1/2} \mathbb{S}_\nu b^{1/2} = a \#_\nu b$$

for any  $\nu \in \mathbb{R}$  that shows that the quadratic weighted mean can be seen as an extension of the weighted geometric mean for positive elements considered in the introduction.

Let  $x, y \in \text{Inv}(A)$ . If we take in the definition of " $\#_\nu$ " the elements  $a = |x|^2 > 0$  and  $b = |y|^2 > 0$  we also have for real  $\nu$

$$|x|^2 \#_\nu |y|^2 = |x| \left( |x|^{-1} |y|^2 |x|^{-1} \right)^\nu |x| = |x| \left| |y| |x|^{-1} \right|^{2\nu} |x| = \left| |y| |x|^{-1} \right|^\nu |x|^2.$$

It is then natural to ask how the positive elements  $x \mathbb{S}_\nu y$  and  $|x|^2 \#_\nu |y|^2$  do compare, when  $x, y \in \text{Inv}(A)$  and  $\nu \in \mathbb{R}$  ?

We need the following lemma that provides a slight generalization of Lemma 1.

**Lemma 2.** *If  $0 < c, d \in \text{Inv}(A)$  and  $\lambda$  is a real number, then*

$$(2.5) \quad (dcd^*)^\lambda = dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{\lambda-1} c^{1/2} d^*.$$

*Proof.* We provide an argument along the lines in the proof of Lemma 7 from [13].

Consider the functions  $F(\lambda) := (dcd^*)^\lambda$  and  $G(\lambda) := dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{\lambda-1} c^{1/2} d^*$  defined for  $\lambda \in \mathbb{R}$ . It is obvious that  $F(1) = G(1)$ .

We have

$$\begin{aligned}
 G^2 \left( \frac{1}{2} \right) &= dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-1/2} c^{1/2} d^* dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-1/2} c^{1/2} d^* \\
 &= dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-1/2} c^{1/2} |d|^2 c^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-1/2} c^{1/2} d^* \\
 &= dcd^* = F^2 \left( \frac{1}{2} \right)
 \end{aligned}$$

and

$$\begin{aligned}
G^{2^2} \left( \frac{1}{2^2} \right) &= \left( dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{\frac{1-2^2}{2^2}} c^{1/2} d^* \right)^{2^2} \\
&= dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} d^* dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} d^* \\
&dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} d^* dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} d^* \\
&= dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} |d|^2 c^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} d^* \\
&dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} |d|^2 c^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{3}{4}} c^{1/2} d^* \\
&= dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{1}{2}} c^{1/2} d^* dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{1}{2}} c^{1/2} d^* \\
&= dc^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{1}{2}} c^{1/2} |d|^2 c^{1/2} \left( c^{1/2} |d|^2 c^{1/2} \right)^{-\frac{1}{2}} c^{1/2} d^* \\
&= dcd^* = F^{2^2} \left( \frac{1}{2^2} \right).
\end{aligned}$$

By induction we can conclude that  $G^{2^n} \left( \frac{1}{2^n} \right) = F^{2^n} \left( \frac{1}{2^n} \right)$  for any natural number  $n \geq 0$ . Since for any  $a > 0$  we have  $(a^2)^{1/2} = a$ , [13, Lemma 6], hence  $G \left( \frac{1}{2^n} \right) = F \left( \frac{1}{2^n} \right)$  for any natural number  $n \geq 0$ .

Since  $F(\lambda)$ ;  $G(\lambda)$  are analytic on the real line  $\mathbb{R}$  and  $\frac{1}{2^n} \rightarrow 0$  for  $n \rightarrow \infty$ , we deduce that  $F(\lambda) = G(\lambda)$  for any  $\lambda \in \mathbb{R}$ .  $\square$

**Remark 1.** *The identity (2.5) was proved by T. Furuta in [6] for positive operator  $c$  and invertible operator  $d$  in the Banach algebra of all bounded linear operators on a Hilbert space by using the polar decomposition of the invertible operator  $dc^{1/2}$ .*

**Theorem 1.** *If  $x, y \in \text{Inv}(A)$  and  $\lambda$  is a real number, then*

$$(2.6) \quad x \otimes_{\nu} y = |x|^2 \sharp_{\nu} |y|^2$$

*Proof.* If we take  $d = (x^*)^{-1}$  and  $c = |y|^2 > 0$  in (2.5), then we get

$$\begin{aligned}
\left( (x^*)^{-1} |y|^2 x^{-1} \right)^{\lambda} &= (x^*)^{-1} |y| \left( |y| \left| (x^*)^{-1} \right|^2 |y| \right)^{\lambda-1} |y| x^{-1} \\
&= (x^*)^{-1} |y| \left( |y| \left( (x^*)^{-1} \right)^* (x^*)^{-1} |y| \right)^{\lambda-1} |y| x^{-1} \\
&= (x^*)^{-1} |y| \left( |y| x^{-1} (x^*)^{-1} |y| \right)^{\lambda-1} |y| x^{-1} \\
&= (x^*)^{-1} |y| \left( |y| (x^* x)^{-1} |y| \right)^{\lambda-1} |y| x^{-1} \\
&= (x^*)^{-1} |y| \left( |y| |x|^{-2} |y| \right)^{\lambda-1} |y| x^{-1}.
\end{aligned}$$



Therefore, if we multiply the inequality (2.11) at left with  $x^*$  and at right with  $x$ , then we get

$$(2.12) \quad x^* \left(1 - \nu + \nu |yx^{-1}|^2\right) x \geq x^* |yx^{-1}|^{2\nu} x \geq x^* \left(1 - \nu + \nu |yx^{-1}|^{-2}\right)^{-1} x$$

for any  $\nu \in [0, 1]$ .

Observe that

$$\begin{aligned} x^* \left(1 - \nu + \nu |yx^{-1}|^2\right) x &= x^* \left(1 - \nu + \nu (x^*)^{-1} y^* y x^{-1}\right) x \\ &= x^* \left(1 - \nu + \nu (x^*)^{-1} y^* y x^{-1}\right) x \\ &= (1 - \nu) |x|^2 + \nu |y|^2 = |x|^2 \nabla_\nu |y|^2 \end{aligned}$$

and

$$\begin{aligned} x^* \left(1 - \nu + \nu |yx^{-1}|^{-2}\right)^{-1} x &= x^* \left(1 - \nu + \nu \left((x^*)^{-1} y^* y x^{-1}\right)^{-1}\right)^{-1} x \\ &= x^* \left(1 - \nu + \nu x y^{-1} (y^*)^{-1} x^*\right)^{-1} x \\ &= x^* \left(x \left((1 - \nu) x^{-1} (x^*)^{-1} + \nu y^{-1} (y^*)^{-1}\right) x^*\right)^{-1} x \\ &= x^* \left(x \left((1 - \nu) (x^* x)^{-1} + \nu (y^* y)^{-1}\right) x^*\right)^{-1} x \\ &= x^* (x^*)^{-1} \left((1 - \nu) (x^* x)^{-1} + \nu (y^* y)^{-1}\right)^{-1} x^{-1} x \\ &= \left((1 - \nu) |x|^{-2} + \nu |y|^{-2}\right)^{-1} = |x|^2 \mathbb{1}_\nu |y|^2. \end{aligned}$$

Therefore by (2.12) we get the desired result (2.9).  $\square$

We can define the weighted means for  $\nu \in [0, 1]$  and the elements  $x, y \in \text{Inv}(A)$  and  $\nu \in [0, 1]$  by

$$x \nabla_\nu^{1/2} y := \left(|x|^2 \nabla_\nu |y|^2\right)^{1/2} = \left((1 - \nu) |x|^2 + \nu |y|^2\right)^{1/2}$$

and

$$x \mathbb{1}_\nu^{1/2} y := \left(|x|^2 \mathbb{1}_\nu |y|^2\right)^{1/2} = \left((1 - \nu) |x|^{-2} + \nu |y|^{-2}\right)^{-1/2}.$$

**Corollary 1.** *Let  $A$  be a Hermitian unital Banach  $*$ -algebra with continuous involution. Then for any  $x, y \in \text{Inv}(A)$  and  $\nu \in [0, 1]$  we have*

$$(2.13) \quad x \nabla_\nu^{1/2} y \geq x \mathbb{S}_\nu^{1/2} y \geq x \mathbb{1}_\nu^{1/2} y.$$

*Proof.* It follows by taking the square root in the inequality (2.9) and by using Okayasu's result from the introduction.  $\square$

Recall that a  $C^*$ -algebra  $A$  is a Banach  $*$ -algebra such that the norm satisfies the condition

$$\|a^* a\| = \|a\|^2 \text{ for any } a \in A.$$

If a  $C^*$ -algebra  $A$  has a unit 1, then automatically  $\|1\| = 1$ .

It is well know that, if  $A$  is a  $C^*$ -algebra, then (see for instance [10, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \|b\| \geq \|a\|.$$



**Corollary 2.** *Let  $A$  be a unital  $C^*$ -algebra. Then for any  $x, y \in \text{Inv}(A)$  and  $\nu \in [0, 1]$  we have*

$$(2.14) \quad (1 - \nu) \|x\|^2 + \nu \|y\|^2 \geq \left\| (1 - \nu) |x|^2 + \nu |y|^2 \right\| \geq \left\| |yx^{-1}|^\nu x \right\|^2.$$

### 3. REFINEMENTS AND REVERSES

If  $X$  is a linear space and  $C \subseteq X$  a convex subset in  $X$ , then for any convex function  $f : C \rightarrow \mathbb{R}$  and any  $z_i \in C, r_i \geq 0$  for  $i \in \{1, \dots, k\}, k \geq 2$  with  $\sum_{i=1}^k r_i = R_k > 0$  one has the *weighted Jensen's inequality*:

$$(J) \quad \frac{1}{R_k} \sum_{i=1}^k r_i f(z_i) \geq f\left(\frac{1}{R_k} \sum_{i=1}^k r_i z_i\right).$$

If  $f : C \rightarrow \mathbb{R}$  is strictly convex and  $r_i > 0$  for  $i \in \{1, \dots, k\}$  then the equality case holds in (J) if and only if  $z_1 = \dots = z_n$ .

By  $\mathcal{P}_n$  we denote the set of all nonnegative  $n$ -tuples  $(p_1, \dots, p_n)$  with the property that  $\sum_{i=1}^n p_i = 1$ . Consider the *normalised Jensen functional*

$$\mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geq 0,$$

where  $f : C \rightarrow \mathbb{R}$  be a convex function on the convex set  $C$  and  $\mathbf{x} = (x_1, \dots, x_n) \in C^n$  and  $\mathbf{p} \in \mathcal{P}_n$ .

The following result holds [4]:

**Lemma 4.** *If  $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n, q_i > 0$  for each  $i \in \{1, \dots, n\}$  then*

$$(3.1) \quad \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \mathcal{J}_n(f, \mathbf{x}, \mathbf{q}) \geq \mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) \geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \mathcal{J}_n(f, \mathbf{x}, \mathbf{q}) (\geq 0).$$

In the case  $n = 2$ , if we put  $p_1 = 1 - p, p_2 = p, q_1 = 1 - q$  and  $q_2 = q$  with  $p \in [0, 1]$  and  $q \in (0, 1)$  then by (3.1) we get

$$(3.2) \quad \begin{aligned} & \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q)f(x) + qf(y) - f((1-q)x + qy)] \\ & \geq (1-p)f(x) + pf(y) - f((1-p)x + py) \\ & \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q)f(x) + qf(y) - f((1-q)x + qy)] \end{aligned}$$

for any  $x, y \in C$ .

If we take  $q = \frac{1}{2}$  in (3.2), then we get

$$(3.3) \quad \begin{aligned} & 2 \max \{t, 1-t\} \left[ \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right] \\ & \geq (1-t)f(x) + tf(y) - f((1-t)x + ty) \\ & \geq 2 \min \{t, 1-t\} \left[ \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \right] \end{aligned}$$

for any  $x, y \in C$  and  $t \in [0, 1]$ .

We consider the scalar weighted arithmetic, geometric and harmonic means defined by  $A_\nu(a, b) := (1 - \nu)a + \nu b, G_\nu(a, b) := a^{1-\nu}b^\nu$  and  $H_\nu(a, b) = A_\nu^{-1}(a^{-1}, b^{-1})$  where  $a, b > 0$  and  $\nu \in [0, 1]$ .

If we take the convex function  $f : \mathbb{R} \rightarrow (0, \infty)$ ,  $f(x) = \exp(\alpha x)$ , with  $\alpha \neq 0$ , then we have from (3.2) that

$$(3.4) \quad \begin{aligned} & \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a, b))] \\ & \geq A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a, b)) \\ & \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a, b))] \end{aligned}$$

for any  $p \in [0, 1]$  and  $q \in (0, 1)$  and any  $x, y \in \mathbb{R}$ .

For  $q = \frac{1}{2}$  we have by (3.4) that

$$(3.5) \quad \begin{aligned} & 2 \max \{p, 1-p\} [A(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A(a, b))] \\ & \geq A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a, b)) \\ & \geq 2 \min \{p, 1-p\} [A(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A(a, b))] \end{aligned}$$

for any  $p \in [0, 1]$  and any  $x, y \in \mathbb{R}$ .

If we take  $x = \ln a$  and  $y = \ln b$  in (3.4), then we get

$$(3.6) \quad \begin{aligned} & \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [A_q(a^\alpha, b^\alpha) - G_q^\alpha(a, b)] \geq A_p(a^\alpha, b^\alpha) - G_p^\alpha(a, b) \\ & \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [A_q(a^\alpha, b^\alpha) - G_q^\alpha(a, b)] \end{aligned}$$

for any  $a, b > 0$ , for any  $p \in [0, 1]$ ,  $q \in (0, 1)$  and  $\alpha \neq 0$ .

For  $q = \frac{1}{2}$  we have by (3.6) that

$$(3.7) \quad \begin{aligned} & \max \{p, 1-p\} (b^{\frac{\alpha}{2}} - a^{\frac{\alpha}{2}})^2 \geq A_p(a^\alpha, b^\alpha) - G_p^\alpha(a, b) \\ & \geq \min \{p, 1-p\} (b^{\frac{\alpha}{2}} - a^{\frac{\alpha}{2}})^2 \end{aligned}$$

for any  $a, b > 0$ , for any  $p \in [0, 1]$  and  $\alpha \neq 0$ .

For  $\alpha = 1$  we get from (3.7) that

$$(3.8) \quad \begin{aligned} & \max \{p, 1-p\} (\sqrt{b} - \sqrt{a})^2 \geq A_p(a, b) - G_p(a, b) \\ & \geq \min \{p, 1-p\} (\sqrt{b} - \sqrt{a})^2 \end{aligned}$$

for any  $a, b > 0$  and for any  $p \in [0, 1]$ , which are the inequalities obtained by Kittaneh and Manasrah in [8] and [9].

For  $\alpha = 1$  in (3.6) we obtain

$$(3.9) \quad \begin{aligned} & \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [A_q(a, b) - G_q(a, b)] \geq A_p(a, b) - G_p(a, b) \\ & \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [A_q(a, b) - G_q(a, b)], \end{aligned}$$

for any  $a, b > 0$ , for any  $p \in [0, 1]$ , which is the inequality (2.1) from [1] in the particular case  $\lambda = 1$  in a slightly more general form for the weights  $p, q$ .

We have the following refinement and reverse for the inequality (2.1):

**Theorem 3.** For any  $x, y \in \text{Inv}(A)$  we have for  $p \in [0, 1]$  and  $q \in (0, 1)$  that

$$(3.10) \quad \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} \left( |x|^2 \nabla_q |y|^2 - x \mathbb{S}_q y \right) \geq |x|^2 \nabla_p |y|^2 - x \mathbb{S}_p y \\ \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} \left( |x|^2 \nabla_q |y|^2 - x \mathbb{S}_q y \right).$$

In particular, we have

$$(3.11) \quad 2 \max \{p, 1-p\} \left( |x|^2 \nabla |y|^2 - x \mathbb{S} y \right) \geq |x|^2 \nabla_p |y|^2 - x \mathbb{S}_p y \\ \geq 2 \min \{p, 1-p\} \left( |x|^2 \nabla |y|^2 - x \mathbb{S} y \right),$$

for any  $p \in [0, 1]$ .

*Proof.* From the inequality (3.9) for  $a = 1$  and  $b = t > 0$  we have

$$(3.12) \quad \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (1 - q + qt - t^q) \geq 1 - p + pt - t^p \\ \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (1 - q + qt - t^q),$$

where  $p \in [0, 1]$  and  $q \in (0, 1)$ .

Consider the functions  $f(z) := \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (1 - q + qz - z^q)$ ,  $g(z) := 1 - p + pz - z^p$  and  $h(z) = \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (1 - q + qt - t^q)$  where  $z^\nu$ ,  $\nu \in \{p, q\}$ , is the principal of the power function. Then  $f(z)$ ,  $g(z)$  and  $h(z)$  are analytic in the right half open plane  $\{\text{Re } z > 0\}$  of the complex plane and and by (3.12) we have  $f(z) \geq g(z) \geq h(z)$  for any  $z > 0$ .

If  $0 < u \in \text{Inv}(A)$  and  $\nu \in [0, 1]$ , then by Lemma 3 we get

$$(3.13) \quad \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (1 - q + qu - u^q) \geq 1 - p + pu - u^p \\ \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (1 - q + qu - u^q),$$

where  $p \in [0, 1]$  and  $q \in (0, 1)$ .

If  $x, y \in \text{Inv}(A)$ , then by taking  $u = |yx^{-1}|^2 \in \text{Inv}(A)$  in (3.13) we have

$$(3.14) \quad \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} \left( 1 - q + q |yx^{-1}|^2 - \left( |yx^{-1}|^2 \right)^q \right) \\ \geq 1 - p + p |yx^{-1}|^2 - \left( |yx^{-1}|^2 \right)^p \\ \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} \left( 1 - q + q |yx^{-1}|^2 - \left( |yx^{-1}|^2 \right)^q \right),$$

where  $p \in [0, 1]$  and  $q \in (0, 1)$ .

By multiplying the inequality (3.14) at left with  $x^*$  and at right with  $x$  we get the desired result (3.10).  $\square$

**Remark 2.** If  $0 < a, b \in A$ , then by taking  $x = a^{1/2}$  and  $y = b^{1/2}$  in (3.10) and (3.11) we get

$$(3.15) \quad \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (a\nabla_q b - a\sharp_q b) \geq a\nabla_p b - a\sharp_p b \\ \geq \min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} (a\nabla_q b - a\sharp_q b),$$

for any  $p \in [0, 1]$  and  $q \in (0, 1)$ .

In particular, for  $q = 1/2$  we have

$$(3.16) \quad 2 \max \{p, 1-p\} (a\nabla b - a\sharp b) \geq a\nabla_p b - a\sharp_p b \\ \geq 2 \min \{p, 1-p\} (a\nabla b - a\sharp b),$$

for any  $p \in [0, 1]$ .

#### 4. INEQUALITIES UNDER BOUNDEDNESS CONDITIONS

We consider the function  $f_\nu : [0, \infty) \rightarrow [0, \infty)$  defined for  $\nu \in (0, 1)$  by

$$f_\nu(t) = 1 - \nu + \nu t - t^\nu = A_\nu(1, t) - G_\nu(1, t),$$

where  $A_\nu(\cdot, \cdot)$  and  $G_\nu(\cdot, \cdot)$  are the scalar arithmetic and geometric means.

The following lemma holds.

**Lemma 5.** For any  $t \in [k, K] \subset [0, \infty)$  we have

$$(4.1) \quad \max_{t \in [k, K]} f_\nu(x) = \Delta_\nu(k, K) := \begin{cases} A_\nu(1, k) - G_\nu(1, k) & \text{if } K < 1, \\ \max \{A_\nu(1, k) - G_\nu(1, k), A_\nu(1, K) - G_\nu(1, K)\} & \text{if } k \leq 1 \leq K, \\ A_\nu(1, K) - G_\nu(1, K) & \text{if } 1 < k \end{cases}$$

and

$$(4.2) \quad \min_{t \in [k, K]} f_\nu(x) = \delta_\nu(k, K) := \begin{cases} A_\nu(1, K) - G_\nu(1, K) & \text{if } K < 1, \\ 0 & \text{if } k \leq 1 \leq K, \\ A_\nu(1, k) - G_\nu(1, k) & \text{if } 1 < K. \end{cases}$$

*Proof.* The function  $f_\nu$  is differentiable and

$$f'_\nu(t) = \nu(1 - t^{\nu-1}) = \nu \frac{t^{1-\nu} - 1}{t^{1-\nu}}, \quad t > 0,$$

which shows that the function  $f_\nu$  is decreasing on  $[0, 1]$  and increasing on  $[1, \infty)$ ,  $f_\nu(0) = 1 - \nu$ ,  $f_\nu(1) = 0$ ,  $\lim_{t \rightarrow \infty} f_\nu(t) = \infty$  and the equation  $f_\nu(t) = 1 - \nu$  for  $t > 0$  has the unique solution  $t_\nu = \nu^{\frac{1}{\nu-1}} > 1$ .

Therefore, by considering the 3 possible situations for the location of the interval  $[k, K]$  and the number 1 we get the desired bounds (4.1) and (4.2).  $\square$

**Remark 3.** We have the inequalities

$$0 \leq f_\nu(t) \leq 1 - \nu \text{ for any } t \in \left[0, \nu^{\frac{1}{\nu-1}}\right]$$

and

$$1 - \nu \leq f_\nu(t) \text{ for any } t \in \left[ \nu^{\frac{1}{\nu-1}}, \infty \right).$$

Assume that  $x, y \in \text{Inv}(A)$  and the constants  $M > m > 0$  are such that

$$(4.3) \quad M \geq |yx^{-1}| \geq m.$$

The inequality (4.3) is equivalent to

$$M^2 \geq (x^*)^{-1} |y|^2 x^{-1} \geq m^2.$$

If we multiply at left with  $x^*$  and at right with  $x$  we get the equivalent relation

$$(4.4) \quad M^2 |x|^2 \geq |y|^2 \geq m^2 |x|^2.$$

We have:

**Theorem 4.** *Assume that  $x, y \in \text{Inv}(A)$  and the constants  $M > m > 0$  are such that either (4.3), or, equivalently (4.4) is true. Then we have the inequalities*

$$(4.5) \quad \Delta_\nu(m^2, M^2) |x|^2 \geq |x|^2 \nabla_\nu |y|^2 - x \mathbb{S}_\nu y \geq \delta_\nu(m^2, M^2) |x|^2,$$

for any  $\nu \in [0, 1]$ , where  $\Delta_\nu(\cdot, \cdot)$  and  $\delta_\nu(\cdot, \cdot)$  are defined by (4.1) and (4.2), respectively.

*Proof.* From Lemma 5 we have the double inequality

$$\Delta_\nu(k, K) \geq 1 - \nu + \nu t - t^\nu \geq \delta_\nu(k, K)$$

for any  $x \in [k, K] \subset (0, \infty)$  and  $\nu \in [0, 1]$ .

If  $u \in A$  is an element such that  $0 < k \leq u \leq K$ , then  $\sigma(u) \subset [k, K]$  and by Lemma 3 we have in the order of  $A$  that

$$(4.6) \quad \Delta_\nu(k, K) \geq 1 - \nu + \nu u - u^\nu \geq \delta_\nu(k, K)$$

for any  $\nu \in [0, 1]$ .

If we take  $u = |yx^{-1}|^2$ , then by (4.3) we have  $0 < m^2 \leq u \leq M^2$  and by (4.6) we get in the order of  $A$  that

$$(4.7) \quad \Delta_\nu(m^2, M^2) \geq 1 - \nu + \nu |yx^{-1}|^2 - |yx^{-1}|^{2\nu} \geq \delta_\nu(m^2, M^2)$$

for any  $\nu \in [0, 1]$ .

If we multiply at left with  $x^*$  and at right with  $x$  we get

$$(4.8) \quad \begin{aligned} \Delta_\nu(m^2, M^2) |x|^2 &\geq (1 - \nu) |x|^2 + \nu x^* |yx^{-1}|^2 x - x^* |yx^{-1}|^{2\nu} x \\ &\geq \delta_\nu(m^2, M^2) |x|^2 \end{aligned}$$

and since  $x^* |yx^{-1}|^2 x = x^* (x^*)^{-1} |y|^2 x^{-1} x = |y|^2$  and  $x^* |yx^{-1}|^{2\nu} x = x \mathbb{S}_\nu y$  we get from (4.8) the desired result (4.5).  $\square$

**Corollary 3.** *With the assumptions of Theorem 4 we have*

$$(4.9) \quad R \times \begin{cases} (1 - m)^2 |x|^2 & \text{if } M < 1, \\ \max \left\{ (1 - m)^2, (M - 1)^2 \right\} |x|^2 & \text{if } m \leq 1 \leq M, \\ (M - 1)^2 |x|^2 & \text{if } 1 < m, \end{cases}$$

$$\geq |x|^2 \nabla_\nu |y|^2 - x \mathbb{S}_\nu y \geq r \times \begin{cases} (1-M)^2 |x|^2 & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ (m-1)^2 |x|^2 & \text{if } 1 < m, \end{cases},$$

where  $\nu \in [0, 1]$ ,  $r = \min \{1 - \nu, \nu\}$  and  $R = \max \{1 - \nu, \nu\}$ .

*Proof.* From the inequality (3.8) we have for  $b = t$  and  $a = 1$  that

$$R \left( \sqrt{t} - 1 \right)^2 \geq f_\nu(t) = 1 - \nu + \nu t - t^\nu \geq r \left( \sqrt{t} - 1 \right)^2$$

for any  $t \in [0, 1]$ .

Then we have

$$\Delta_\nu(m^2, M^2) \leq R \times \begin{cases} (1-m)^2 & \text{if } M < 1, \\ \max \left\{ (1-m)^2, (M-1)^2 \right\} & \text{if } m \leq 1 \leq M, \\ (M-1)^2 & \text{if } 1 < m \end{cases}$$

and

$$\delta_\nu(m^2, M^2) \geq r \times \begin{cases} (1-M)^2 & \text{if } M < 1, \\ 0 & \text{if } m \leq 1 \leq M, \\ (m-1)^2 & \text{if } 1 < m, \end{cases}$$

which by Theorem 4 proves the corollary.  $\square$

We observe that, with the assumptions of Theorem 4 and if  $A$  is a unital  $C^*$ -algebra, then by taking the norm in (4.5), we get

$$(4.10) \quad \Delta_\nu(m^2, M^2) \|x\|^2 \geq \left\| |x|^2 \nabla_\nu |y|^2 - x \mathbb{S}_\nu y \right\| \geq \delta_\nu(m^2, M^2) \|x\|^2,$$

for any  $\nu \in [0, 1]$ , which, by triangle inequality also implies that

$$(4.11) \quad \Delta_\nu(m^2, M^2) \|x\|^2 \geq \left\| (1-\nu)|x|^2 + \nu|y|^2 \right\| - \left\| |yx^{-1}|^\nu x \right\|^2 \geq 0$$

for any  $\nu \in [0, 1]$ . This provides a reverse for the second inequality in (2.14).

**Remark 4.** If  $0 < a, b \in A$  and there exists the constants  $0 < k < K$  such that

$$(4.12) \quad Ka \geq b \geq ka > 0,$$

then by (4.5) we get

$$(4.13) \quad \Delta_\nu(k, K) a \geq a \nabla_\nu b - a \sharp_\nu b \geq \delta_\nu(k, K) a,$$

while by (4.9) we get

$$(4.14) \quad R \times \begin{cases} (1 - \sqrt{k})^2 a \text{ if } K < 1, \\ \max \left\{ (1 - \sqrt{k})^2, (\sqrt{K} - 1)^2 \right\} a \text{ if } m \leq 1 \leq M, \\ (\sqrt{K} - 1)^2 a \text{ if } 1 < k, \end{cases}$$

$$\geq a \nabla_{\nu} b - a \sharp_{\nu} b \geq r \times \begin{cases} (1 - \sqrt{K})^2 a \text{ if } K < 1, \\ 0 \text{ if } k \leq 1 \leq K, \\ (\sqrt{k} - 1)^2 a \text{ if } 1 < k \end{cases},$$

where  $\nu \in [0, 1]$ ,  $r = \min \{1 - \nu, \nu\}$  and  $R = \max \{1 - \nu, \nu\}$ .

REFERENCES

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, **63** (2015), Issue 3, 622-635.  
 [2] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.  
 [3] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.  
 [4] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.  
 [5] B. Q. Feng, The geometric means in Banach \*-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.  
 [6] T. Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization. *Linear Algebra Appl.* **219** (1995), 139-155.  
 [7] F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators. *Publ. Math. Debrecen* **80** (2012), no. 3-4, 465-478.  
 [8] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.  
 [9] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra*, **59** (2011), 1031-1037.  
 [10] G. J. Murphy, *C\*-Algebras and Operator Theory*, Academic Press, 1990.  
 [11] T. Okayasu, The Löwner-Heinz inequality in Banach \*-algebra, *Glasgow Math. J.* **42** (2000), 243-246.  
 [12] S. Shiralı and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.  
 [13] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach \*-algebras, *Proc. Amer. Math. Soc.* **128** (2000), 1691-1695.

<sup>1</sup>MATHEMATICS, COLLEGE OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.

E-mail address: sever.dragomir@vu.edu.au

URL: <http://rgmia.org/dragomir>

<sup>2</sup>DST-NRF CENTRE OF EXCELLENCE, IN THE MATHEMATICAL AND STATISTICAL SCIENCES, SCHOOL OF COMPUTER SCIENCE & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA