SOME GENERALIZED HERMITE-HADAMARD TYPE INEQUALITIES INVOLVING FRACTIONAL INTEGRAL OPERATOR FOR FUNCTIONS WHOSE SECOND DERIVATIVES IN ABSOLUTE VALUE ARE $s$-CONVEX

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Abstract. In this article, a general integral identity for twice differentiable mapping involving fractional integral operators is derived. As a second, by using this identity we obtained some new generalized Hermite-Hadamards type inequalities for functions whose absolute values of second derivatives are $s$-convex and concave. The main results generalize the existing Hermite-Hadamard type inequalities involving the Riemann-Liouville fractional integral. Also we pointed out, some results in this study in some special cases, such as setting $s = 1, \lambda = \alpha, \sigma(0) = 1$ and $w = 0$, more reasonable than those obtained in [10].

1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $I$ in the set of real numbers $\mathbb{R}$. Then, for $a, b \in I$ with $a < b$, the following so-called Hermite-Hadamard inequality (see, e.g., [14])

$$f \left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

(1.1)

holds true. Since its discovery in 1983, Hermite-Hadamard’s inequality has been considered the most useful inequality in mathematical analysis. A number of the papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications, see and references therein [8, 9, 14, 17].

Two definitions of $s$-convexity ($0 < s \leq 1$) of real-valued functions are well known in the literature.

Definition 1.1. Let $0 < s \leq 1$. A function $f : [0, \infty) \to \mathbb{R}$, is said to be $s$-Orlicz convex or $s$-convex in the first sense, if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have:

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y).$$

(1.2)

We denote the set of all s-convex functions in the first sense by $K_1^s$. This definition of s-convexity was introduced by Orlicz in [16] and was used in the theory of Orlicz spaces. Then, s-convex function in the second sense was introduced in Breckner’s paper [6] and a number of properties and connections with s-convexity in the first sense are discussed in paper [12].

Definition 1.2. [6] A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

We denote this by $K_2^s$. It is obvious that the s-convexity means just the convexity when $s = 1$.

In [8] Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for s-convex functions in the second sense as follows:

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**Theorem 1.1.** Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an $s$-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty), a < b$. If $f \in L^1[a, b]$ then the following inequality hold:

$$2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1}. \quad (1.3)$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3). For more study, see ([4], [5], [8], [13]).

In the following, we will give some necessary definitions and preliminary results which are used and referred to throughout this paper.

**Definition 1.3.** Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) \, dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) \, dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1} \, du$. Here is $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Some recent results and properties concerning this operator can be found [7, 11, 18, 21–23, 26]. The beta function is defined as follows:

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} = \int_0^1 t^{a-1}(1 - t)^{b-1} \, dt, \quad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function. In [26], Sarıkaya et al. gave a remarkable integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals as follows:

**Theorem 1.2.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in [a, b]$. If $f$ is convex function on $[a, b]$, then the following inequality for fractional integrals hold:

$$f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a+}^\alpha f(b) + (J_{b-}^\alpha f)(a)] \leq \frac{f(a) + f(b)}{2}. \quad (1.4)$$

It is obviously seen that, if we take $\alpha = 1$ in Theorem 1.2, then the inequality (1.4) reduces to well known Hermite-Hadamard’s inequality as (1.1).

Hermite-Hadamard type inequality for $s$-convex functions on Riemann-Liouville fractional integral is given in [21] as follows:

**Theorem 1.3.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If $f$ is $s$-convex mapping in the second sense on $[a, b]$, then the following inequality for fractional integral with $\alpha > 0$ and $s \in (0, 1]$ hold:

$$2^{s-1} f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a+}^\alpha f(b) + (J_{b-}^\alpha f)(a)] \leq \alpha \left[ \frac{1}{\alpha + s} + B(\alpha, s + 1) \right] \frac{f(a) + f(b)}{2}, \quad (1.5)$$

where $B(a, b)$ is beta function.
In [10] Dragonir et al proved the following identity and by using this identity they established new results involving Riemann-Liouville fractional integrals for twice differentiable convex mappings.

Lemma 1.1. [10] Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on $I^\circ$, the interior of $I$. Assume that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following identity for fractional integral with $\alpha > 0$ holds:

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)\alpha} [J_{b+}^\alpha f(a) + J_{a+}^\alpha f(b)] = \frac{(b - a)^2}{2(\alpha + 1)} \int_0^1 t(1 - t^\alpha) [f''(ta + (1-t)b) + f''(tb + (1-t)a)] \, dt. \tag{1.6}$$

In [24], Raina introduced a class of functions defined formally by

$$\mathcal{F}_{\rho, \lambda}^\sigma(x) = \mathcal{F}_{\rho, \lambda}^{\sigma(0), \ldots, \sigma(n)}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; \, |x| < R), \tag{1.7}$$

where the coefficients $\sigma(k) (k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $R$ is the set of real numbers. With the help of (1.7), Raina [24] and Agarwal et al. [3] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\mathcal{J}_{\rho, \lambda, a+}^\sigma \varphi(x) = \int_a^x (x - t)^{\lambda - 1} \mathcal{F}_{\rho, \lambda}^\sigma[w(x - t)^\rho] \varphi(t) \, dt \quad (x > a > 0), \tag{1.8}$$

$$\mathcal{J}_{\rho, \lambda, b-}^\sigma \varphi(x) = \int_x^b (t - x)^{\lambda - 1} \mathcal{F}_{\rho, \lambda}^\sigma[w(t - x)^\rho] \varphi(t) \, dt \quad (0 < x < b), \tag{1.9}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits. In recently some new integral inequalities this operator involving have appeared in the literature (see, e.g., [1–3, 19, 20, 28]).

It is easy to verify that $\mathcal{J}_{\rho, \lambda, a+}^\sigma \varphi(x)$ and $\mathcal{J}_{\rho, \lambda, b-}^\sigma \varphi(x)$ are bounded integral operators on $L(a, b)$, if

$$\mathfrak{M} := \mathcal{F}_{\rho, \lambda, 1}^\sigma[w(b - a)^\rho] < \infty. \tag{1.10}$$

In fact, for $\varphi \in L(a, b)$, we have

$$||\mathcal{J}_{\rho, \lambda, a+}^\sigma \varphi(x)||_1 \leq \mathfrak{M}(b - a)^\lambda ||\varphi||_1 \tag{1.11}$$

and

$$||\mathcal{J}_{\rho, \lambda, b-}^\sigma \varphi(x)||_1 \leq \mathfrak{M}(b - a)^\lambda ||\varphi||_1 \tag{1.12}$$

where

$$||\varphi||_p := \left( \int_a^b |\varphi(t)|^p \, dt \right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals $J_{a+}^\alpha$ and $J_{b-}^\alpha$ of order $\alpha$ follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.8) and (1.9).

In [25] generalized Hermite-Hadamard’s inequality for s-convex mapping involving fractional integral operators as follows;
**Theorem 1.4.** Let \( f : [a, b] \to \mathbb{R} \) be a function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is an \( s \)-convex function on \([a, b]\) then we have the following inequalities for generalized fractional integral operators:

\[
2^s f \left( \frac{a+b}{2} \right) \leq \frac{1}{(b-a)\lambda} \mathcal{F}_{\rho,\lambda}^\sigma [w(b-a)^\rho] \left( \left( \mathcal{J}_{\rho,\lambda,b^{-},w}^\sigma f \right) (a) + \left( \mathcal{J}_{\rho,\lambda,a^{+},w}^\sigma f \right) (b) \right) 
\]

\[
= \frac{1}{\mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho]} \left[ A_1(\lambda, s) + \mathcal{F}_{\rho,\lambda}^\sigma [w(b-a)^\rho] \right] [f(a) + f(b)], \tag{1.13}
\]

where

\[
\sigma_{0,\lambda}(k) = \frac{\sigma(k)}{\lambda + \rho k + s}, \quad k = 0, 1, 2, \ldots \quad \text{and}
\]

\[
A_1(\lambda, s) = \int_0^1 t^{\lambda-1} (1-t)^s \mathcal{F}_{\rho,\lambda}^\sigma [w(b-a)^\rho] dt.
\]

In this paper, first aim is to establish a new integral identity for a twice differentiable function via fractional integral operators. Using this new identity, we next present some Hermite-Hadamard type inequalities for functions whose second-order derivatives absolute values are \( s \)-convex and concave in the second sense.

## 2. Main Results

**Lemma 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable mapping on \((a, b)\) with \( a < b \) and \( \lambda > 0 \). If \( f'' \in L[a, b] \), then the following equality for generalized fractional integrals holds:

\[
\mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{2(b-a)^\lambda} \left( \mathcal{J}_{\rho,\lambda,b^{-},w}^\sigma f (a) + \mathcal{J}_{\rho,\lambda,a^{+},w}^\sigma f (b) \right) = \frac{(b-a)^2}{2}
\]

\[
\times \int_0^1 t^{\lambda-1} \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \frac{f'' ((1-t) b + f' ((1-t) a + tb))}{b-a} dt.
\]

**Proof.** Integrating by parts and changing variables with \( x = (ta + (1-t)b) \) we get,

\[
I_1 = \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \int_0^1 tf'' ((ta + (1-t)b)) dt
\]

\[
= \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \left\{ \frac{1}{a-b} \left( f' (ta + (1-t)b) \right) \right\} - \frac{1}{a-b} \int_0^1 f' ((ta + (1-t)b)) dt
\]

\[
\mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \left\{ \frac{f'(a)}{b-a} - \frac{f(a) - f(b)}{(b-a)^2} \right\},
\]

by using same method

\[
I_2 = \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \int_0^1 tf'' ((1-t)a + tb) dt
\]

\[
= \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \left\{ \frac{f(b)}{b-a} - \frac{f(b) - f(a)}{(b-a)^2} \right\}.
\]
where

Theorem 2.1.

a differentiable function whose absolute value is s-convex and s-concave.

Remark 2.1.

I

Thus combining (2.2), (2.3), (2.4) and (2.5) as

and

\[ \rho, \lambda > 0 \]

\[ f''(ta + (1-t)b) \]

\[ f''((1-t)a + tb) \]

\[ F^{(k)}_{\rho,\lambda+1}[(b-a)^{\rho}f'(a)] \]

\[ \frac{1}{a-b} \]

\[ \int_0^1 \left[ \frac{F^{(k)}_{\rho,\lambda+1}[(b-a)^{\rho}f'(a)]}{a-b} - \frac{F^{(k)}_{\rho,\lambda+1}[(b-a)^{\rho}f(b)]}{(b-a)^2} + \frac{1}{b-a} \right] dt \]

and

\[ I_4 = \int_0^1 \left[ \frac{F^{(k)}_{\rho,\lambda+1}[(b-a)^{\rho}f'(a)]}{a-b} - \frac{F^{(k)}_{\rho,\lambda+1}[(b-a)^{\rho}f(b)]}{(b-a)^2} + \frac{1}{b-a} \right] dt \]

Thus combining (2.2), (2.3), (2.4) and (2.5) as \( I_1 + I_2 - I_3 - I_4 \) and multiplying both sides of the obtained equality with \( \frac{(b-a)^2}{2} \), which proof is completed.

\[ \square \]

Remark 2.1. Setting \( \lambda = \alpha, \sigma(0) = 1 \) and \( w = 0 \) in Lemma 2.1 found to yield the same identity as Lemma 2.1 in [10].

Using this lemma, we can get the following results via fractional integral operator for twice differentiable function whose absolute value is s-convex and s-concave.

Theorem 2.1. Let \( f: [a, b] \to \mathbb{R} \) be a twice differentiable function on \( [a, b] \) with \( a < b \) and \( \lambda > 0 \). If \( f'' \) is s-convex in the second sense on \( (a, b) \) then the following inequality for generalized fractional integral operators holds:

\[
\left[ \frac{F_{\rho,\lambda+1}^{(k)}[w(b-a)^{\rho}]}{2} \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{2(b-a)^{\lambda}} \left[ \left( \int_{a}^{\infty} \frac{1}{\rho,\lambda,b^{-\rho}f} \right) (a) + \left( \int_{a}^{\infty} \frac{1}{\rho,\lambda,a^{\rho}f} \right) (b) \right] \right] \leq \frac{(b-a)^2}{2} \left[ \frac{F^{(k+1)}_{\rho,\lambda+2}[(b-a)^{\rho}f'']}{2} \right] \]

where

\[ \sigma_{1,s}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{(2 + s)(\lambda + \rho k + s + 2)} + B(2, s + 1) - B(\lambda + \rho k + 2, s + 1) \right] \]

\( \rho, \lambda > 0, \ w \in \mathbb{R}, \ s \in (0, 1) \) and \( B(\ldots) \), is Euler beta function.
Proof. From Lemma 2.1 with properties of modulus, we get
\[
\left| F_{\rho, \lambda+1}^\sigma[w(b - a)\rho] \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{2(b - a)^{\lambda}} \left[ \left( F_{\rho, \lambda}^\sigma w^\rho f \right)(a) + \left( F_{\rho, \lambda}^\sigma w^\rho f \right)(b) \right] \right|
\leq \frac{(b - a)^2}{2} \int_0^1 \left| t F_{\rho, \lambda+2}^\sigma[(b - a)\rho] - t^{\lambda+1} F_{\rho, \lambda+2}^\sigma[(b - a)\rho]\right| dt
\]
\[
\leq \frac{(b - a)^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(b - a)^{\rho k}}{\Gamma(\rho k + \lambda + 2)} \int_0^1 \left| t - t^{\lambda+\rho k+1} \right| |f''(ta + (1 - t)b)| dt
\]
\[
+ \frac{(b - a)^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(b - a)^{\rho k}}{\Gamma(\rho k + \lambda + 2)} \int_0^1 \left| t - t^{\lambda+\rho k+1} \right| |f''((1 - t)a + tb)| dt.
\]
Since $|f''|$ is s-convex, we have
\[
\int_0^1 (t - t^{\lambda+\rho k+1}) |f''(ta + (1 - t)b)| dt + \int_0^1 (t - t^{\lambda+\rho k+1}) |f''((1 - t)a + tb)| dt
\]
\[
\leq \int_0^1 (t^{\lambda+\rho k}) dt + \int_0^1 (t - t^{\lambda+\rho k}) dt \left[ |f''(a)| + |f''(b)| \right]
\]
\[
= \frac{(\lambda + \rho k)}{(2 + s)(\lambda + \rho k + s + 2)} + B(2, s + 1) - B(\lambda + \rho k + 2, s + 1) \left[ |f''(a)| + |f''(b)| \right].
\]
Thus combining the inequalities (2.6) with (2.7), the requested result is obtained.

\[
\Box
\]

Corollary 2.1. If we take $s = 1$ in Theorem 2.1, we get the following inequality:
\[
\left| F_{\rho, \lambda+1}^\sigma[w(b - a)\rho] \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{2(b - a)^{\lambda}} \left[ \left( F_{\rho, \lambda}^\sigma w^\rho f \right)(a) + \left( F_{\rho, \lambda}^\sigma w^\rho f \right)(b) \right] \right|
\leq \frac{(b - a)^2}{2} F_{\rho, \lambda+2}^\sigma[w(b - a)\rho] \left[ |f''(b)| + |f''(b)| \right],
\]
where
\[
\sigma_{1,1}(k) = \sigma(k) \left[ \frac{(\lambda + \rho k)}{2(\lambda + \rho k + 2)} \right], \quad \rho, \lambda > 0 \quad \text{and} \quad w \in \mathbb{R}.
\]

Corollary 2.2. If we take $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Corollary 2.1, we get the following inequality:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left[ J_{(0)}^\alpha f(a) + J_{(0)}^\alpha f(b) \right] \right|
\leq \frac{(b - a)^2 \alpha}{4(\alpha + 1)(\alpha + 2)} \left[ |f''(b)| + |f''(b)| \right],
\]
which is more reasonable than the result obtained Theorem 2 in [10] under the same assumptions.

Remark 2.2. Setting $s = 1$, $\lambda = \alpha = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 2.1 found to yield the same result as Proposition 2 in [27].
Theorem 2.2. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on $(a, b)$ with $a < b$. If $|f''|^q$ is s-convex in the second sense and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for generalized fractional integral operators holds:

$$
\left| \mathcal{F}_{\rho, \lambda, 1}^{\sigma} [w(b-a)^q] \left( \frac{f''(a) + f''(b)}{2} \right) \right| - \frac{1}{2(b-a)^2} \left[ \left( \mathcal{J}_{\rho, \lambda, b^{-w}}^{\sigma} f \right)(a) + \left( \mathcal{J}_{\rho, \lambda, a^{+w}}^{\sigma} f \right)(b) \right] \\
\leq \frac{(b-a)^2}{2} \mathcal{F}_{\rho, \lambda, 2}^{\sigma} [w(b-a)^q] \left[ \frac{|f''(a)|^q + |f''(b)|^q}{s+1} \right]^{\frac{1}{2}},
$$

where

$$\sigma_2(k) = 2\sigma(k) \left[ \frac{1}{\lambda + \rho k} B \left( \frac{p+1}{\lambda + \rho k}, p + 1 \right) \right]^{\frac{1}{2}},$$

$\rho, \lambda > 0$, $w \in \mathbb{R}$, $s \in (0, 1]$ and $B(\cdot, \cdot)$ is Euler beta function.

Proof. From Lemma 2.1 have,

$$
\left| \mathcal{F}_{\rho, \lambda, 1}^{\sigma} [w(b-a)^q] \left( \frac{f''(a) + f''(b)}{2} \right) \right| - \frac{1}{2(b-a)^2} \left[ \left( \mathcal{J}_{\rho, \lambda, b^{-w}}^{\sigma} f \right)(a) + \left( \mathcal{J}_{\rho, \lambda, a^{+w}}^{\sigma} f \right)(b) \right] \\
\leq \frac{(b-a)^2}{2} \int_0^1 \left| t \mathcal{F}_{\rho, \lambda, 2}^{\sigma} [(b-a)^q] \right| \left[ f''(ta + (1-t)b) + f''((1-t)a + tb) \right] dt \tag{2.8}
$$

Using Hölder Inequality and the s-convexity of $|f''|^q$ we get the following inequality,

$$
\int_0^1 \left| t(1 - t^{\lambda + \rho k}) \left[ f''(ta + (1-t)b) + f''((1-t)a + tb) \right] \right| dt \tag{2.9}
$$

$$
\leq \frac{1}{\lambda + \rho k} \left\{ \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 |f''((1-t)a + tb)|^q dt \right)^{\frac{1}{q}} \right\}
$$

$$
\leq \frac{1}{\lambda + \rho k} \left( \frac{1}{s+1} \right)^{\frac{1}{2}} \left[ \left( \frac{1}{s+1} \right)^{\frac{1}{q}} \left[ |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{2}} \right]
$$

$$
= 2 \left( \frac{1}{\lambda + \rho k} B \left( \frac{p+1}{\lambda + \rho k}, p + 1 \right) \right)^{\frac{1}{2}} \left( \frac{1}{s+1} \right)^{\frac{1}{2}} \left[ \left( \frac{1}{s+1} \right)^{\frac{1}{q}} \left[ |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{2}} \right].
$$

By changing $x = t^{\lambda + \rho k}$ and a simple calculation we get,

$$
\int_0^1 t^{p} (1 - t^{\lambda + \rho k}) dt = \frac{1}{\lambda + \rho k} B \left( \frac{p+1}{\lambda + \rho k}, p + 1 \right).
$$

Thus combining (2.8) with (2.9), the desired result is obtained. \hfill \Box

Remark 2.3. Setting $\lambda = \alpha = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 2.2 found to yield the same result as Theorem 10 in [13].
Corollary 2.3. Taking \( s = 1 \) in Theorem 2.2, the following inequality holds:

\[
\left| \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^{\lambda}} \left( \left( \mathcal{F}^{\sigma}_{\rho,\lambda,w} f \right)(a) + \left( \mathcal{F}^{\sigma}_{\rho,\lambda+1,w} f \right)(b) \right) \right| \\
\leq \frac{(b-a)^{2}}{2} \mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-a)^{\rho}] \left[ \left| f''(a) \right|^{q} + \left| f''(b) \right|^{q} \right]^{\frac{1}{q}}
\]

where

\[
\sigma_{2}(k) = 2\sigma(k) \left[ \frac{1}{(\lambda + k)} B \left( \frac{p + 1}{\lambda + k}, \frac{p + 1}{\lambda} \right) \right]^{\frac{1}{q}},
\]

\( \rho, \lambda > 0, \ w \in \mathbb{R} \) and \( B(\ldots,\ldots) \) is Euler Gamma function.

Corollary 2.4. Taking \( \lambda = \alpha, \ \sigma(0) = 1 \) and \( w = 0 \) in Corollary 2.3, the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ J^{\alpha}_{(b)} f(a) + J^{\alpha}_{(a)} f(b) \right] \right| \\
\leq \frac{(b-a)^{2}}{(\alpha + 1)} \left[ \frac{1}{\alpha} B \left( \frac{p + 1}{\alpha}, \frac{p + 1}{\alpha} \right) \right]^{\frac{1}{q}} \left[ \left| f''(a) \right|^{q} + \left| f''(b) \right|^{q} \right]^{\frac{1}{q}},
\]

which is more reasonable than obtained Theorem 3 in [10] under the same assumptions.

Theorem 2.3. Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\) with \( a < b \). If \( f'' \) is \( s \)-convex in the second sense and \( q \geq 1 \), then the following inequality for generalized fractional integral operators holds:

\[
\left| \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^{\lambda}} \left( \left( \mathcal{F}^{\sigma}_{\rho,\lambda,w} f \right)(a) + \left( \mathcal{F}^{\sigma}_{\rho,\lambda+1,w} f \right)(b) \right) \right| \\
\leq \frac{(b-a)^{2}}{2} \mathcal{F}^{\sigma}_{\rho,\lambda+2}[w(b-a)^{\rho}] \left[ \left| f''(a) \right|^{q} + \left| f''(b) \right|^{q} \right]^{\frac{1}{q}}
\]

where

\[
\sigma_{3,s}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{s}{q}} \times \left\{ \frac{\lambda + \rho k}{(s+2)(\lambda + \rho k + s + 2)} \left[ \left| f''(a) \right|^{q} + (B(2,s+1) - B(\lambda + \rho k + 2, s + 1)) \left| f''(b) \right|^{q} \right]^{\frac{1}{q}} + \left[ (B(2,s+1) - B(\lambda + \rho k + 2, s + 1)) \left| f''(a) \right|^{q} + \frac{\lambda + \rho k}{(s+2)(\lambda + \rho k + s + 2)} \left| f''(b) \right|^{q} \right]^{\frac{1}{q}} \right\}
\]

\( \rho, \lambda > 0, \ w \in \mathbb{R}, \ s \in (0,1] \) and \( B(\ldots,\ldots) \) is Euler beta function.
Proof. From Lemma 2.1 with properties of modulus we get,

\[ \left| F_{\rho, \lambda+1}^\sigma \left[ w(b-a)^\rho \right] \left( \frac{f(a) + f(b)}{2} \right) - \frac{1}{2(b-a)^\lambda} \left( \left( J_{\rho, \lambda,b-w}^\sigma f \right)(a) + \left( J_{\rho, \lambda,a+w}^\sigma f \right)(b) \right) \right| \]

\[ \leq \frac{(b-a)^2}{2} \int_0^1 \left| tF_{\rho, \lambda+2}^\sigma [(b-a)^\rho] - t^{\lambda+1} F_{\rho, \lambda+2}^\sigma [(b-a)^\rho \rho^\rho] \right| dt \]

\[ \leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)w^k(b-a)^\rho^k}{\Gamma(pk + \lambda + 2)} \int_0^1 \left| t - t^{\lambda+pk+1} \right| f''(ta + (1-t)b) |dt| \]

Using Power-mean Inequality and s-convexity of \( |f''|^q \), we obtain the following inequality;

\[ \int_0^1 \left( 1 - t^{\lambda+pk} \right) \left[ f''(ta + (1-t)b) + f''((1-t)a + tb) \right] dt \]

\[ \leq \left[ \int_0^1 \left( 1 - t^{\lambda+pk+1} \right) \right]^{1/2} \left[ \left( \int_0^1 \left| f''(ta + (1-t)b) \right|^q dt \right) \right]^{1/2} \]

\[ + \left[ \int_0^1 \left( 1 - t^{\lambda+pk+1} \right) \right]^{1/2} \left[ \left( \int_0^1 \left| f''((1-t)a + tb) \right|^q dt \right) \right]^{1/2} \]

\[ \leq \left[ \frac{\lambda + pk}{2(\lambda + pk + 2)} \right]^{1/2} \times \left\{ \frac{\lambda + pk}{(s+2)(\lambda + pk + s + 2)} \left[ f''(a)^q + (B(2, s+1) - B(\lambda + pk + 2, s + 1)) |f''(b)|^q \right]^{1/2} \right\} \]

Combining the inequalities (2.10) with (2.11) we have

\[ \left| F_{\rho, \lambda+1}^\sigma \left[ w(b-a)^\rho \right] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^\lambda} \left( \left( J_{\rho, \lambda,b-w}^\sigma f \right)(a) + \left( J_{\rho, \lambda,a+w}^\sigma f \right)(b) \right) \right| \]

\[ \leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k)w^k(b-a)^\rho^k}{\Gamma(pk + \lambda + 2)} \left[ \frac{\lambda + pk}{2(\lambda + pk + 2)} \right]^{1/2} \times \left\{ \frac{\lambda + pk}{(s+2)(\lambda + pk + s + 2)} \left[ f''(a)^q + (B(2, s+1) - B(\lambda + pk + 2, s + 1)) |f''(b)|^q \right]^{1/2} \right\} \]

\[ = \frac{(b-a)^2}{2} F_{\rho, \lambda+2}^\sigma \left[ w(b-a)^\rho \right]. \]

Thus the proof is completed. \( \square \)
Corollary 2.5. Taking $s = 1$ with $\rho, \lambda > 0$ and $w \in \mathbb{R}$ in Theorem 2.3, the following inequality holds:

$$
\left| F_{\rho,\lambda+1}^{s} \left[ w(b-a)^{\sigma} \right] f(a) + f(b) \right| \leq \frac{1}{2(b-a)^{\lambda}} \left[ \left( F_{\rho,\lambda,b-wf}^{s} \right)(a) + \left( F_{\rho,\lambda,a+wf}^{s} \right)(b) \right] \left( b-a \right)^{2} \sigma_{4,1}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{1}{q}} \left\{ \begin{array}{l}
\lambda + \rho k \\
3(\lambda + \rho k + 3)
\end{array} \begin{array}{l}
|f''(a)|^{q} + (\lambda + \rho k)(\lambda + \rho k + 3) |f''(b)|^{q} \\
6(\lambda + \rho k + 2)(\lambda + \rho k + 3) |f''(a)|^{q} + \frac{\lambda + \rho k}{3(\lambda + \rho k + 3)} |f''(b)|^{q}
\end{array} \right\}^{\frac{1}{q}}
$$

where

$$
\sigma_{3,1}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{1}{q}} \left( \frac{\lambda + \rho k}{3(\lambda + \rho k + 3)} \right)^{\frac{1}{q}} + \left( \frac{\lambda + \rho k}{6(\lambda + \rho k + 2)(\lambda + \rho k + 3)} \right)^{\frac{1}{q}}
$$

Remark 2.4. Setting $\lambda = \alpha = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 2.3 founds to yield the same result as Theorem 8 in [13].

Remark 2.5. Setting $s = 1$, $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 2.3 founds to yield the same result as Theorem 4 in [10].

Theorem 2.4. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on $(a, b)$ with $a < b$. If $|f''|^q$ is $s$-concave in the second sense and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for generalized fractional integral operators holds:

$$
\left| F_{\rho,\lambda+1}^{s} \left[ w(b-a)^{\sigma} \right] f(a) + f(b) \right| \leq \frac{1}{2(b-a)^{\lambda}} \left[ \left( F_{\rho,\lambda,b-wf}^{s} \right)(a) + \left( F_{\rho,\lambda,a+wf}^{s} \right)(b) \right] \left( b-a \right)^{2} \sigma_{4,1}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{1}{q}} \left( \frac{\lambda + \rho k}{3(\lambda + \rho k + 3)} \right)^{\frac{1}{q}} + \left( \frac{\lambda + \rho k}{6(\lambda + \rho k + 2)(\lambda + \rho k + 3)} \right)^{\frac{1}{q}}
$$

where

$$
\sigma_{3,1}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{1}{q}} \left( \frac{\lambda + \rho k}{3(\lambda + \rho k + 3)} \right)^{\frac{1}{q}} + \left( \frac{\lambda + \rho k}{6(\lambda + \rho k + 2)(\lambda + \rho k + 3)} \right)^{\frac{1}{q}}
$$

where $\sigma_{4,1}(k) = \sigma(k) \left[ \frac{\lambda + \rho k}{2(\lambda + \rho k + 2)} \right]^{1-\frac{1}{q}} \left( \frac{\lambda + \rho k}{3(\lambda + \rho k + 3)} \right)^{\frac{1}{q}} + \left( \frac{\lambda + \rho k}{6(\lambda + \rho k + 2)(\lambda + \rho k + 3)} \right)^{\frac{1}{q}}$, $\rho, \lambda > 0$, $s \in (0, 1]$, and $w \in \mathbb{R}$.
Proof. From Lemma 2.1 and Hölder inequality with properties of modulus, we have

\[ \left| \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^\lambda} \left[ \left( \mathcal{F}_{\rho,\lambda,b,-w}^\sigma \right)(a) + \left( \mathcal{F}_{\rho,\lambda,\alpha^+,w}^\sigma \right)(b) \right] \right| \]

\[ \leq \frac{(b-a)^2}{2} \int_0^1 \left| t \mathcal{F}_{\rho,\lambda+2}^\sigma (b-a)^\rho - t^{\lambda+1} \mathcal{F}_{\rho,\lambda+2}^\sigma (b-a)^\rho \right| dt \]

\[ \leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k (b-a)^{\rho k}}{\Gamma(\rho k + \lambda + 2)} \]

\[ \times \int_0^1 \left| t - t^{\lambda + \rho k + 1} \right| \left[ \left| f''(ta + (1-t)b) \right| + \left| f''((1-t)a + tb) \right| \right] dt \]

\[ \leq \frac{(b-a)^2}{2} \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k (b-a)^{\rho k}}{\Gamma(\rho k + \lambda + 2)} \]

\[ \times \left( \int_0^1 (t - t^{\lambda + \rho k + 1})^p dt \right)^{\frac{1}{p}} \left\{ \left( \int_0^1 \left| f''(ta + (1-t)b) \right|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 \left| f''((1-t)a + tb) \right|^q dt \right)^{\frac{1}{q}} \right\}. \]

Since \( |f''|^2 \) is s-concave, we can write

\[ \int_0^1 \left| f''((1-t)a + tb) \right|^q dt \leq 2^{s-1} \left| f'' \left( \frac{a + b}{2} \right) \right|^q \]

(2.13)

and

\[ \int_0^1 \left| f''(ta + (1-t)b) \right|^q dt \leq 2^{s-1} \left| f'' \left( \frac{a + b}{2} \right) \right|^q, \]

(2.14)

On the other hand, by simple calculating we establish,

\[ \int_0^1 (t - t^{\lambda + \rho k + 1})^p dt = \frac{1}{\lambda + \rho k} B \left( \frac{p + 1}{\lambda + \rho k}, p + 1 \right). \]

Thus combining (2.13) and (2.14) with (2.12) the requested result is obtained. \( \Box \)

Corollary 2.6. Taking \( s = 1 \) in Theorem 2.4, the following inequality holds;

\[ \left| \mathcal{F}_{\rho,\lambda+1}^\sigma [w(b-a)^\rho] \frac{f(a) + f(b)}{2} - \frac{1}{2(b-a)^\lambda} \left[ \left( \mathcal{F}_{\rho,\lambda,b,-w}^\sigma \right)(a) + \left( \mathcal{F}_{\rho,\lambda,\alpha^+,w}^\sigma \right)(b) \right] \right| \]

\[ \leq \frac{(b-a)^2}{2} \mathcal{F}_{\rho,\alpha^+,\lambda+2}^\sigma [w(b-a)^\rho] \left| f'' \left( \frac{a + b}{2} \right) \right| \]

where \( \rho, \lambda > 0 \) and \( w \in \mathbb{R} \),

\[ \sigma_{1,1}(k) = \sigma(k) 2^s \left[ \frac{2}{\lambda + \rho k} B \left( \frac{p + 1}{\lambda + \rho k}, p + 1 \right) \right]^s. \]

Corollary 2.7. If we take \( \lambda = \alpha, \sigma(0) = 1 \) and \( w = 0 \) in Corollary 2.6, the following inequality holds;

\[ \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\lambda} \left[ J_{(b)}^\alpha f(a) + J_{(a)}^\alpha f(b) \right] \right| \]

\[ \leq \frac{(b-a)^2}{(\alpha + 1)} \left[ \frac{1}{\alpha} B \left( \frac{p + 1}{\alpha}, p + 1 \right) \right]^s \left| f'' \left( \frac{a + b}{2} \right) \right|, \]

which is more reasonable than Theorem 5 in [10] under the same assumptions.
Remark 2.6. Setting $s = 1$, $\lambda = \alpha = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 2.3 found to yield the same result as Theorem 9 in [13].

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