A NEW HERMITE-HADAMARD INEQUALITY FOR $h$–CONVEX STOCHASTIC PROCESSES

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Abstract. Firstly, some new definitions which are the special cases of $h$–convex stochastic processes are given. Then, we establish a new refinement of Hermite-Hadamard inequality for $h$–convex stochastic processes and give some special cases of this result.

1. Introduction

The classical Hermite-Hadamard inequality which was first published in [5] gives us an estimate of the mean value of a convex function $f : I \rightarrow \mathbb{R}$,

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

An account the history of this inequality can be found in [3]. Surveys on various generalizations and developments can be found in [2] and [9].


Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \rightarrow \mathbb{R}$ is called a random variable if it is $\mathcal{A}$–measurable. A function $X : I \times \Omega \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t,.)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is called

(i) continuous in probability in interval $I$, if for all $t_0 \in I$ we have

$$P - \lim_{t \rightarrow t_0} X(t,.) = X(t_0,.)$$

where $P - \lim$ denotes the limit in probability.

(ii) mean-square continuous in the interval $I$, if for all $t_0 \in I$

$$\lim_{t \rightarrow t_0} E \left[ (X(t) - X(t_0))^2 \right] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t,.)$.

Obviously, mean-square continuity implies continuity in probability, but the converse implication is not true.

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Definition 1. Suppose we are given a sequence \( \{ \Delta^m \} \) of partitions, \( \Delta^m = \{ a_{m,0}, \ldots, a_{m,n_m} \} \). We say that the sequence \( \{ \Delta^m \} \) is a normal sequence of partitions if the length of the greatest interval in the \( n \)-th partition tends to zero, i.e.,
\[
\lim_{m \to \infty} \sup_{1 \leq i \leq n_m} |a_{m,i} - a_{m,i-1}| = 0.
\]

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [15].

Let \( X : I \times \Omega \to \mathbb{R} \) be a stochastic process with \( \mathbb{E}[X(t)^2] < \infty \) for all \( t \in I \). Let \([a, b] \subset I\), \( a = t_0 < t_1 < t_2 < \ldots < t_n = b \) be a partition of \([a, b]\) and \( \Theta_k \in [t_{k-1}, t_k] \) for all \( k = 1, \ldots, n \). A random variable \( Y : \Omega \to \mathbb{R} \) is called the mean-square integral of the process \( X \) on \([a, b]\), if we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y \right)^2 \right] = 0
\]
for all normal sequence of partitions of the interval \([a, b]\) and for all \( \Theta_k \in [t_{k-1}, t_k] \), \( k = 1, \ldots, n \). Then, we write
\[
Y(\cdot) = \int_a^b X(t, \cdot) \, dt \quad (a.e.).
\]

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process \( X \).

Throughout the paper we will frequently use the monotonicity of the mean-square integral. If \( X(t, \cdot) \leq Y(t, \cdot) \quad (a.e.) \) in some interval \([a, b]\), then
\[
\int_a^b X(t, \cdot) \, dt \leq \int_a^b Y(t, \cdot) \, dt \quad (a.e.).
\]

Of course, this inequality is the immediate consequence of the definition of the mean-square integral.

Definition 2. We say that a stochastic processes \( X : I \times \Omega \to \mathbb{R} \) is convex, if for all \( \lambda \in [0, 1] \) and \( u, v \in I \) the inequality
\[
X(\lambda u + (1 - \lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot) \quad (a.e.)
\]
is satisfied. If the above inequality is assumed only for \( \lambda = \frac{1}{2} \), then the process \( X \) is Jensen-convex or \( \frac{1}{2} \)-convex. A stochastic process \( X \) is concave if \((-X) \) is convex.

Some interesting properties of convex and Jensen-convex processes are presented in [10, 15].

Now, we present some results proved by Kotrys [6] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 1. If \( X : I \times \Omega \to \mathbb{R} \) is a stochastic process of the form \( X(t, \cdot) = A(\cdot)t + B(\cdot) \), where \( A, B : \Omega \to \mathbb{R} \) are random variables, such that \( \mathbb{E}[A^2] < \infty, \mathbb{E}[B^2] < \infty \) and \([a, b] \subset I\), then
\[
\int_a^b X(t, \cdot) \, dt = A(\cdot) \frac{b^2 - a^2}{2} + B(\cdot)(b - a) \quad (a.e.).
\]
Proposition 1. Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in \text{int}I$. Then there exist a random variable $A : \Omega \to \mathbb{R}$ such that $X$ is supported at $t_0$ by the process $A(\cdot)(t - t_0) + X(t_0, \cdot)$. That is

$$X(t, \cdot) \geq A(\cdot)(t - t_0) + X(t_0, \cdot) \text{ (a.e.)}$$

for all $t \in I$.

Theorem 1. Let $X : I \times \Omega \to \mathbb{R}$ be Jensen-convex, mean-square continuous in the interval $I$ stochastic process. Then for any $u, v \in I$ we have

$$X \left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot)\ dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \text{ (a.e.)} \quad (1.3)$$

In [11], Sarskaya et al. proved the following refinement of the inequality (1.3):

Theorem 2. If $X : I \times \Omega \to \mathbb{R}$ be Jensen-convex, mean-square continuous in the interval $I$ stochastic process. Then for any $u, v \in I$ and for all $\lambda \in [0, 1]$, we have

$$X \left(\frac{u + v}{2}, \cdot\right) \leq h(\lambda) \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot)\ dt \leq H(\lambda) \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}, \quad (1.4)$$

where

$$h(\lambda) := \lambda X \left(\frac{\lambda v + (2 - \lambda) u}{2}, \cdot\right) + (1 - \lambda) X \left(\frac{(1 + \lambda) v + (1 - \lambda) u}{2}, \cdot\right)$$

and

$$H(\lambda) := \frac{1}{2} \left( X(\lambda v + (1 - \lambda) u, \cdot) + \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot) \right).$$

In [1], Barraez et al. introduced the concept of $h-$convex stochastic process with following definition.

Definition 3. Let $h : (0, 1) \to \mathbb{R}$ be a non-negative function, $h \neq 0$. we say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is an $h-$convex stochastic process if, for every $t_1, t_2 \in I$, $\lambda \in (0, 1)$, the following inequality is satisfied

$$X(\lambda u + (1 - \lambda) v, \cdot) \leq h(\lambda) X(u, \cdot) + h(1 - \lambda) X(v, \cdot) \text{ (a.e.)} \quad (1.5)$$

Obviously, if we take $h(\lambda) = \lambda$ and $h(\lambda) = \lambda^*$ in (1.5), then the definition of $h-$convex stochastic process reduces to the definition of classical convex stochastic process [10] and s-convex stochastic process in the second sense [12] respectively. Moreover, A stochastic process $X : I \times \Omega \to \mathbb{R}$ is:

1) Godunova-Levin stochastic process if, we take $h(\lambda) = \frac{1}{\lambda}$ in (1.5),

$$X(\lambda u + (1 - \lambda) v, \cdot) \leq \frac{X(u, \cdot)}{\lambda} + \frac{X(v, \cdot)}{1 - \lambda} \text{ (a.e.)} \quad (1.6)$$

2) $P-$stochastic process if, we take $h(\lambda) = 1$ in (1.5),

$$X(\lambda u + (1 - \lambda) v, \cdot) \leq X(u, \cdot) + X(v, \cdot) \text{ (a.e.)} \quad (1.7)$$

Authors proved the following Hermite-Hadamard inequality for $h-$convex stochastic process in [1]:
Theorem 3. If \( X : I \times \Omega \to \mathbb{R} \) Let be \( h : (0,1) \to \mathbb{R} \) a non-negative function, \( h \neq 0 \) and \( X : I \times \Omega \to \mathbb{R} \) a non negative, \( h \)-convex, mean square integrable stochastic process. For every \( u,v \in I, (u < v) \), the following inequality is satisfied almost everywhere

\[
\frac{1}{2h\left(\frac{1}{2}\right)}X\left(\frac{u + v}{2}, \cdot \right) \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot) \, dt \leq [X(u, \cdot) + X(v, \cdot)] \int_{0}^{1} h(\lambda) \, d\lambda.
\]

For more information and recent developments on Hermite-Hadamard type inequalities for stochastic process, please refer to ([1], [4], [6]-[8], [11]-[13], [16]).

The aim of this paper is to establish an improvement of Hermite-Hadamard inequality for \( h \)-convex stochastic process.

2. Main Results

Theorem 4. If \( X : I \times \Omega \to \mathbb{R} \) Let be \( h : (0,1) \to \mathbb{R} \) a non-negative function, \( h \neq 0 \) and \( X : I \times \Omega \to \mathbb{R} \) a non negative, \( h \)-convex, mean square integrable stochastic process. For every \( u,v \in I, (u < v) \), we have the following inequality

\[
\frac{1}{4h\left(\frac{1}{2}\right)^2}X\left(\frac{u + v}{2}, \cdot \right) \leq \Delta_1 \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot) \, dt \leq \Delta_2 \leq [X(u, \cdot) + X(v, \cdot)] \left[\frac{1}{2} + h\left(\frac{1}{2}\right)\right] \int_{0}^{1} h(\lambda) \, d\lambda,
\]

where

\[
\Delta_1 := \frac{1}{4h\left(\frac{1}{2}\right)} \left[X\left(\frac{3u + v}{4}, \cdot \right) + \left(\frac{u + 3v}{4}, \cdot \right)\right]
\]

and

\[
\Delta_2 := [X(u, \cdot) + X(v, \cdot)] \left[\frac{1}{2} + h\left(\frac{1}{2}\right)\right] \int_{0}^{1} h(\lambda) \, d\lambda.
\]

Proof. Since \( X : I \times \Omega \to \mathbb{R} \) is a \( h \)-convex stochastic process, we have

\[
X\left(\frac{u + \frac{u+v}{2}}{2}, \cdot \right) = X\left(\frac{\lambda u + (1 - \lambda) \frac{u+v}{2} + (1 - \lambda)u + \lambda \frac{u+v}{2}}{2}, \cdot \right)
\]

\[
\leq h\left(\frac{1}{2}\right) \left[X\left(\frac{\lambda u + (1 - \lambda) \frac{u+v}{2}}{2}, \cdot \right) + X\left(\frac{(1 - \lambda)u + \lambda \frac{u+v}{2}}{2}, \cdot \right)\right].
\]
Integrating (2.2) from 0 to 1 with respect to \( \lambda \), we get

\[
X \left( \frac{3u + v}{4}, \cdot \right)
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ \int_0^1 X \left( \lambda u + (1 - \lambda) \frac{u + v}{2}, \cdot \right) \, d\lambda + \int_0^1 X \left( (1 - \lambda)u + \lambda \frac{u + v}{2}, \cdot \right) \, d\lambda \right]
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ \frac{2}{v - u} \int_u^v X (t, \cdot) \, dt + \frac{2}{v - u} \int_u^v X (t, \cdot) \, dt \right]
\]

\[
= \frac{4h}{v - u} \int_u^v X (t, \cdot) \, dt.
\]

That is,

\[
(2.4) \quad \frac{1}{4h} \left( \frac{3}{2} \right) X \left( \frac{3u + v}{4}, \cdot \right) \leq \frac{1}{v - u} \int_u^v X (t, \cdot) \, dt.
\]

Since \( X \) is a \( h \)-convex stochastic process, we also have

\[
X \left( \frac{u + v}{2}, \cdot \right) = X \left( \frac{\lambda u + v}{2} + (1 - \lambda)v, \cdot \right),
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ X \left( \lambda \frac{u + v}{2} + (1 - \lambda)v, \cdot \right) + X \left( (1 - \lambda) \frac{u + v}{2} + \lambda v, \cdot \right) \right].
\]

Integrating (2.5) from 0 to 1 with respect to \( \lambda \), we get

\[
X \left( \frac{u + 3v}{4}, \cdot \right)
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ \int_0^1 X \left( \lambda \frac{u + v}{2} + (1 - \lambda)v, \cdot \right) \, d\lambda + \int_0^1 X \left( (1 - \lambda) \frac{u + v}{2} + \lambda v, \cdot \right) \, d\lambda \right]
\]

\[
\leq h \left( \frac{1}{2} \right) \left[ \frac{2}{v - u} \int_u^v X (t, \cdot) \, dt + \frac{2}{v - u} \int_u^v X (t, \cdot) \, dt \right]
\]

\[
= \frac{4h}{v - u} \int_u^v X (t, \cdot) \, dt,
\]

i.e.

\[
(2.6) \quad \frac{1}{4h} \left( \frac{3}{2} \right) X \left( \frac{u + 3v}{4}, \cdot \right) \leq \frac{1}{v - u} \int_u^v X (t, \cdot) \, dt.
\]
Summing inequalities (2.4) and (2.6), we obtain
\[ \Delta_1 = \frac{1}{4h \left( \frac{1}{2} \right)^2} \left[ X \left( \frac{3u + v}{4}, \cdot \right) + \left( \frac{u + 3v}{4}, \cdot \right) \right] \leq \frac{1}{v - u} \int_u^v X (t, \cdot) \, dt \]

which finishes the proof of second inequality in (2.1).

Applying the Hermite-Hadamard inequality for \( h \)-convex stochastic process (Theorem 3), we have
\[
\frac{1}{v - u} \int_u^v X (t, \cdot) \, dt = \frac{1}{2} \left[ \frac{2}{v - u} \int_u^v X (t, \cdot) \, dt + \frac{2}{v - u} \int_{\frac{u + v}{2}}^v X (t, \cdot) \, dt \right]
\]
\[ \leq \frac{1}{2} \left[ X (u, \cdot) + X \left( \frac{u + v}{2}, \cdot \right) \right] \int_0^1 h(\lambda) \, d\lambda + \frac{1}{2} \left[ X \left( \frac{u + v}{2}, \cdot \right) + X (v, \cdot) \right] \int_0^1 h(\lambda) \, d\lambda \]
\[ = \left[ X (u, \cdot) + X (v, \cdot) \right] \frac{1}{2} + \frac{1}{2} \left[ \frac{1}{2} X (u, \cdot) + \frac{1}{2} X (v, \cdot) \right] \int_0^1 h(\lambda) \, d\lambda = \Delta_2. \]

This completes the proof of third inequality in (2.1).

For the first inequality, using the \( h \)-convexity of \( X \), we have
\[
\frac{1}{4 \left[ h \left( \frac{1}{2} \right) \right]^2} X \left( \frac{u + v}{2}, \cdot \right)
\]
\[ = \frac{1}{4 \left[ h \left( \frac{1}{2} \right) \right]^2} X \left( \frac{3u + v}{2}, \cdot \right) + \frac{1}{2} X (\frac{u + 3v}{4}, \cdot) \]
\[ \leq \frac{1}{4 \left[ h \left( \frac{1}{2} \right) \right]^2} \left[ h \left( \frac{1}{2} \right) X \left( \frac{3u + v}{2}, \cdot \right) + h \left( \frac{1}{2} \right) X \left( \frac{u + 3v}{4}, \cdot \right) \right] \]
\[ = \Delta_1. \]

Finally,
\[
\Delta_2 = \left[ X (u, \cdot) + X (v, \cdot) \right] \frac{1}{2} + \frac{1}{2} \left[ \frac{1}{2} X (u, \cdot) + X (v, \cdot) \right] \int_0^1 h(\lambda) \, d\lambda \]
\[ \leq \left[ X (u, \cdot) + X (v, \cdot) \right] + h \left( \frac{1}{2} \right) \left[ X (u, \cdot) + X (v, \cdot) \right] \int_0^1 h(\lambda) \, d\lambda \]
\[ = \left[ X (u, \cdot) + X (v, \cdot) \right] \frac{1}{2} + h \left( \frac{1}{2} \right) \int_0^1 h(\lambda) \, d\lambda. \]
This completes completely the proof of the Theorem.

**Remark 1.** Under assumption of Theorem 4 with \( h(t) = t \), we have

\[
X \left( \frac{u + v}{2}, \cdot \right) \leq \Delta_1 \leq \frac{1}{v - u} \int \limits_{v}^{u} X (t, \cdot) \, dt \leq \Delta_2 \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}
\]

where

\[
\Delta_1 := \frac{1}{2} \left[ X \left( \frac{3u + v}{4}, \cdot \right) + \left( \frac{u + 3v}{4}, \cdot \right) \right]
\]

and

\[
\Delta_2 := \frac{1}{2} \left[ \frac{X(u, \cdot) + X(v, \cdot)}{2} + X \left( \frac{u + v}{2}, \cdot \right) \cdot \right].
\]

This inequality is a special case of the Theorem 2 with \( \lambda = \frac{1}{2} \).

**Corollary 1.** Under assumption of Theorem 4 with \( h(t) = t^s \), we have the refinement Hermite-Hadamard inequality for \( s \)-convex stochastic processes in the second sense

\[
2^{2s-2} X \left( \frac{u + v}{2}, \cdot \right) \leq \Delta_1 \leq \frac{1}{v - u} \int \limits_{v}^{u} X (t, \cdot) \, dt \leq \Delta_2 \leq \left[ X(u, \cdot) + X(v, \cdot) \right] \left[ \frac{1}{2} + \frac{1}{2s} \right] \frac{1}{s + 1}
\]

where

\[
\Delta_1 = 2^{s-2} \left[ X \left( \frac{3u + v}{4}, \cdot \right) + \left( \frac{u + 3v}{4}, \cdot \right) \right]
\]

and

\[
\Delta_2 = \left[ \frac{X(u, \cdot) + X(v, \cdot)}{2} + X \left( \frac{u + v}{2}, \cdot \right) \cdot \right] \frac{1}{s + 1}.
\]

**Corollary 2.** Under assumption of Theorem 4 with \( h(t) = 1 \), we have the following Hermite-Hadamard type inequality for \( P \)-stochastic processes

\[
\frac{1}{4} X \left( \frac{u + v}{2}, \cdot \right) \leq \Delta_1 \leq \frac{1}{v - u} \int \limits_{v}^{u} X (t, \cdot) \, dt \leq \Delta_2 \leq \frac{3}{2} \left[ X(u, \cdot) + X(v, \cdot) \right]
\]

where

\[
\Delta_1 = \frac{1}{4} \left[ X \left( \frac{3u + v}{4}, \cdot \right) + \left( \frac{u + 3v}{4}, \cdot \right) \right]
\]

and

\[
\Delta_2 = \left[ \frac{X(u, \cdot) + X(v, \cdot)}{2} + X \left( \frac{u + v}{2}, \cdot \right) \cdot \right].
\]

**Corollary 3.** Under assumption of Theorem 4 with \( h(t) = \frac{1}{t} \), we have the following Hermite-Hadamard type inequality for Godunova-Levin stochastic processes

\[
\frac{1}{16} X \left( \frac{u + v}{2}, \cdot \right) \leq \Delta \leq \frac{1}{v - u} \int \limits_{v}^{u} X (t, \cdot) \, dt
\]

where

\[
\Delta = \frac{1}{8} \left[ X \left( \frac{3u + v}{4}, \cdot \right) + \left( \frac{u + 3v}{4}, \cdot \right) \right].
\]
References


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