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INEQUALITIES FOR HYPO-q-NORMS ON A CARTESIAN PRODUCT OF INNER PRODUCT SPACES

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Abstract. In this paper we introduce the hypo-q-norms on a Cartesian product of inner product spaces. A representation of these norms in terms of inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al., Grüss type inequalities, Boas-Bellman and Bombieri type inequalities are also given.

1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}\left(\|\cdot\|_n\right) := \left\{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \middle| \, \|\lambda\|_n \leq 1\right\}.$$

As an example of such norms we should mention the usual *p-norms*

$$\|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\left\{|\lambda_1|, \ldots, |\lambda_n|\right\} & \text{if } \ p = \infty; \\ \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } \ p \in [1, \infty). \end{cases} \tag{1.1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\left\{\|x_1\|, \ldots, \|x_n\|\right\} & \text{if } \ p = \infty; \\ \\ \left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } \ p \in [1, \infty); \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$.

Following [9], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in B\left(\|\cdot\|_n\right)} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$.

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It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
(ii) $\|\mathbf{x}+\mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}, \mathbf{y} \in E^n$;
(iii) $\|\alpha\mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^n \lambda_j x_j = 0$ for any $(\lambda_1, \ldots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \ldots, \lambda_n^0 \neq 0$ such that $\left(\lambda_1^0, 0, \ldots, 0\right), \left(0, \lambda_2^0, \ldots, 0\right), \ldots, \left(0, 0, \ldots, \lambda_n^0\right) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.4}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in $\mathbb{K}^n$, which we denote by $\mathbb{B}_n$, $\mathbb{B}_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \,\middle|\, \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.5}$$

Moreover, if $E = H$, H is a inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.6}$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the $n-$tuples of vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n \langle x_j, y_j \rangle, \qquad \mathbf{x},\ \mathbf{y} \in H^n, \tag{1.7}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad \mathbf{x} \in H^n. \tag{1.8}$$

The following result established in [9] connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1 (Dragomir, 2007, [9])**.** *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2, \tag{1.9}$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2 (Dragomir, 2007, [9]). *For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$\|\mathbf{x}\|_e = \sup_{\|y\| \leq 1} \left(\sum_{j=1}^{n} |\langle x_j, y \rangle|^2 \right)^{\frac{1}{2}}. \tag{1.10}$$

Motivated by the above results, in this paper we introduce the hypo-q-norms on a Cartesian product of inner product spaces. A representation of these norms in terms of inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

2. General Results

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. We have the following representation result for the *hypo-q-norms* on H^n.

Theorem 3. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|y\| \leq 1} \left\{ \left(\sum_{j=1}^{n} |\langle x_j, y \rangle|^q \right)^{1/q} \right\} \tag{2.1}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{x}\|_{h,n,1} = \sup_{\|y\| \leq 1} \left\{ \sum_{j=1}^{n} |\langle x_j, y \rangle| \right\} \tag{2.2}$$

and

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1,\ldots,n\}} \{\|x_j\|\}. \tag{2.3}$$

In particular,

$$\|\mathbf{x}\|_{h,e} = \sup_{\|y\| \leq 1} \left\{ \left(\sum_{j=1}^{n} |\langle x_j, y \rangle|^2 \right)^{1/2} \right\}. \tag{2.4}$$

Proof. Using Hölder's discrete inequality for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \leq \left(\sum_{j=1}^{n} |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \leq \|\beta\|_q \tag{2.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with

$$\alpha_j:=\frac{\overline{\beta_j}\left|\beta_j\right|^{q-2}}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)^{1/p}}$$

for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We observe that

$$\left|\sum_{j=1}^n\alpha_j\beta_j\right|=\left|\sum_{j=1}^n\frac{\overline{\beta_j}\left|\beta_j\right|^{q-2}}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)^{1/p}}\beta_j\right|=\frac{\sum_{j=1}^n\left|\beta_j\right|^q}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)^{1/p}}$$
$$=\left(\sum_{j=1}^n\left|\beta_j\right|^q\right)^{1/q}=\left\|\beta\right\|_q$$

and

$$\left\|\alpha\right\|_p^p=\sum_{j=1}^n\left|\alpha_j\right|^p=\sum_{j=1}^n\frac{\left|\overline{\beta_j}\left|\beta_j\right|^{q-2}\right|^p}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)}=\sum_{j=1}^n\frac{\left(\left|\beta_j\right|^{q-1}\right)^p}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)}$$
$$=\sum_{j=1}^n\frac{\left|\beta_j\right|^{qp-p}}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)}=\sum_{j=1}^n\frac{\left|\beta_j\right|^q}{\left(\sum_{k=1}^n\left|\beta_k\right|^q\right)}=1.$$

Therefore, by (2.5) we have the representation

$$\sup_{\left\|\alpha\right\|_p\leq 1}\left|\sum_{j=1}^n\alpha_j\beta_j\right|=\left\|\beta\right\|_q \tag{2.6}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By the properties of inner product, we have for any $u\in H$, $u\neq 0$ that

$$\left\|u\right\|=\sup_{\left\|y\right\|\leq 1}\left|\left\langle u,y\right\rangle\right|. \tag{2.7}$$

Let $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{K}^n$ and $\mathbf{x}\in H^n$ with $\mathbf{x}=(x_1,\dots,x_n)$. Then by (2.7) we have

$$\left\|\sum_{j=1}^n\alpha_jx_j\right\|=\sup_{\left\|y\right\|\leq 1}\left|\left\langle\sum_{j=1}^n\alpha_jx_j,y\right\rangle\right|=\sup_{\left\|y\right\|\leq 1}\left|\sum_{j=1}^n\alpha_j\left\langle x_j,y\right\rangle\right|. \tag{2.8}$$

By taking the supremum in this equality we have

$$\sup_{\left\|\alpha\right\|_p\leq 1}\left\|\sum_{j=1}^n\alpha_jx_j\right\|=\sup_{\left\|\alpha\right\|_p\leq 1}\left(\sup_{\left\|y\right\|\leq 1}\left|\sum_{j=1}^n\alpha_j\left\langle x_j,y\right\rangle\right|\right)$$
$$=\sup_{\left\|y\right\|\leq 1}\left(\sup_{\left\|\alpha\right\|_p\leq 1}\left|\sum_{j=1}^n\alpha_j\left\langle x_j,y\right\rangle\right|\right)=\sup_{\left\|y\right\|\leq 1}\left(\sum_{j=1}^n\left|\left\langle x_j,y\right\rangle\right|^q\right)^{1/2},$$

where for the last equality we used the representation (2.6).

This proves (2.1).

Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \leq \max_{j\in\{1,\ldots,n\}}|\alpha_j|\sum_{j=1}^{n}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_\infty\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \leq \|\beta\|_1, \tag{2.9}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$.

For $(\beta_1,\ldots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\ldots,\alpha_n)$ with $\alpha_j:=\frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{|\beta_j|}\beta_j\right| = \sum_{j=1}^{n}|\beta_j| = \|\beta\|_1$$

and

$$\|\alpha\|_\infty = \max_{j\in\{1,\ldots,n\}}|\alpha_j| = \max_{j\in\{1,\ldots,n\}}\left|\frac{\overline{\beta_j}}{|\beta_j|}\right| = 1$$

and by (2.9) we get the representation

$$\sup_{\|\alpha\|_\infty\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_1 \tag{2.10}$$

for any $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (2.8) we have

$$\begin{aligned}\sup_{\|\alpha\|_\infty\leq 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_\infty\leq 1}\left(\sup_{\|y\|\leq 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right)\\ &= \sup_{\|y\|\leq 1}\left(\sup_{\|\alpha\|_\infty\leq 1}\left|\sum_{j=1}^{n}\alpha_j\langle x_j,y\rangle\right|\right) = \sup_{\|y\|\leq 1}\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|\right),\end{aligned}$$

where for the last equality we used the equality (2.10), which proves the representation (2.2).

Finally, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \leq \sum_{j=1}^{n}|\alpha_j|\max_{j\in\{1,\ldots,n\}}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_1\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \leq \|\beta\|_\infty, \tag{2.11}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, let $j_0\in\{1,...,n\}$ such that $\|\beta\|_\infty=\max_{j\in\{1,...,n\}}\left|\beta_j\right|=\left|\beta_{j_0}\right|$. Consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0}=\frac{\overline{\beta_{j_0}}}{\left|\beta_{j_0}\right|}$ and $\alpha_j=0$ for $j\neq j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j|=\frac{\left|\overline{\beta_{j_0}}\right|}{\left|\beta_{j_0}\right|}=1 \text{ and } \left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\frac{\overline{\beta_{j_0}}}{\left|\beta_{j_0}\right|}\beta_{j_0}\right|=\left|\beta_{j_0}\right|=\|\beta\|_\infty,$$

therefore by (2.11) we obtain the representation

$$\sup_{\|\alpha\|_1\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_\infty \tag{2.12}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (2.8) and by using the equality (2.12), we have

$$\begin{aligned}\sup_{\|\alpha\|_1\leq 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_1\leq 1}\left(\sup_{\|y\|\leq 1}\left|\sum_{j=1}^{n}\alpha_j\left\langle x_j,y\right\rangle\right|\right)\\ &= \sup_{\|y\|\leq 1}\left(\sup_{\|\alpha\|_1\leq 1}\left|\sum_{j=1}^{n}\alpha_j\left\langle x_j,y\right\rangle\right|\right)=\sup_{\|y\|\leq 1}\left(\max_{j\in\{1,...,n\}}\left|\left\langle x_j,y\right\rangle\right|\right)\\ &= \max_{j\in\{1,...,n\}}\left(\sup_{\|y\|\leq 1}\left|\left\langle x_j,y\right\rangle\right|\right)=\max_{j\in\{1,...,n\}}\left\{\|x_j\|\right\},\end{aligned}$$

which proves (2.3). For the last equality we used the property (2.7). □

Corollary 1. *With the assumptions of Theorem 3 we have for $q\geq 1$ that*

$$\frac{1}{n^{1/q}}\|\mathbf{x}\|_{n,q}\leq\|\mathbf{x}\|_{h,n,q}\leq\|\mathbf{x}\|_{n,q} \tag{2.13}$$

for any $\mathbf{x}\in H^n$.

In particular, we have

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2\leq\|\mathbf{x}\|_{h,e}\leq\|\mathbf{x}\|_2 \tag{2.14}$$

for any $\mathbf{x}\in H^n$.

Proof. Let $\mathbf{x}\in H^n$ with $\mathbf{x}=(x_1,\dots,x_n)$ and $y\in H$ with $\|y\|\leq 1$, then for $q\geq 1$

$$\left(\sum_{j=1}^{n}\left|\left\langle x_j,y\right\rangle\right|^q\right)^{1/q}\leq\left(\sum_{j=1}^{n}\left(\|y\|\,\|x_i\|\right)^q\right)^{1/q}=\|y\|\left(\sum_{j=1}^{n}\|x_i\|^q\right)^{1/q}=\|y\|\,\|\mathbf{x}\|_{n,q}$$

and by taking the supremum over $\|y\|\leq 1$, we get the second inequality in (2.13).

By the properties of complex numbers, we have

$$\max_{j\in\{1,...,n\}}\left\{\left|\left\langle x_j,y\right\rangle\right|\right\}\leq\left(\sum_{j=1}^{n}\left|\left\langle x_j,y\right\rangle\right|^q\right)^{1/q}$$

and by taking the supremum over $\|y\| \leq 1$, we get

$$\sup_{\|y\|\leq 1}\left(\max_{j\in\{1,\ldots,n\}}\{|\langle x_j,y\rangle|\}\right) \leq \sup_{\|y\|\leq 1}\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|^q\right)^{1/q} \tag{2.15}$$

and since

$$\begin{aligned}\sup_{\|y\|\leq 1}\left(\max_{j\in\{1,\ldots,n\}}\{|\langle x_j,y\rangle|\}\right) &= \max_{j\in\{1,\ldots,n\}}\left\{\sup_{\|y\|\leq 1}|\langle x_j,y\rangle|\right\}\\ &= \max_{j\in\{1,\ldots,n\}}\{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty},\end{aligned}$$

then by (2.15) we get

$$\|\mathbf{x}\|_{n,\infty} \leq \|\mathbf{x}\|_{h,n,q} \text{ for any } \mathbf{x}\in H^n. \tag{2.16}$$

Since

$$\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{1/q} \leq \left(n\|\mathbf{x}\|_{n,\infty}^q\right)^{1/q} = n^{1/q}\|\mathbf{x}\|_{n,\infty}$$

then also

$$\frac{1}{n^{1/q}}\|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{n,\infty} \text{ for any } \mathbf{x}\in H^n. \tag{2.17}$$

By utilising the inequalities (2.16) and (2.17) we obtain the first inequality in (2.13). □

Remark 1. *In the case of inner product spaces the inequality (2.14) has been obtained in a different and more difficult way in* [9] *by employing the rotation-invariant normalised positive Borel measure on the unit sphere.*

Corollary 2. *With the assumptions of Theorem 3 we have for $r \geq q \geq 1$ that*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}}\|\mathbf{x}\|_{h,n,r} \tag{2.18}$$

for any any $\mathbf{x}\in H^n$.

In particular, for $q \geq 2$ we have

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,e} \leq n^{\frac{q-2}{2q}}\|\mathbf{x}\|_{h,n,q} \tag{2.19}$$

and for $1 \leq q \leq 2$ we have

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{2-q}{2q}}\|\mathbf{x}\|_{h,e} \tag{2.20}$$

for any $\mathbf{x}\in H^n$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1, ..., n$ and $r \geq q > 0$ (see for instance [16])

$$\left(\sum_{j=1}^{n}a_j^r\right)^{1/r} \leq \left(\sum_{j=1}^{n}a_j^q\right)^{1/q} \leq n^{\frac{r-q}{rq}}\left(\sum_{j=1}^{n}a_j^r\right)^{1/r}. \tag{2.21}$$

Let $\mathbf{x}\in H^n$ with $\mathbf{x} = (x_1,\ldots,x_n)$ and $y\in H$ with $\|y\| \leq 1$, then for $r \geq q \geq 1$ we have

$$\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|^r\right)^{1/r} \leq \left(\sum_{j=1}^{n}|\langle x_j,y\rangle|^q\right)^{1/q} \leq n^{\frac{r-q}{rq}}\left(\sum_{j=1}^{n}|\langle x_j,y\rangle|^r\right)^{1/r}. \tag{2.22}$$

By taking the supremum over $y \in H$ with $\|y\| \leq 1$ and using Theorem 3, we get (2.18). $\square$

Remark 2. *If we take $q = 1$ in (2.18), then we get for $r \geq 1$ that*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|\mathbf{x}\|_{h,n,r} \tag{2.23}$$

for any $\mathbf{x} \in H^n$.

In particular, for $r = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,1} \leq \sqrt{n} \|\mathbf{x}\|_{h,e} \tag{2.24}$$

for any $\mathbf{x} \in H^n$.

3. Some Reverse Inequalities

Recall the following additive reverse of Cauchy-Buniakowski-Schwarz inequality [7] (see also [8, Theorem 5. 14])

Lemma 1. *Let a, $A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ be two sequences of real numbers with the property that:*

$$ay_j \leq z_j \leq Ay_j \quad \text{for each} \quad j \in \{1, \ldots, n\}. \tag{3.1}$$

Then for any $\mathbf{w} = (w_1, \ldots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^{n} w_j z_j^2 \sum_{j=1}^{n} w_j y_j^2 - \left(\sum_{j=1}^{n} w_j z_j y_j \right)^2 \leq \frac{1}{4} (A - a)^2 \left(\sum_{j=1}^{n} w_j y_j^2 \right)^2. \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [17]) the following counterparts of (CBS)- inequality (see also [8, Theorem 5.20 & 5.21])

Lemma 2. *Assume that $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ are such that there exists a, A, b, B with the property that:*

$$0 \leq a \leq a_j \leq A \quad \text{and} \quad 0 < b \leq b_j \leq B \quad \text{for any } j \in \{1, \ldots, n\}, \tag{3.3}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2, \tag{3.4}$$

and

Lemma 3. *Assume that $\mathbf{a}$, $\mathbf{b}$ are nonnegative sequences and there exists γ, Γ with the property that*

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \text{for any} \quad j \in \{1, \ldots, n\}. \tag{3.5}$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{3.6}$$

We have the following result:

Theorem 4. *Let $(H, \|\cdot\|)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then we have*

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^2 , \tag{3.7}$$

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \|\mathbf{x}\|_{h,n,1} \|\mathbf{x}\|_{n,\infty} \tag{3.8}$$

and

$$0 \leq \|\mathbf{x}\|_{h,e} - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|\mathbf{x}\|_{n,\infty} . \tag{3.9}$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and put $R = \max_{j \in \{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $y \in H$ with $\|y\| \leq 1$ then $|\langle x_j, y\rangle| \leq \|y\| \|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

If we write the inequality (3.2) for $z_j = |\langle x_j, y\rangle|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^{n} |\langle x_j, y\rangle|^2 - \left(\sum_{j=1}^{n} |\langle x_j, y\rangle| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\sum_{j=1}^{n} |\langle x_j, y\rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |\langle x_j, y\rangle| \right)^2 + \frac{1}{4} n R^2 \tag{3.10}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.10) over $y \in H$ with $\|y\| \leq 1$ we get (3.7).

If we write the inequality (3.4) for $a_j = |\langle x_j, y\rangle|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^{n} |\langle x_j, y\rangle|^2 - \left(\sum_{j=1}^{n} |\langle x_j, y\rangle| \right)^2 \leq nR \sum_{j=1}^{n} |\langle x_j, y\rangle| ,$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\sum_{j=1}^{n} |\langle x_j, y\rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |\langle x_j, y\rangle| \right)^2 + R \sum_{j=1}^{n} |\langle x_j, y\rangle| , \tag{3.11}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.11) over $y \in H$ with $\|y\| \leq 1$ we get (3.8).

Finally, if we write the inequality (3.6) for $a_j = |\langle x_j, y\rangle|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$, then we have

$$0 \leq \left(n \sum_{j=1}^{n} |\langle x_j, y\rangle|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} |\langle x_j, y\rangle| \leq \frac{1}{4} nR,$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\left(\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right| + \frac{1}{4}\sqrt{n}R, \tag{3.12}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.12) over $y \in H$ with $\|y\| \leq 1$ we get (3.9). □

Further, we recall the *Čebyšev's inequality* for *synchronous* n-tuples of vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, namely if $(a_j - a_k)(b_j - b_k) \geq 0$ for any j, $k \in \{1, ..., n\}$, then

$$\frac{1}{n}\sum_{j=1}^{n} a_j b_j \geq \frac{1}{n}\sum_{j=1}^{n} a_j \frac{1}{n}\sum_{j=1}^{n} b_j. \tag{3.13}$$

In 1950, Biernacki et al. [2] obtained the following discrete version of Grüss' inequality:

Lemma 4. *Assume that* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *are such that there exists real numbers* a, A, b, B *with the property that:*

$$a \leq a_j \leq A \quad \text{and} \quad b \leq b_j \leq B \quad \text{for any } j \in \{1, \ldots, n\}. \tag{3.14}$$

Then

$$\begin{aligned} &\left|\frac{1}{n}\sum_{j=1}^{n} a_j b_j - \frac{1}{n}\sum_{j=1}^{n} a_j \frac{1}{n}\sum_{j=1}^{n} b_j\right| \\ &\leq \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)(A-a)(B-b) \\ &= \frac{1}{n^2}\left\lceil\frac{n^2}{4}\right\rceil(A-a)(B-a) \leq \frac{1}{4}(A-a)(B-b), \end{aligned} \tag{3.15}$$

where $\lceil x \rceil$ *gives the largest integer less than or equal to* x.

The following result also holds:

Theorem 5. *Let* $(H, \|\cdot\|)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$. *Then for* q, $r \geq 1$ *we have*

$$\begin{aligned} \|\mathbf{x}\|_{h,n,q+r}^{q+r} &\leq \frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{q+r} \\ &\leq \frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{q+r}. \end{aligned} \tag{3.16}$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and put $R = \max_{j\in\{1,\ldots,n\}}\{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $y \in H$ with $\|y\| \leq 1$ then $|\langle x_j, y\rangle| \leq \|y\|\,\|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

If we take into the inequality (3.15) $a_j = |\langle x_j, y\rangle|^q$, $b_j = |\langle x_j, y\rangle|^r$, $a = 0$, $A = R^q$, $b = 0$ and $B = R^r$, then we get

$$\left|\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q+r} - \frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{q}\frac{1}{n}\sum_{j=1}^{n}|\langle x_j,y\rangle|^{r}\right| \leq \frac{1}{n^2}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r}. \tag{3.17}$$

On the other hand, since the sequences $\{a_j\}_{j=1,\dots,n}$, $\{b_j\}_{j=1,\dots,n}$ are synchronous, then by (3.13) we have

$$0 \leq \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q+r} - \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q} \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{r}.$$

Using (3.17) we then get

$$\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q+r} \leq \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q} \sum_{j=1}^{n} |\langle x_j, y\rangle|^{r} + \frac{1}{n}\left\lceil \frac{n^2}{4} \right\rceil R^{q+r} \tag{3.18}$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.18), we get

$$\begin{aligned}
&\sup_{\|y\|\leq 1} \left\{ \sum_{j=1}^{n} |\langle x_j, y\rangle|^{q+r} \right\} \\
&\leq \frac{1}{n} \sup_{\|y\|\leq 1} \left\{ \sum_{j=1}^{n} |\langle x_j, y\rangle|^{q} \sum_{j=1}^{n} |\langle x_j, y\rangle|^{r} \right\} + \frac{1}{n}\left\lceil \frac{n^2}{4} \right\rceil R^{q+r} \\
&\leq \frac{1}{n} \sup_{\|y\|\leq 1} \left\{ \sum_{j=1}^{n} |\langle x_j, y\rangle|^{q} \right\} \sup_{\|y\|\leq 1} \left\{ \sum_{j=1}^{n} |\langle x_j, y\rangle|^{r} \right\} + \frac{1}{n}\left\lceil \frac{n^2}{4} \right\rceil R^{q+r},
\end{aligned}$$

which proves the first inequality in (3.16).

The second part of (3.16) is obvious. □

Corollary 3. *With the assumptions of Theorem 5 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \frac{1}{n}\left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n,\infty}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{2r}. \tag{3.19}$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^{2} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^{2} + \frac{1}{n}\left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n,\infty}^{2} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^{2} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{2}. \tag{3.20}$$

The first inequality in (3.20) is better than the second inequality in (3.7).

For an n-tuple of complex numbers $\mathbf{a} = (a_1, \dots, a_n)$ with $n \geq 2$ consider the $(n-1)$-tuple built by the aid of forward differences $\Delta\mathbf{a} = (\Delta a_1, ..., \Delta a_{n-1})$ where $\Delta a_k := a_{k+1} - a_k$ where $k \in \{1, ..., n-1\}$. Similarly, if $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ is an n-tuple of vectors we also can consider in a similar way the $(n-1)$-tuple $\Delta\mathbf{x} = (\Delta x_1, ..., \Delta x_{n-1})$.

We obtained the following Grüss' type inequalities in terms of forward differences:

Lemma 5. *Assume that* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *are* n*-tuples of complex numbers. Then*

$$\left| \frac{1}{n} \sum_{j=1}^{n} a_j b_j - \frac{1}{n} \sum_{j=1}^{n} a_j \frac{1}{n} \sum_{j=1}^{n} b_j \right| \tag{3.21}$$
$$\leq \begin{cases} \frac{1}{12} \left(n^2 - 1\right) \|\Delta \mathbf{a}\|_{n-1,\infty} \|\Delta \mathbf{b}\|_{n-1,\infty}, & [12], \\ \frac{1}{6} \frac{n^2-1}{n} \|\Delta \mathbf{a}\|_{n-1,\alpha} \|\Delta \mathbf{b}\|_{n-1,\beta} \ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, & [5], \\ \frac{1}{2} \left(1 - \frac{1}{n}\right) \|\Delta \mathbf{a}\|_{n-1,1} \|\Delta \mathbf{b}\|_{n-1,1}, & [6]. \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ *and* $\frac{1}{2}$ *are best possible in (3.21).*

The following result also holds:

Theorem 6. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$. *Then for* $q,\ r \geq 1$ *we have*

$$\|\mathbf{x}\|_{h,n,q+r}^{q+r} \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^{q} \|\mathbf{x}\|_{h,n,r}^{r} \tag{3.22}$$
$$+ \begin{cases} \frac{1}{12} qr \left(n^2 - 1\right) n \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta \mathbf{x}\|_{n-1,\infty}^{2}, \\ \frac{1}{6} \left(n^2 - 1\right) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta \mathbf{x}\|_{h,n-1,\alpha} \|\Delta \mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} (n-1) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta \mathbf{x}\|_{h,n-1,1}^{2}. \end{cases}$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$ with $\|y\| \leq 1$. If we take into the inequality (3.21) $a_j = |\langle x_j, y \rangle|^q$, $b_j = |\langle x_j, y \rangle|^r$, then we get

$$\left| \frac{1}{n} \sum_{j=1}^{n} |\langle x_j, y \rangle|^{q+r} - \frac{1}{n} \sum_{j=1}^{n} |\langle x_j, y \rangle|^{q} \frac{1}{n} \sum_{j=1}^{n} |\langle x_j, y \rangle|^{r} \right| \tag{3.23}$$
$$\leq \begin{cases} \frac{1}{12} \left(n^2 - 1\right) \max_{j=1,\ldots,n-1} \left|\Delta \left|\langle x_j, y \rangle\right|^q\right| \max_{j=1,\ldots,n-1} \left|\Delta \left|\langle x_j, y \rangle\right|^r\right|, \\ \frac{1}{6} \frac{n^2-1}{n} \left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y \rangle\right|^q\right|^{\alpha} \right)^{1/\alpha} \left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y \rangle\right|^r\right|^{\beta} \right)^{1/\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y \rangle\right|^q\right| \sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y \rangle\right|^r\right|. \end{cases}$$

We use the following elementary inequality for powers $p \geq 1$

$$|a^p - b^p| \leq p R^{p-1} |a - b|$$

where $a,\ b \in [0, R]$.

Put $R = \max_{j \in \{1,\ldots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. Then for any $y \in H$ with $\|y\| \leq 1$ we have $|\langle x_j, y \rangle| \leq \|y\| \|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

Therefore

$$|\Delta \left|\langle x_j, y \rangle\right|^q| = \left| |\langle x_{j+1}, y \rangle|^q - |\langle x_j, y \rangle|^q \right| \leq q R^{q-1} \left| |\langle x_{j+1}, y \rangle| - |\langle x_j, y \rangle| \right| \tag{3.24}$$
$$\leq q R^{q-1} |\langle x_{j+1}, y \rangle - \langle x_j, y \rangle| = q R^{q-1} |\langle \Delta x_j, y \rangle|$$

for any $j = 1, ..., n-1$, where $\Delta x_j = x_{j+1} - x_j$ is the forward difference.

On the other hand, since the sequences $\{a_j\}_{j=1,...,n}$, $\{b_j\}_{j=1,...,n}$ are synchronous, then we have

$$0 \leq \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q+r} - \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q} \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{r} \tag{3.25}$$

and by the first inequality in (3.23) we get

$$\begin{aligned} &\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q+r} \\ &\leq \frac{1}{n}\sum_{j=1}^{n} |\langle x_j, y\rangle|^{q} \sum_{j=1}^{n} |\langle x_j, y\rangle|^{r} \\ &+ \frac{1}{12}\left(n^2-1\right) nqR^{q-1} \max_{j=1,...,n-1} |\langle \Delta x_j, y\rangle| \, rR^{r-1} \max_{j=1,...,n-1} |\langle \Delta x_j, y\rangle| \\ &= \frac{1}{n}\sum_{j=1}^{n} |\langle \Delta x_j, y\rangle|^{q} \sum_{j=1}^{n} |\langle \Delta x_j, y\rangle|^{r} \\ &+ \frac{1}{12}\left(n^2-1\right) nqrR^{q+r-2} \left(\max_{j=1,...,n-1} |\langle \Delta x_j, y\rangle|\right)^2 \end{aligned} \tag{3.26}$$

for any $y \in H$ with $\|y\| \leq 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.26) we get the first branch in the inequality (3.22).

We also have, by (3.24), that

$$\begin{aligned} \left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^{q}\right|^{\alpha}\right)^{1/\alpha} &\leq \left[\left(qR^{q-1}\right)^{\alpha} \sum_{j=1}^{n-1} |\langle \Delta x_j, y\rangle|^{\alpha}\right]^{1/\alpha} \\ &= qR^{q-1}\left(\sum_{j=1}^{n-1} |\langle \Delta x_j, y\rangle|^{\alpha}\right)^{1/\alpha} \end{aligned}$$

and, similarly,

$$\left(\sum_{j=1}^{n-1} \left|\Delta \left|\langle x_j, y\rangle\right|^{r}\right|^{\beta}\right)^{1/\beta} \leq rR^{r-1}\left(\sum_{j=1}^{n-1} |\langle \Delta x_j, y\rangle|^{\beta}\right)^{1/\beta}$$

where α, $\beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

By the second inequality in (3.23) and by (3.25) we have

$$
\begin{aligned}
(3.27)\qquad & \sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q+r} \\
& \leq \frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{r} \\
& + \frac{1}{6}\left(n^2-1\right)\left(\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^{q}\right|^{\alpha}\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^{r}\right|^{\beta}\right)^{1/\beta} \\
& \leq \frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{r} \\
& + \frac{1}{6}\left(n^2-1\right)qrR^{q+r-2}\left(\sum_{j=1}^{n-1}\left|\langle \Delta x_j,y\rangle\right|^{\alpha}\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}\left|\langle \Delta x_j,y\rangle\right|^{\beta}\right)^{1/\beta}
\end{aligned}
$$

for any $y\in H$ with $\|y\|\leq 1$, where $\alpha,\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Taking the supremum over $y\in H$ with $\|y\|\leq 1$ in (3.27) we get the second branch in the inequality (3.22).

We also have, by (3.24), that

$$
\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^{q}\right|\leq qR^{q-1}\sum_{j=1}^{n-1}\left|\langle \Delta x_j,y\rangle\right|
$$

and

$$
\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^{r}\right|\leq rR^{r-1}\sum_{j=1}^{n-1}\left|\langle \Delta x_j,y\rangle\right|.
$$

By the third inequality in (3.23) and by (3.25) we have

$$
\begin{aligned}
(3.28)\qquad \sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q+r} & \leq \frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{r} \\
& + \frac{1}{2}\left(n-1\right)\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^{q}\right|\sum_{j=1}^{n-1}\left|\Delta\left|\langle x_j,y\rangle\right|^{r}\right| \\
& \leq \frac{1}{n}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{q}\sum_{j=1}^{n}\left|\langle x_j,y\rangle\right|^{r} \\
& + \frac{1}{2}\left(n-1\right)qrR^{q+r-2}\sum_{j=1}^{n-1}\left|\langle \Delta x_j,y\rangle\right|\sum_{j=1}^{n-1}\left|\langle \Delta x_j,y\rangle\right|
\end{aligned}
$$

for any $y\in H$ with $\|y\|\leq 1$.

Taking the supremum over $y\in H$ with $\|y\|\leq 1$ in (3.28) we get the third branch in the inequality (3.22). □

Corollary 4. *With the assumptions of Theorem 6 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \begin{cases} \frac{1}{12}r^2\left(n^2-1\right)n\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \frac{1}{6}r^2\left(n^2-1\right)\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \frac{1}{2}r^2(n-1)\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{3.29}$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^2 \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^2 + \begin{cases} \frac{1}{12}\left(n^2-1\right)n\|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \frac{1}{6}\left(n^2-1\right)\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \frac{1}{2}(n-1)\|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{3.30}$$

4. Further Inequalities

In 1992, J. Pečarić [15] proved the following general inequality in inner product spaces:

Lemma 6. *Let $y,\ x_1,\ldots,\ x_n \in H$ and $c_1,\ldots,\ c_n \in \mathbb{K}$. Then*

$$\left|\sum_{i=1}^n c_i\langle x_i, y\rangle\right|^2 \leq \|y\|^2\sum_{i=1}^n|c_i|^2\left(\sum_{j=1}^n|\langle x_i,x_j\rangle|\right) \leq \|y\|^2\sum_{i=1}^n|c_i|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^n|\langle x_i,x_j\rangle|\right\}. \tag{4.1}$$

He showed that the Bombieri inequality [4] may be obtained from (4.1) for the choice $c_i = \overline{\langle x_i, y\rangle}$ (using the second inequality), the Selberg inequality [14, p. 394] may be obtained from the first part of (4.1) for the choice

$$c_i = \frac{\overline{\langle x_i, y\rangle}}{\sum_{j=1}^n|\langle x_i,x_j\rangle|}, \quad i \in \{1,\ldots,n\};$$

while the Heilbronn inequality [13] may be obtained from the first part of (4.1) if one chooses $c_i = \frac{\overline{\langle x_i,y\rangle}}{|\langle x_i,y\rangle|}$, for any $i \in \{1,\ldots,n\}$.

Theorem 7. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1,\ldots,x_n)$. Then for $q \geq 1$ we have*

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q}\left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^n|\langle x_i,x_j\rangle|\right\}\right]^{\frac{1}{2q}}. \tag{4.2}$$

In particular, for $q = 1$ we get

$$\|\mathbf{x}\|_{h,n,1} \leq \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right\}\right]^{\frac{1}{2}} \tag{4.3}$$

while for $q = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right\}\right]^{\frac{1}{2}}. \tag{4.4}$$

Proof. If we take in (4.1) $c_i = \overline{\langle x_i, y\rangle}\,|\langle x_i, y\rangle|^{q-2}$, $i \in \{1, ..., n\}$, then we get

$$\left(\sum_{i=1}^{n}|\langle x_i, y\rangle|^q\right)^2 \leq \|y\|^2\sum_{i=1}^{n}|\langle x_i, y\rangle|^{2(q-1)}\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right\} \tag{4.5}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$.

By taking the square root in (4.5) we get

$$\begin{aligned}&\sum_{i=1}^{n}|\langle x_i, y\rangle|^q \\ &\leq \|y\|\left(\sum_{i=1}^{n}|\langle x_i, y\rangle|^{2(q-1)}\right)^{1/2}\left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right\}\right]^{1/2},\end{aligned} \tag{4.6}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$.

If we take the supremum in (4.6) for $\|y\| \leq 1$, then we get

$$\|\mathbf{x}\|_{h,n,q}^q \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{q-1}\left[\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|\langle x_i, x_j\rangle|\right\}\right]^{1/2}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$.

This proves (4.2). □

In 1941, R. P. Boas [3] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality

Lemma 7. *If y, $x_1, \ldots, x_n$ are elements of an inner product space $(H; \langle\cdot,\cdot\rangle)$, then the following inequality:*

$$\sum_{i=1}^{n}|\langle x_i, y\rangle|^2 \leq \|y\|^2\left[\max_{1\leq i\leq n}\|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right], \tag{4.7}$$

holds.

A generalization of the Boas-Bellman result was given in Mitrinović-Pečarić-Fink [14, p. 392] where they proved the following:

Lemma 8. *If y, $x_1, \dots, x_n$ are as in Lemma 7 and $c_1, \dots, c_n \in \mathbb{K}$, then one has the inequality:*

$$\left|\sum_{i=1}^{n} c_i \langle x_i, y\rangle\right|^2 \leq \|y\|^2 \sum_{i=1}^{n} |c_i|^2 \left[\max_{1\leq i\leq n} \|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{4.8}$$

They also noted that if in (4.8) one chooses $c_i = \overline{(x, y_i)}$, then this inequality becomes (4.7).

Using a similar argument to the one in Theorem 7 and Lemma 8 we have:

Theorem 8. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q \geq 1$ we have*

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q} \left[\max_{1\leq i\leq n} \|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2q}}. \tag{4.9}$$

In particular, for $q = 1$ we get

$$\|\mathbf{x}\|_{h,n,1} \leq \left[\max_{1\leq i\leq n} \|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{4.10}$$

while for $q = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \left[\max_{1\leq i\leq n} \|x_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{4.11}$$

In [10] we obtained the following result that provides some companions to the Boas-Bellman inequality above:

Lemma 9. *Let y, $x_1, \dots, x_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then*

$$\begin{aligned}
&\left|\sum_{i=1}^{n} c_i \langle x_i, y\rangle\right|^2 \\
&\leq \|y\|^2 \times \begin{cases}
\max\limits_{1\leq i\leq n}\left\{|c_i|^2\right\}\left\{\sum_{i=1}^{n}\|x_i\|^2 + \sum_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|\right\}; \\
\left(\sum_{i=1}^{n}|c_i|^{2p}\right)^{\frac{1}{p}}\left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\
\left. + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|^r\right)^{\frac{1}{r}}\right\}, \\
\text{where } \ p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\
\sum_{i=1}^{n}|c_i|^2\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|\right\}.
\end{cases}
\end{aligned} \tag{4.12}$$

We have:

Theorem 9. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\mathbf{x}\in H^n$ with $\mathbf{x}=(x_1,\dots,x_n)$. Then for $q\geq 1$ we have*

$$\|\mathbf{x}\|_{h,n,q}\leq \begin{cases} \max\limits_{1\leq i\leq n}\left\{\|x_i\|\right\}^{1-1/q}\left\{\sum_{i=1}^{n}\|x_i\|^2+\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2q}}; \\ \|\mathbf{x}\|_{h,n,2p(q-1)}^{1-1/q}\left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\ \left.+(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{\frac{1}{2q}}, \\ \text{where } \ p>1,\frac{1}{p}+\frac{1}{r}=1; \\ \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2+(n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2q}}. \end{cases} \tag{4.13}$$

In particular, we have for $q=1$ that

$$\|\mathbf{x}\|_{h,n,1}\leq \begin{cases} \left\{\sum_{i=1}^{n}\|x_i\|^2+\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}; \\ \left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\ \left.+(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{\frac{1}{2}}, \\ \text{where } \ p>1,\frac{1}{p}+\frac{1}{r}=1; \\ \left\{\max\limits_{1\leq i\leq n}\|x_i\|^2+(n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}, \end{cases} \tag{4.14}$$

while for $q=2$ that

$$\|\mathbf{x}\|_{h,e}^2\leq \begin{cases} \max\limits_{1\leq i\leq n}\left\{\|x_i\|\right\}\left\{\sum_{i=1}^{n}\|x_i\|^2+\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}; \\ \|\mathbf{x}\|_{h,n,2p}\left\{\left(\sum_{i=1}^{n}\|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\ \left.+(n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{\frac{1}{2}}, \\ \text{where } \ p>1,\frac{1}{p}+\frac{1}{r}=1; \\ \|\mathbf{x}\|_{h,e}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2+(n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i,x_j\rangle|\right\}^{\frac{1}{2}}. \end{cases} \tag{4.15}$$

Proof. If we take in (4.12) $c_i = \overline{\langle x_i, y\rangle}\,|\langle x_i, y\rangle|^{q-2}$, $i \in \{1, ..., n\}$, then we get

$$
\begin{aligned}
(4.16)\quad & \left(\sum_{i=1}^{n} |\langle x_i, y\rangle|^q\right)^2 \\
& \leq \|y\|^2 \times \begin{cases}
\max\limits_{1\leq i\leq n}\left\{|\langle x_i, y\rangle|^{2(q-1)}\right\}\left\{\sum_{i=1}^n \|x_i\|^2 + \sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|\right\}; \\
\left(\sum_{i=1}^n |\langle x_i, y\rangle|^{2p(q-1)}\right)^{\frac{1}{p}}\left\{\left(\sum_{i=1}^n \|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\
\left. + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^r\right)^{\frac{1}{r}}\right\}, \\
\text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\
\sum_{i=1}^n |\langle x_i, y\rangle|^{2(q-1)}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|\right\},
\end{cases}
\end{aligned}
$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$.

If we take the square root in this inequality, then we get

$$
\begin{aligned}
(4.17)\quad & \sum_{i=1}^{n} |\langle x_i, y\rangle|^q \\
& \leq \|y\| \times \begin{cases}
\max\limits_{1\leq i\leq n}\left\{|\langle x_i, y\rangle|^{q-1}\right\}\left\{\sum_{i=1}^n \|x_i\|^2 + \sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|\right\}^{1/2}; \\
\left(\sum_{i=1}^n |\langle x_i, y\rangle|^{2p(q-1)}\right)^{\frac{1}{2p}}\left\{\left(\sum_{i=1}^n \|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\
\left. + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{1/2}, \\
\text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\
\left(\sum_{i=1}^n |\langle x_i, y\rangle|^{2(q-1)}\right)^{1/2}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|\right\}^{1/2},
\end{cases}
\end{aligned}
$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $y \in H$.

If we take the supremum over $y \in H$, $\|y\| \leq 1$, then we get

$$
\begin{aligned}
\|\mathbf{x}\|_{h,n,q}^q \leq & \begin{cases}
\max\limits_{1\leq i\leq n}\left\{\|x_i\|^{q-1}\right\}\left\{\sum_{i=1}^n \|x_i\|^2 + \sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|\right\}^{1/2}; \\
\|\mathbf{x}\|_{h,n,2p(q-1)}^{q-1}\left\{\left(\sum_{i=1}^n \|x_i\|^{2r}\right)^{\frac{1}{r}}\right. \\
\left. + (n-1)^{\frac{1}{p}}\left(\sum_{1\leq i\neq j\leq n} |\langle x_i, x_j\rangle|^r\right)^{\frac{1}{r}}\right\}^{1/2}, \\
\text{where } \ p>1, \frac{1}{p}+\frac{1}{r}=1; \\
\|\mathbf{x}\|_{h,n,2(q-1)}^{q-1}\left\{\max\limits_{1\leq i\leq n}\|x_i\|^2 + (n-1)\max\limits_{1\leq i\neq j\leq n}|\langle x_i, x_j\rangle|\right\}^{1/2},
\end{cases}
\end{aligned}
$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$, which produces the desired result (4.13). $\square$

The interested reader may obtain other similar results by utilising the following inequalities obtained in [11]

Lemma 10. *Let y, $x_1, \dots, x_n$ be vectors of an inner product space $(H; \langle \cdot, \cdot \rangle)$ and $c_1, \dots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left| \sum_{i=1}^{n} c_i \langle x_i, y \rangle \right|^2 \leq \|y\|^2 \times \begin{cases} D \\ E \\ F \end{cases}, \tag{4.18}$$

where

$$D := \begin{cases} \max\limits_{1 \leq k \leq n} |c_k|^2 \sum\limits_{i,j=1}^{n} |\langle x_i, x_j \rangle| ; \\ \max\limits_{1 \leq k \leq n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^r \right)^{\frac{1}{r}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |\langle x_i, x_j \rangle| \right)^s \right]^{\frac{1}{s}}, \quad r > 1, \ \frac{1}{r} + \frac{1}{s} = 1; \\ \max\limits_{1 \leq k \leq n} |c_k| \sum\limits_{k=1}^{n} |c_k| \max\limits_{1 \leq i \leq n} \left(\sum\limits_{j=1}^{n} |\langle x_i, x_j \rangle| \right); \end{cases}$$

$$E := \begin{cases} \left(\sum\limits_{k=1}^{n} |c_k|^p \right)^{\frac{1}{p}} \max\limits_{1 \leq i \leq n} |c_i| \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |\langle x_i, x_j \rangle| \right)^q \right)^{\frac{1}{q}}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |c_i|^t \right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |\langle x_i, x_j \rangle|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ & t > 1, \ \frac{1}{t} + \frac{1}{u} = 1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p \right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |c_i| \max\limits_{1 \leq i \leq n} \left\{ \left(\sum\limits_{j=1}^{n} |\langle x_i, x_j \rangle|^q \right)^{\frac{1}{q}} \right\}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

and

$$F := \begin{cases} \sum_{k=1}^{n} |c_k| \ \max\limits_{1 \leq i \leq n} |c_i| \ \sum_{i=1}^{n} \left[\max\limits_{1 \leq j \leq n} |\langle x_i, x_j \rangle| \right]; \\ \sum_{k=1}^{n} |c_k| \ \left(\sum_{i=1}^{n} |c_i|^m \right)^{\frac{1}{m}} \left(\sum_{i=1}^{n} \left[\max\limits_{1 \leq j \leq n} |\langle x_i, x_j \rangle| \right]^l \right)^{\frac{1}{l}}, \\ \qquad\qquad m > 1, \ \frac{1}{m} + \frac{1}{l} = 1; \\ \left(\sum_{k=1}^{n} |c_k| \right)^2 \max\limits_{i, 1 \leq j \leq n} |\langle x_i, x_j \rangle| . \end{cases}$$

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