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SOME INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF FUNCTIONS UNDER COMPLEX BOUNDEDNESS CONDITIONS

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Abstract. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some inequalities for the k-g-fractional integrals of integrable functions satisfying some boundedness conditions. Further bounds for absolutely continuous functions whose derivatives also satisfy some boundedness conditions are given as well. Examples for a general exponential fractional integral are also provided.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\ge1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

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and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ \ x\in[a,b). \tag{1.2}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x [g(x)-g(t)]^{\alpha-1}g'(t)\,f(t)\,dt \\ &=: I_{a+,g}^{\alpha}f(x),\ \ a<x\le b \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} S_{k,g,b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b [g(t)-g(x)]^{\alpha-1}g'(t)\,f(t)\,dt \\ &=: I_{b-,g}^{\alpha}f(x),\ \ a\le x<b, \end{aligned} \tag{1.4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [22, p. 100]

For $g(t)=t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [22, p. 111]

$$H_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x\le b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ \ 0\le a<x\le b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ \ 0\le a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.9}$$

for $a<x\le b$ and

$$E_{b-,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b[\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.10}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)\,f(t)\,dt,\ \ x\in(a,b] \tag{1.11}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)f(t)\,dt,\ x\in[a,b). \tag{1.12}$$

In [25], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x):=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ |x|<R,\ \text{with } R>0 \tag{1.13}$$

for $\rho,\ \lambda>0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f(x):=\int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right)f(t)\,dt,\ x>a \tag{1.14}$$

where $\rho,\ \lambda>0$, $w\in\mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f(x):=\int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right)f(t)\,dt,\ x<b \tag{1.15}$$

where $\rho,\ \lambda>0$, $w\in\mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t)=t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [23], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)\,dt,\ x>a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)\,dt,\ x<b \tag{1.17}$$

where $\alpha\in(0,1)$.

We observe that for $k(t)=\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t\in\mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b \tag{1.19}$$

where $\alpha\in(0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha>0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+}f(x):=\int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt, \tag{1.20}$$

for $0 < a < x \leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha} f(x) := \int_x^b (g(t) - g(x))^{\alpha-1} \ln(g(t) - g(x)) g'(t) f(t) \, dt, \tag{1.21}$$

for $0 < a \leq x < b$, where $\alpha > 0$. These are obtained from (1.11) and (1.12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) \, dt, \; 0 < a < x \leq b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) \, dt, \; 0 < a \leq x < b. \tag{1.23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) \, dt, \; 0 < a < x \leq b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) \, dt, \; 0 < a \leq x < b, \tag{1.25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) \, dt, \; 0 < a < x \leq b \tag{1.26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) \, dt, \; 0 < a \leq x < b. \tag{1.27}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [20]-[33] and the references therein.

In this paper we establish some inequalities for the k-g-fractional integrals of integrable functions satisfying some boundedness conditions. Further bounds for absolutely continuous functions whose derivatives also satisfy some boundedness conditions are given as well. Examples for a general exponential fractional integral are also provided.

2. Inequalities for Bounded Functions

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} & S_{k,g,a+,b-} f(x) \\ & := \frac{1}{2} \left[S_{k,g,a+} f(x) + S_{k,g,b-} f(x) \right] \\ & = \frac{1}{2} \left[\int_a^x k(g(x) - g(t)) g'(t) f(t) \, dt + \int_x^b k(g(t) - g(x)) g'(t) f(t) \, dt \right] \end{aligned} \tag{2.1}$$

for the Lebesgue integrable function $f : (a,b) \to \mathbb{C}$ and $x \in (a,b)$.

Observe that

$$S_{k,g,x+} f(b) = \int_x^b k(g(b) - g(t)) g'(t) f(t) \, dt, \; x \in [a,b) \tag{2.2}$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt,\ x\in(a,b]. \tag{2.3}$$

We can define also the mixed operator

$$\begin{aligned}
&\breve{S}_{k,g,a+,b-}f(x) \\
&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right] \\
&=\frac{1}{2}\left[\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt+\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt\right]
\end{aligned} \tag{2.4}$$

for any $x\in(a,b)$.

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold:

Lemma 1. *With the above assumptions for k, g and f we have*

$$\begin{aligned}
S_{k,g,a+,b-}f(x)&=\frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right] \\
&+\frac{1}{2}\int_a^x k(g(x)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt \\
&+\frac{1}{2}\int_x^b k(g(t)-g(x))\,g'(t)\,[f(t)-\gamma]\,dt
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\breve{S}_{k,g,a+,b-}f(x)&=\frac{1}{2}\left[\lambda K(g(b)-g(x))+\gamma K(g(x)-g(a))\right] \\
&+\frac{1}{2}\int_a^x k(g(t)-g(a))\,g'(t)\,[f(t)-\gamma]\,dt \\
&+\frac{1}{2}\int_x^b k(g(b)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt
\end{aligned} \tag{2.6}$$

for $x\in(a,b)$ and for any λ, $\gamma\in\mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$[K(g(x)-g(t))]'=K'(g(x)-g(t))\,(g(x)-g(t))'=-k(g(x)-g(t))\,g'(t)$$

for $t\in(a,x)$ and

$$[K(g(t)-g(x))]'=K'(g(t)-g(x))\,(g(t)-g(x))'=k(g(t)-g(x))\,g'(t)$$

for $t\in(x,b)$.

Therefore, for any λ, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned}
&\int_a^x k(g(x)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt \\
&=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt-\lambda\int_a^x k(g(x)-g(t))\,g'(t)\,dt \\
&=S_{k,g,a+}f(x)+\lambda\int_a^x [K(g(x)-g(t))]'\,dt \\
&=S_{k,g,a+}f(x)+\lambda\left[K(g(x)-g(t))\right]\Big|_a^x=S_{k,g,a+}f(x)-\lambda K(g(x)-g(a))
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
(2.8) \quad & \int_x^b k(g(t)-g(x))\,g'(t)\,[f(t)-\gamma]\,dt \\
&= \int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt - \gamma\int_x^b k(g(t)-g(x))\,g'(t)\,dt \\
&= S_{k,g,b-}f(x) - \gamma\int_x^b [K(g(t)-g(x))]'\,dt \\
&= S_{k,g,b-}f(x) - \gamma\,[K(g(t)-g(x))]\big|_x^b = S_{k,g,b-}f(x) - \gamma K(g(b)-g(x))
\end{aligned}$$

for $x\in(a,b)$.

If we add the equalities (2.7) and (2.8) and divide by 2 then we get the desired result (2.5).

Moreover, by taking the derivative over t and using the chain rule, we have that

$$[K(g(b)-g(t))]' = K'(g(b)-g(t))\,(g(b)-g(t))' = -k(g(b)-g(t))\,g'(t)$$

for $t\in(x,b)$ and

$$[K(g(t)-g(a))]' = K'(g(t)-g(a))\,(g(t)-g(a))' = k(g(t)-g(a))\,g'(t)$$

for $t\in(a,x)$.

For any $\lambda,\ \gamma\in\mathbb{C}$ we have

$$\begin{aligned}
(2.9) \quad & \int_x^b k(g(b)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt \\
&= \int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt - \lambda\int_x^b k(g(b)-g(t))\,g'(t)\,dt \\
&= S_{k,g,x+}f(b) + \lambda\int_x^b [K(g(b)-g(t))]'\,dt \\
&= S_{k,g,x+}f(b) - \lambda K(g(b)-g(x))
\end{aligned}$$

and

$$\begin{aligned}
(2.10) \quad & \int_a^x k(g(t)-g(a))\,g'(t)\,[f(t)-\gamma]\,dt \\
&= \int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt - \gamma\int_a^x k(g(t)-g(a))\,g'(t)\,dt \\
&= \int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt - \gamma\int_a^x [K(g(t)-g(a))]'\,dt \\
&= \int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt - \gamma K(g(x)-g(a))
\end{aligned}$$

for $x\in(a,b)$.

If we add the equalities (2.9) and (2.10) and divide by 2 then we get the desired result (2.6). $\square$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\ b\in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the g-mean of two numbers we can introduce

$$\begin{aligned}P_{k,g,a+,b-}f&:=S_{k,g,a+,b-}f\left(M_g(a,b)\right)\\&=\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)f(t)\,dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)f(t)\,dt.\end{aligned}\tag{2.11}$$

Using the representation (2.5) we have

$$\begin{aligned}P_{k,g,a+,b-}f&=K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2}\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-\lambda\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-\gamma\right]dt\end{aligned}\tag{2.12}$$

for any $\lambda,\gamma\in\mathbb{C}$.

Also, if

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f&:=\breve{S}_{k,g,a+,b-}f\left(M_g(a,b)\right)\\&=\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(b)-g(t)\right)g'(t)f(t)\,dt\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(g(t)-g(a)\right)g'(t)f(t)\,dt.\end{aligned}\tag{2.13}$$

then by (2.6) we get

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f&=K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2}\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(g(t)-g(a)\right)g'(t)\left[f(t)-\gamma\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(b)-g(t)\right)g'(t)\left[f(t)-\lambda\right]dt\end{aligned}\tag{2.14}$$

for any $\lambda,\gamma\in\mathbb{C}$.

Now, for $\phi,\ \Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\phi,\Phi) := \left\{f:[a,b]\to\mathbb{C} \,|\, \operatorname{Re}\left[(\Phi-f(t))\left(\overline{f(t)}-\overline{\phi}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi) := \left\{f:[a,b]\to\mathbb{C} \,|\, \left|f(t)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_{[a,b]}(\phi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\phi,\Phi)=\bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{2.15}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi-z)(\bar{z}-\phi)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|z-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[(\Phi-z)(\bar{z}-\phi)\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (2.15) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\phi,\Phi)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Phi-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\phi)\\ &+(\operatorname{Im}\Phi-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\phi)\geq 0 \textit{ for a.e. } t\in[a,b]\}.\end{aligned} \tag{2.16}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\phi,\Phi):=\{f:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\phi)\\ &\text{and }\operatorname{Im}(\Phi)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\phi)\ \text{ for a.e. } t\in[a,b]\}.\end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\phi,\Phi)\subseteq\bar{U}_{[a,b]}(\phi,\Phi).$$

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}(t):=\begin{cases}\int_0^t|k(s)|\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

We observe that if k takes nonnegative values on $(0,\infty)$, as it does in some of the examples in Introduction, then $\mathbf{K}(t)=K(t)$ for $t\in[0,\infty)$.

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a measurable function on $[a,b]$ such that $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$ for some $\phi,\Phi\in\mathbb{C}$, $\phi\neq\Phi$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \le\frac{1}{4}\left|\Phi-\phi\right|\left[\mathbf{K}(g(x)-g(a))+\mathbf{K}(g(b)-g(x))\right] \tag{2.17}$$

and

$$\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \le\frac{1}{4}\left|\Phi-\phi\right|\left[\mathbf{K}(g(x)-g(a))+\mathbf{K}(g(b)-g(x))\right] \tag{2.18}$$

for $x\in(a,b)$.

Proof. Since $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$, then from (2.5) we have for $x\in(a,b)$ that

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \\
&\le\frac{1}{2}\left|\int_a^x k(g(x)-g(t))\,g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)dt\right| \\
&+\frac{1}{2}\left|\int_x^b k(g(t)-g(x))\,g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)dt\right| \\
&\le\frac{1}{2}\int_a^x\left|k(g(x)-g(t))\,g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)\right|dt \\
&+\frac{1}{2}\int_x^b\left|k(g(t)-g(x))\,g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)\right|dt \\
&\le\frac{1}{4}\left|\Phi-\phi\right|\left[\int_a^x\left|k(g(x)-g(t))\right|g'(t)\,dt+\int_x^b\left|k(g(t)-g(x))\right|g'(t)\,dt\right] \\
&:=B(x)
\end{aligned} \tag{2.19}$$

We have, by taking the derivative over t and using the chain rule, that

$$\left[\mathbf{K}(g(x)-g(t))\right]'=\mathbf{K}'(g(x)-g(t))\,(g(x)-g(t))'=-\left|k(g(x)-g(t))\right|g'(t)$$

for $t\in(a,x)$ and

$$\left[\mathbf{K}(g(t)-g(x))\right]'=\mathbf{K}'(g(t)-g(x))\,(g(t)-g(x))'=\left|k(g(t)-g(x))\right|g'(t)$$

for $t\in(x,b)$.

Then

$$\int_a^x\left|k(g(x)-g(t))\right|g'(t)\,dt=-\int_a^x\left[\mathbf{K}(g(x)-g(t))\right]'dt=\mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b\left|k(g(t)-g(x))\right|g'(t)\,dt=\int_x^b\left[\mathbf{K}(g(t)-g(x))\right]'dt=\mathbf{K}(g(b)-g(x)).$$

Therefore,

$$B(x) \leq \frac{1}{4}|\Phi-\phi|\left[\mathbf{K}(g(x)-g(a))+\mathbf{K}(g(b)-g(x))\right]$$

for $x \in (a,b)$, which proves (2.17).

Also, by the equality (2.6) we have

$$\begin{aligned}
(2.20)\quad & \left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \\
& \leq \frac{1}{2}\left|\int_a^x k(g(t)-g(a))g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)dt\right| \\
& + \frac{1}{2}\left|\int_x^b k(g(b)-g(t))g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)dt\right| \\
& \leq \frac{1}{2}\int_a^x \left|k(g(t)-g(a))g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)\right|dt \\
& + \frac{1}{2}\int_x^b \left|k(g(b)-g(t))g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)\right|dt \\
& \leq \frac{1}{2}\int_a^x \left|k(g(t)-g(a))g'(t)\left(f(t)-\frac{\phi+\Phi}{2}\right)\right|dt \\
& \leq \frac{1}{4}|\Phi-\phi|\left[\int_x^b |k(g(b)-g(t))|g'(t)\,dt+\int_a^x |k(g(t)-g(a))|g'(t)\,dt\right] \\
& := C(x)
\end{aligned}$$

for $x \in (a,b)$.

We have, by taking the derivative over t and using the chain rule, that

$$\left[\mathbf{K}(g(b)-g(t))\right]' = \mathbf{K}'(g(b)-g(t))(g(b)-g(t))' = -|k(g(b)-g(t))|g'(t)$$

for $t \in (x,b)$ and

$$\left[\mathbf{K}(g(t)-g(a))\right]' = \mathbf{K}'(g(t)-g(a))(g(t)-g(a))' = |k(g(t)-g(a))|g'(t)$$

for $t \in (a,x)$.

Therefore

$$\int_x^b |k(g(b)-g(t))|g'(t)\,dt = -\int_x^b \left[\mathbf{K}(g(b)-g(t))\right]'dt = \mathbf{K}(g(b)-g(x))$$

and

$$\int_a^x |k(g(t)-g(a))|g'(t)\,dt = \int_a^x \left[\mathbf{K}(g(t)-g(a))\right]'dt = \mathbf{K}(g(x)-g(a))$$

and by (2.20) we get (2.18). □

Corollary 2. *With the assumptions of Theorem 1 we have*

$$(2.21)\quad \left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}|\Phi-\phi|\,\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)$$

and

$$(2.22)\quad \left|\breve{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{\phi+\Phi}{2}\right| \leq \frac{1}{2}|\Phi-\phi|\,\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right).$$

Remark 1. *By Hölder's integral inequality we have for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ that*

$$\mathbf{K}(t)=\int_0^t |k(s)|\,ds \le \begin{cases} t\,\mathrm{essup}_{s\in[0,t]}|k(s)| \\ t^{1/p}\left(\int_0^t |k(s)|^q\,ds\right)^{1/q}\end{cases} = \begin{cases} t\,\|k\|_{[0,t],\infty} \\ t^{1/p}\,\|k\|_{[0,t],q}\end{cases} \tag{2.23}$$

for $t\ge 0$.

By (2.17) and (2.18) we then have

$$\begin{aligned}&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \\ &\le \frac{1}{4}|\Phi-\phi|\left[(g(x)-g(a))\|k\|_{[0,g(x)-g(a)],\infty}+(g(b)-g(x))\|k\|_{[0,g(b)-g(x)],\infty}\right] \\ &\le \frac{1}{4}|\Phi-\phi|(g(b)-g(a))\|k\|_{[0,g(b)-g(a)],\infty}\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \\ &\le \frac{1}{4}|\Phi-\phi|\left[(g(x)-g(a))\|k\|_{[0,g(x)-g(a)],\infty}+(g(b)-g(x))\|k\|_{[0,g(b)-g(x)],\infty}\right] \\ &\le \frac{1}{4}|\Phi-\phi|(g(b)-g(a))\|k\|_{[0,g(b)-g(a)],\infty}\end{aligned} \tag{2.25}$$

for $x\in(a,b)$.

In particular, we have from (2.24) and (2.25) that

$$\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{\phi+\Phi}{2}\right|\le\frac{1}{4}|\Phi-\phi|(g(b)-g(a))\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],\infty}$$

and

$$\left|\breve{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{\phi+\Phi}{2}\right|\le\frac{1}{4}|\Phi-\phi|(g(b)-g(a))\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],\infty}.$$

By utilising the second branch in (2.23), then we also have

$$\begin{aligned}&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \\ &\le \frac{1}{4}|\Phi-\phi|\left[(g(x)-g(a))^{1/p}\|k\|_{[0,g(x)-g(a)],q}+(g(b)-g(x))^{1/p}\|k\|_{[0,g(b)-g(x)],q}\right] \\ &\le \frac{1}{4}|\Phi-\phi|(g(b)-g(a))^{1/p}\left[\|k\|^q_{[0,g(x)-g(a)],q}+\|k\|^q_{[0,g(b)-g(x)],q}\right]^{1/q} \\ &\le \frac{1}{2^{1+1/p}}|\Phi-\phi|(g(b)-g(a))^{1/p}\|k\|_{[0,g(b)-g(a)],q}\end{aligned} \tag{2.26}$$

and

$$\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]\frac{\phi+\Phi}{2}\right| \tag{2.27}$$
$$\leq\frac{1}{4}|\Phi-\phi|\left[(g(x)-g(a))^{1/p}\|k\|_{[0,g(x)-g(a)],q}+(g(b)-g(x))^{1/p}\|k\|_{[0,g(b)-g(x)],q}\right]$$
$$\leq\frac{1}{4}|\Phi-\phi|(g(b)-g(a))^{1/p}\left[\|k\|_{[0,g(x)-g(a)],q}^{q}+\|k\|_{[0,g(b)-g(x)],q}^{q}\right]^{1/q}$$
$$\leq\frac{1}{2^{1+1/p}}|\Phi-\phi|(g(b)-g(a))^{1/p}\|k\|_{[0,g(b)-g(a)],q}$$

for $x\in(a,b)$.

Finally, from (2.21) and (2.22) we derive the simple inequalities

$$\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{\phi+\Phi}{2}\right| \tag{2.28}$$
$$\leq\frac{1}{2^{1+1/p}}|\Phi-\phi|(g(b)-g(a))^{1/p}\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],q}$$

and

$$\left|\breve{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{\phi+\Phi}{2}\right| \tag{2.29}$$
$$\leq\frac{1}{2^{1+1/p}}|\Phi-\phi|(g(b)-g(a))^{1/p}\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],q},$$

where $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

3. Inequalities for Bounded Derivatives

We start with the following two parameters representations incorporated in:

Lemma 2. *With the above assumptions for* k, g *and if* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$, *then we have for* $x\in(a,b)$ *that*

$$S_{k,g,a+,b-}f(x)=\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right] \tag{3.1}$$
$$+\frac{1}{2}\lambda\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}\gamma\int_x^b K(g(t)-g(x))\,dt$$
$$+\frac{1}{2}\int_a^x K(g(x)-g(t))\left[f'(t)-\lambda\right]dt+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[\gamma-f'(t)\right]dt$$

and

$$\breve{S}_{k,g,a+,b-}f(x)=\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \tag{3.2}$$
$$+\frac{1}{2}\gamma\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}\lambda\int_a^x K(g(t)-g(a))\,dt$$
$$+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt+\frac{1}{2}\int_a^x K(g(t)-g(a))\left[\lambda-f'(t)\right]dt$$

for any $\lambda,\ \gamma\in\mathbb{C}$.

Proof. Using the integration by parts formula, we have

$$\begin{aligned}
&\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=-\int_a^x \left[K\left(g\left(x\right)-g\left(t\right)\right)\right]'f\left(t\right)dt \\
&=-\left[K\left(g\left(x\right)-g\left(t\right)\right)f\left(t\right)\Big|_a^x-\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)f'\left(t\right)dt\right] \\
&=K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)f'\left(t\right)dt
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=\int_x^b \left[K\left(g\left(t\right)-g\left(x\right)\right)\right]'f\left(t\right)dt \\
&=\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]f\left(t\right)\Big|_x^b-\int_x^b \left[K\left(g\left(t\right)-g\left(x\right)\right)\right]f'\left(t\right)dt \\
&=\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)-\int_x^b \left[K\left(g\left(t\right)-g\left(x\right)\right)\right]f'\left(t\right)dt
\end{aligned} \tag{3.4}$$

for any $x\in(a,b)$.

From (3.3) and (3.4) we get

$$\begin{aligned}
&\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\lambda\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)dt \\
&+\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)\left[f'\left(t\right)-\lambda\right]dt
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)-\gamma\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)dt \\
&-\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)\left[f'\left(t\right)-\gamma\right]dt
\end{aligned} \tag{3.6}$$

for any $x\in(a,b)$.

If we add the equalities (3.5) and (3.6) and divide by 2 then we get the desired result (3.1).

Using the integration by parts formula, we have

$$
\begin{aligned}
(3.7)\qquad & \int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= -\int_x^b \left[K\left(g\left(b\right)-g\left(t\right)\right)\right]' f\left(t\right)dt \\
&= -\left[K\left(g\left(b\right)-g\left(t\right)\right)f\left(t\right)\Big|_x^b - \int_x^b K\left(g\left(b\right)-g\left(t\right)\right)f'\left(t\right)dt\right] \\
&= K\left(g\left(b\right)-g\left(x\right)\right)f\left(x\right) + \int_x^b K\left(g\left(b\right)-g\left(t\right)\right)f'\left(t\right)dt
\end{aligned}
$$

and

$$
\begin{aligned}
(3.8)\qquad & \int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= \int_a^x \left[K\left(g\left(t\right)-g\left(a\right)\right)\right]' f\left(t\right)dt \\
&= K\left(g\left(t\right)-g\left(a\right)\right)f\left(t\right)\Big|_a^x - \int_a^x K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)dt \\
&= K\left(g\left(x\right)-g\left(a\right)\right)f\left(x\right) - \int_a^x K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)dt
\end{aligned}
$$

for any $x \in (a,b)$.

From (3.7) and (3.8) we have

$$
\begin{aligned}
(3.9)\qquad & \int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= K\left(g\left(b\right)-g\left(x\right)\right)f\left(x\right) + \gamma\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)dt \\
&+ \int_x^b K\left(g\left(b\right)-g\left(t\right)\right)\left[f'\left(t\right)-\gamma\right]dt
\end{aligned}
$$

and

$$
\begin{aligned}
(3.10)\qquad & \int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= K\left(g\left(x\right)-g\left(a\right)\right)f\left(x\right) - \lambda\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)dt \\
&- \int_a^x K\left(g\left(t\right)-g\left(a\right)\right)\left[f'\left(t\right)-\lambda\right]dt
\end{aligned}
$$

for any $x \in (a,b)$.

If we add the equalities (3.9) and (3.10) and divide by 2 then we get the desired result (3.2). □

Corollary 3. *With the assumptions of Lemma 2 we have*

$$P_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2} \tag{3.11}$$
$$+\frac{1}{2}\lambda\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt - \frac{1}{2}\gamma\int_{M_g(a,b)}^{b} K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt$$
$$+\frac{1}{2}\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)[f'(t)-\lambda]\,dt$$
$$+\frac{1}{2}\int_{M_g(a,b)}^{b} K\left(g(t)-\frac{g(a)+g(b)}{2}\right)[\gamma-f'(t)]\,dt$$

and

$$\breve{P}_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b)) \tag{3.12}$$
$$+\frac{1}{2}\gamma\int_{M_g(a,b)}^{b} K(g(b)-g(t))\,dt - \frac{1}{2}\lambda\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt$$
$$+\frac{1}{2}\int_{M_g(a,b)}^{b} K(g(b)-g(t))[f'(t)-\gamma]\,dt$$
$$+\frac{1}{2}\int_a^{M_g(a,b)} K(g(t)-g(a))[\lambda-f'(t)]\,dt$$

for any $\lambda,\ \gamma\in\mathbb{C}$.

The following error estimates result can be stated:

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an absolutely function on $[a,b]$ such that $f'\in\bar{\Delta}_{[a,b]}(\psi,\Psi)$ for some $\psi,\ \Psi\in\mathbb{C}$, $\psi\neq\Psi$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have*

$$\Bigg|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+[K(g(b)-g(x))]f(b)\right] \tag{3.13}$$
$$+\frac{1}{2}\left(\int_x^b K(g(t)-g(x))\,dt-\int_a^x K(g(x)-g(t))\,dt\right)\frac{\psi+\Psi}{2}\Bigg|$$
$$\leq\frac{1}{4}|\Psi-\psi|\left[\int_a^x |K(g(x)-g(t))|\,dt+\int_x^b |K(g(t)-g(x))|\,dt\right]$$
$$\leq\frac{1}{4}|\Psi-\psi|\left[\int_x^b \mathbf{K}(g(t)-g(x))\,dt+\int_a^x \mathbf{K}(g(x)-g(t))\,dt\right]$$

and

$$\begin{aligned}(3.14)\quad &\Bigg|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\\ &+\frac{1}{2}\left(\int_a^x K(g(t)-g(a))\,dt-\int_x^b K(g(b)-g(t))\,dt\right)\frac{\psi+\Psi}{2}\Bigg|\\ &\le\frac{1}{4}|\Psi-\psi|\left[\int_x^b |K(g(b)-g(t))|\,dt+\int_a^x |K(g(t)-g(a))|\,dt\right]\\ &\le\frac{1}{4}|\Psi-\psi|\left[\int_x^b \mathbf{K}(g(b)-g(t))\,dt+\int_a^x \mathbf{K}(g(t)-g(a))\,dt\right]\end{aligned}$$

for $x\in(a,b)$.

Proof. Using the identity (3.1) and the fact that $f'\in\bar{\Delta}_{[a,b]}(\psi,\Psi)$, then we have for $x\in(a,b)$ that

$$\begin{aligned}(3.15)\quad &\Bigg|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right]\\ &+\frac{1}{2}\left(\int_x^b K(g(t)-g(x))\,dt-\int_a^x K(g(x)-g(t))\,dt\right)\frac{\psi+\Psi}{2}\Bigg|\\ &\le\frac{1}{2}\left|\int_a^x K(g(x)-g(t))\left(f'(t)-\frac{\psi+\Psi}{2}\right)dt\right|\\ &+\frac{1}{2}\left|\int_x^b K(g(t)-g(x))\left(\frac{\psi+\Psi}{2}-f'(t)\right)dt\right|\\ &\le\frac{1}{2}\int_a^x \left|K(g(x)-g(t))\left(f'(t)-\frac{\psi+\Psi}{2}\right)\right|dt\\ &+\frac{1}{2}\int_x^b \left|K(g(t)-g(x))\left(\frac{\psi+\Psi}{2}-f'(t)\right)\right|dt\\ &\le\frac{1}{4}|\Psi-\psi|\left[\int_a^x |K(g(x)-g(t))|\,dt+\int_x^b |K(g(t)-g(x))|\,dt\right],\end{aligned}$$

which proves the first inequality in (3.13).

The last part follows by the fact that

$$|K(t)|=\left|\int_0^t k(s)\,ds\right|\le\int_0^t |k(s)|\,ds=\mathbf{K}(t)\ \text{ for } t\ge 0.$$

Using the identity (3.2) we also have

$$\begin{aligned}
(3.16)\quad & \Big| \breve{S}_{k,g,a+,b-} f(x) - \frac{1}{2} \left[K(g(b) - g(x)) + K(g(x) - g(a)) \right] f(x) \\
& + \frac{1}{2} \left(\int_a^x K(g(t) - g(a))\, dt - \int_x^b K(g(b) - g(t))\, dt \right) \frac{\psi + \Psi}{2} \Big| \\
& \leq \frac{1}{2} \left| \int_x^b K(g(b) - g(t)) \left(f'(t) - \frac{\psi + \Psi}{2} \right) dt \right| \\
& + \frac{1}{2} \left| \int_a^x K(g(t) - g(a)) \left(\frac{\psi + \Psi}{2} - f'(t) \right) dt \right| \\
& \leq \frac{1}{4} |\Psi - \psi| \left[\int_x^b |K(g(b) - g(t))|\, dt + \int_a^x |K(g(t) - g(a))|\, dt \right]
\end{aligned}$$

for $x \in (a, b)$, which proves (3.14). $\square$

Corollary 4. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
(3.17)\quad & \Big| P_{k,g,a+,b-} f - K\left(\frac{g(b) - g(a)}{2} \right) \frac{f(a) + f(b)}{2} \\
& + \frac{1}{2} \left(\int_{M_g(a,b)}^b K\left(g(t) - \frac{g(a) + g(b)}{2} \right) dt - \int_a^{M_g(a,b)} K\left(\frac{g(a) + g(b)}{2} - g(t) \right) dt \right) \\
& \times \frac{\psi + \Psi}{2} \Big| \\
& \leq \frac{1}{4} |\Psi - \psi| \\
& \times \left[\int_a^{M_g(a,b)} \mathbf{K}\left(\frac{g(a) + g(b)}{2} - g(t) \right) dt + \int_{M_g(a,b)}^b \mathbf{K}\left(g(t) - \frac{g(a) + g(b)}{2} \right) dt \right]
\end{aligned}$$

and

$$\begin{aligned}
(3.18)\quad & \Big| \breve{P}_{k,g,a+,b-} f - K\left(\frac{g(b) - g(a)}{2} \right) f(M_g(a,b)) \\
& + \frac{1}{2} \left(\int_a^{M_g(a,b)} K(g(t) - g(a))\, dt - \int_{M_g(a,b)}^b K(g(b) - g(t))\, dt \right) \frac{\psi + \Psi}{2} \Big| \\
& \leq \frac{1}{4} |\Psi - \psi| \left[\int_{M_g(a,b)}^b \mathbf{K}(g(b) - g(t))\, dt + \int_a^{M_g(a,b)} \mathbf{K}(g(t) - g(a))\, dt \right].
\end{aligned}$$

Remark 2. *Using the first branch in (2.23) we have*

$$\begin{aligned}
\int_a^x \mathbf{K}(g(x) - g(t))\, dt & \leq \int_a^x (g(x) - g(t)) \|k\|_{[0, g(x) - g(t)], \infty}\, dt \\
& \leq \|k\|_{[0, g(x) - g(a)], \infty} \int_a^x (g(x) - g(t))\, dt
\end{aligned}$$

and

$$\int_x^b \mathbf{K}(g(t)-g(x))\,dt \le \int_x^b (g(t)-g(x))\,\|k\|_{[0,g(t)-g(x)],\infty}\,dt$$
$$\le \|k\|_{[0,g(b)-g(x)],\infty}\int_x^b (g(t)-g(x))\,dt.$$

Therefore

$$\begin{aligned}
&\int_a^x \mathbf{K}(g(x)-g(t))\,dt + \int_x^b \mathbf{K}(g(t)-g(x))\,dt \\
&\le \|k\|_{[0,g(x)-g(a)],\infty}\int_a^x (g(x)-g(t))\,dt + \|k\|_{[0,g(b)-g(x)],\infty}\int_x^b (g(t)-g(x))\,dt \\
&\le \left[\int_a^x (g(x)-g(t))\,dt + \int_x^b (g(t)-g(x))\,dt\right]\|k\|_{[0,g(b)-g(a)],\infty} \\
&= \left[g(x)(x-a) - g(x)(b-x) + \int_x^b g(t)\,dt - \int_a^x g(t)\,dt\right]\|k\|_{[0,g(b)-g(a)],\infty} \\
&= \left[g(x)(2x-a-b) + \int_x^b g(t)\,dt - \int_a^x g(t)\,dt\right]\|k\|_{[0,g(b)-g(a)],\infty}
\end{aligned}$$

and by (3.13) we get

$$\begin{aligned}
&\Bigg|S_{k,g,a+,b-}f(x) - \frac{1}{2}\left[K(g(x)-g(a))f(a) + [K(g(b)-g(x))]f(b)\right] \\
&\qquad + \frac{1}{2}\left(\int_x^b K(g(t)-g(x))\,dt - \int_a^x K(g(x)-g(t))\,dt\right)\frac{\psi+\Psi}{2}\Bigg| \\
&\le \frac{1}{2}|\Psi-\psi|\left[g(x)\left(x-\frac{a+b}{2}\right) + \frac{1}{2}\left(\int_x^b g(t)\,dt - \int_a^x g(t)\,dt\right)\right]\|k\|_{[0,g(b)-g(a)],\infty}
\end{aligned} \tag{3.19}$$

for $x\in(a,b)$.

In particular, for $x=\frac{a+b}{2}$ we get

$$\begin{aligned}
&\Bigg|S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \\
&\quad - \frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a) + \left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right]f(b)\right] \\
&+\frac{1}{2}\left(\int_{\frac{a+b}{2}}^b K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt - \int_a^{\frac{a+b}{2}} K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt\right)\frac{\psi+\Psi}{2}\Bigg| \\
&\qquad\le \frac{1}{4}|\Psi-\psi|\left[\int_{\frac{a+b}{2}}^b g(t)\,dt - \int_a^{\frac{a+b}{2}} g(t)\,dt\right]\|k\|_{[0,g(b)-g(a)],\infty}.
\end{aligned} \tag{3.20}$$

Also

$$\begin{aligned}
&\int_x^b \mathbf{K}\left(g\left(b\right)-g\left(t\right)\right)dt+\int_a^x \mathbf{K}\left(g\left(t\right)-g\left(a\right)\right)dt\\
&\leq \int_x^b \left(g\left(b\right)-g\left(t\right)\right)\left\|k\right\|_{[0,g(b)-g(t)],\infty}dt+\int_a^x \left(g\left(t\right)-g\left(a\right)\right)\left\|k\right\|_{[0,g(t)-g(a)],\infty}dt\\
&\leq \left\|k\right\|_{[0,g(b)-g(x)],\infty}\int_x^b \left(g\left(b\right)-g\left(t\right)\right)dt+\left\|k\right\|_{[0,g(x)-g(a)],\infty}\int_a^x \left(g\left(t\right)-g\left(a\right)\right)dt\\
&\leq \left[\int_x^b \left(g\left(b\right)-g\left(t\right)\right)dt+\int_a^x \left(g\left(t\right)-g\left(a\right)\right)dt\right]\left\|k\right\|_{[0,g(b)-g(a)],\infty}\\
&=\left[g\left(b\right)\left(b-x\right)-g\left(a\right)\left(x-a\right)+\int_a^x g\left(t\right)dt-\int_x^b g\left(t\right)dt\right]\left\|k\right\|_{[0,g(b)-g(a)],\infty}
\end{aligned}$$

and by (3.14) we get

$$\begin{aligned}
&\Bigg|\breve{S}_{k,g,a+,b-}f\left(x\right)-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)+K\left(g\left(x\right)-g\left(a\right)\right)\right]f\left(x\right)\\
&\quad+\frac{1}{2}\left(\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)dt-\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)dt\right)\frac{\psi+\Psi}{2}\Bigg|\\
&\qquad\leq \frac{1}{4}\left|\Psi-\psi\right|\left\|k\right\|_{[0,g(b)-g(a)],\infty}\\
&\qquad\quad\times\left[g\left(b\right)\left(b-x\right)-g\left(a\right)\left(x-a\right)+\int_a^x g\left(t\right)dt-\int_x^b g\left(t\right)dt\right],
\end{aligned}\tag{3.21}$$

for $x\in(a,b)$.

In particular, for $x=\frac{a+b}{2}$ we get

$$\begin{aligned}
&\Bigg|\breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)\\
&\quad-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right)\\
&\quad+\frac{1}{2}\left(\int_a^{\frac{a+b}{2}} K\left(g\left(t\right)-g\left(a\right)\right)dt-\int_{\frac{a+b}{2}}^b K\left(g\left(b\right)-g\left(t\right)\right)dt\right)\frac{\psi+\Psi}{2}\Bigg|\\
&\leq \frac{1}{4}\left|\Psi-\psi\right|\left[\frac{g\left(b\right)-g\left(a\right)}{2}\left(b-a\right)+\int_a^{\frac{a+b}{2}} g\left(t\right)dt-\int_{\frac{a+b}{2}}^b g\left(t\right)dt\right]\left\|k\right\|_{[0,g(b)-g(a)],\infty}.
\end{aligned}\tag{3.22}$$

Similar inequalities may be stated on using the second branch of the inequality (2.23). The details are omitted.

4. Example for an Exponential Kernel

The above inequalities may be written for all the particular fractional integrals introduced in the introduction. We consider here only an example for a general exponential kernel that generalizes the transforms (1.16) and (1.17).

For α, $\beta \in \mathbb{R}$ we consider the kernel $k(t) := \exp[(\alpha+\beta i)t]$, $t \in \mathbb{R}$. We have

$$K(t) = \frac{\exp[(\alpha+\beta i)t]-1}{(\alpha+\beta i)},\text{ if } t \in \mathbb{R}$$

for α, $\beta \neq 0$.

Also, we have

$$|k(s)| := |\exp[(\alpha+\beta i)s]| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$\mathbf{K}(t) = \int_0^t \exp(\alpha s)\,ds = \frac{\exp(\alpha t)-1}{\alpha} \text{ if } 0 < t,$$

for $\alpha \neq 0$.

Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We have

$$\mathcal{E}^{\alpha+\beta i}_{g,a+,b-}f(x) = \frac{1}{2}\int_a^x \exp[(\alpha+\beta i)(g(x)-g(t))]\,g'(t)\,f(t)\,dt + \frac{1}{2}\int_x^b \exp[(\alpha+\beta i)(g(t)-g(x))]\,g'(t)\,f(t)\,dt \tag{4.1}$$

for $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}\kappa^{\alpha+\beta i}_{h,a+,b-}f(x) &:= \mathcal{E}^{\alpha+\beta i}_{\ln h,a+,b-}f(x)\\ &= \frac{1}{2}\left[\int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt + \int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned} \tag{4.2}$$

for $x \in (a,b)$.

Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We have

$$\mathcal{G}^{\alpha+\beta i}_{g,a+,b-}f(x) = \frac{1}{2}\int_x^b \exp[(\alpha+\beta i)(g(b)-g(t))]\,g'(t)\,f(t)\,dt + \frac{1}{2}\int_a^x \exp[(\alpha+\beta i)(g(t)-g(a))]\,g'(t)\,f(t)\,dt \tag{4.3}$$

for $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}\mathcal{H}^{\alpha+\beta i}_{h,a+,b-}f(x) &:= \mathcal{G}^{\alpha+\beta i}_{\ln h,a+,b-}f(x)\\ &= \frac{1}{2}\left[\int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt + \int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned} \tag{4.4}$$

for $x \in (a,b)$.

Assume that $\alpha > 0$, then

$$\|k\|_{[0,g(b)-g(a)],\infty} = \sup_{s\in[0,g(b)-g(a)]} \exp(\alpha s) = \exp(\alpha [g(b) - g(a)]) .$$

By using the inequalities (2.24) and (2.25) we have

$$\begin{aligned}
(4.5)\quad & \Bigg| \mathcal{E}^{\alpha+\beta i}_{g,a+,b-} f(x) \\
&- \frac{1}{2}\left[\frac{\exp[(\alpha+\beta i)(g(b)-g(x))] + \exp[(\alpha+\beta i)(g(x)-g(a))] - 2}{(\alpha+\beta i)}\right]\frac{\phi+\Phi}{2}\Bigg| \\
&\le \frac{1}{4}|\Phi-\phi| \left[(g(x)-g(a))\exp(\alpha[g(x)-g(a)]) + (g(b)-g(x))\exp(\alpha[g(b)-g(x)])\right] \\
&\le \frac{1}{4}|\Phi-\phi|(g(b)-g(a))\exp(\alpha[g(b)-g(a)])
\end{aligned}$$

and

$$\begin{aligned}
(4.6)\quad & \Bigg| \mathcal{G}^{\alpha+\beta i}_{g,a+,b-} f(x) \\
&- \frac{1}{2}\left[\frac{\exp[(\alpha+\beta i)(g(b)-g(x))] + \exp[(\alpha+\beta i)(g(x)-g(a))] - 2}{(\alpha+\beta i)}\right]\frac{\phi+\Phi}{2}\Bigg| \\
&\le \frac{1}{4}|\Phi-\phi| \left[(g(x)-g(a))\exp(\alpha[g(x)-g(a)]) + (g(b)-g(x))\exp(\alpha[g(b)-g(x)])\right] \\
&\le \frac{1}{4}|\Phi-\phi|(g(b)-g(a))\exp(\alpha[g(b)-g(a)])
\end{aligned}$$

for $x \in (a,b)$.

If we take in (4.5) and (4.6) $g = \ln h$, where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we

$$\begin{aligned}
(4.7)\quad & \left| \kappa^{\alpha+\beta i}_{h,a+,b-} f(x) - \frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i} + \left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i} - 2}{(\alpha+\beta i)}\right]\frac{\phi+\Phi}{2}\right| \\
&\le \frac{1}{4}|\Phi-\phi|\left[\left(\frac{h(x)}{h(a)}\right)^{\alpha}\ln\left(\frac{h(x)}{h(a)}\right) + \left(\frac{h(b)}{h(x)}\right)^{\alpha}\ln\left(\frac{h(b)}{h(x)}\right)\right] \\
&\le \frac{1}{4}|\Phi-\phi|\left(\frac{h(b)}{h(a)}\right)^{\alpha}\ln\left(\frac{h(b)}{h(a)}\right)
\end{aligned}$$

and

$$\begin{aligned}
(4.8)\quad & \left| \mathcal{H}^{\alpha+\beta i}_{h,a+,b-} f(x) - \frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i} + \left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i} - 2}{(\alpha+\beta i)}\right]\frac{\phi+\Phi}{2}\right| \\
&\le \frac{1}{4}|\Phi-\phi|\left[\left(\frac{h(x)}{h(a)}\right)^{\alpha}\ln\left(\frac{h(x)}{h(a)}\right) + \left(\frac{h(b)}{h(x)}\right)^{\alpha}\ln\left(\frac{h(b)}{h(x)}\right)\right] \\
&\le \frac{1}{4}|\Phi-\phi|\left(\frac{h(b)}{h(a)}\right)^{\alpha}\ln\left(\frac{h(b)}{h(a)}\right)
\end{aligned}$$

for $x \in (a,b)$.

Similar results may be stated for the inequalities (3.19) and (3.21). However, the details are not presented here.

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