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OSTROWSKI TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF BOUNDED VARIATION, HÖLDER AND LIPSCHITZIAN FUNCTIONS

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Abstract. In this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation, of Hölder continuous functions and of Lipschitzian functions. Applications for mid-point inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

1. Introduction

In 1999 we obtained the following inequality of Ostrowski type for functions of bounded variation:

Theorem 1 (Dragomir, 1999 [8])**.** *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$\left|\int_a^b f(t)\,dt - (b-a) f(x)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{1.1}$$

where $\bigvee_a^b (f)$ denotes the total variation of f. The constant $\frac{1}{2}$ is the best possible one.

The following *mid-point inequality* is the best possible one we can get from (1.1).

Corollary 1 (Dragomir, 2000 [9])**.** *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation. Then we have the inequality:*

$$\left|\int_a^b f(t)\,dx - (b-a) f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{1.2}$$

The constant $\frac{1}{2}$ is best possible.

In 2013 we obtained the following improvement of (1.1):

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Theorem 2 (Dragomir, 2013 [11])**.** *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \tag{1.3}$$

$$\leq \int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt \leq (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f)$$

$$\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \\ \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a), \end{cases}$$

for any $x \in [a,b]$.

This provides the following mid-point inequality

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \tag{1.4}$$

$$\leq \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f)\right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f)\right) dt \leq \frac{1}{2}(b-a)\bigvee_a^b (f),$$

which improves (1.2).

For other Ostrowski type inequalities for Lebesgue integral, see [10], [6] and the recent survey [12].

In order to extend these results for the fractional integrals we need the following preparation.

Let $f : [a,b] \to \mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha > 0$ by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

for $a < x \leq b$ and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt$$

for $a \leq x < b$, where Γ is the *Gamma function.* For $\alpha = 0$, they are defined as

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x) \text{ for } x \in (a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [13]-[23] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation, of Hölder continuous functions and of Lipschitzian functions. Applications for mid-point inequalities are provided as well.

2. Some Identities

We start with the following simple identities:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$.*

(i) *For any* $x \in (a,b)$ *we have*

$$J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) = \frac{f(x)}{\Gamma(\alpha+1)} [(x-a)^{\alpha} + (b-x)^{\alpha}] - \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} [f(x) - f(t)] dt + \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} [f(t) - f(x)] dt. \tag{2.1}$$

(ii) *For any* $x \in (a,b)$ *we have*

$$J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) = \frac{f(x)}{\Gamma(\alpha+1)} [(x-a)^{\alpha} + (b-x)^{\alpha}] - \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} [f(x) - f(t)] dt + \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} [f(t) - f(x)] dt. \tag{2.2}$$

(iii) *For any* $x \in [a,b]$ *we have*

$$\frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} = \frac{1}{\Gamma(\alpha+1)} f(x) (b-a)^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} [f(t) - f(x)] dt. \tag{2.3}$$

Proof. (i) We have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} [f(t) - f(x)] dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt - f(x) \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} dt \\ &= J_{a+}^{\alpha} f(x) - f(x) \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\alpha}}{\alpha} = J_{a+}^{\alpha} f(x) - f(x) \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} \end{aligned} \tag{2.4}$$

for $a < x \le b$ and, similarly

$$\frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} [f(t) - f(x)] dt = J_{b-}^{\alpha} f(x) - f(x) \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} \tag{2.5}$$

for $a \le x < b$.

By adding these equalities for $x \in (a,b)$ we get the representation (2.1).

(ii) We have

$$J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t) dt$$

for $a \le x < b$ and

$$J_{x-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t) dt$$

for $a < x \le b$.

Then

$$\frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} [f(t) - f(x)] dt = J_{x+}^{\alpha} f(b) - f(x) \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}$$

for $a \leq x < b$ and

$$\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}[f(t)-f(x)]\,dt = J_{x-}^{\alpha}f(a) - f(x)\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}$$

for $a < x \leq b$.

By adding these equalities for $x \in (a,b)$ we get the representation (2.2).

(iii) We have for any $x \in [a,b]$ that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}[f(t)-f(x)]\,dt \\
&= \frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}f(t)\,dt \\
&- f(x)\frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt \\
&= \frac{1}{2\Gamma(\alpha)}\int_a^b \left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]f(t)\,dt - \frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha} \\
&= \frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2} - \frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha},
\end{aligned}$$

which proves the equality (2.3). □

Corollary 2. *With the assumption of Lemma 1, we have the particular mid-point equalities*

$$\begin{aligned}
&J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right) \\
&- \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\left[f\left(\frac{a+b}{2}\right)-f(t)\right]dt \\
&+ \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^{\alpha-1}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&J_{\frac{a+b}{2}+}^{\alpha}f(b) + J_{\frac{a+b}{2}-}^{\alpha}f(a) \\
&= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right) \\
&- \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\left[f\left(\frac{a+b}{2}\right)-f(t)\right]dt \\
&+ \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b(b-t)^{\alpha-1}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2} &= \frac{1}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\ &+\frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt. \end{aligned} \tag{2.8}$$

3. Inequalities for Functions of Bounded Variation

The following Ostrowski type inequalities for functions of bounded variation hold:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a,b]$.*

(i) *For any $x\in(a,b)$ we have*

$$\begin{aligned}&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x (x-t)^{\alpha-1}\bigvee_t^x(f)\,dt+\int_x^b (t-x)^{\alpha-1}\bigvee_x^t(f)\,dt\right] \\ &\le \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\right] \\ &\le \frac{1}{\Gamma(\alpha+1)} \\ &\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \left((x-a)^{\alpha p}+(b-x)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\left((x-a)^{\alpha}+(b-x)^{\alpha}\right),\end{cases}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left[\int_x^b (b-t)^{\alpha-1}\bigvee_x^t(f)\,dt+\int_a^x (t-a)^{\alpha-1}\bigvee_t^x(f)\,dt\right] \\ &\le \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\right]\end{aligned} \tag{3.2}$$

$$\leq \frac{1}{\Gamma(\alpha+1)}$$

$$\times \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \\ ((x-a)^{\alpha p}+(b-x)^{\alpha p})^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]((x-a)^{\alpha}+(b-x)^{\alpha}). \end{cases}$$

(ii) *For any* $x\in[a,b]$ *we have*

$$\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}\right| \tag{3.3}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^x \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_t^x(f)\,dt$$

$$+\frac{1}{\Gamma(\alpha)}\int_x^b \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_x^t(f)\,dt$$

$$\leq \frac{1}{2\Gamma(\alpha+1)}\left[(b-a)^{\alpha}+(x-a)^{\alpha}-(b-x)^{\alpha}\right]\bigvee_a^x(f)$$

$$+\frac{1}{2\Gamma(\alpha+1)}\left[(b-a)^{\alpha}+(b-x)^{\alpha}-(x-a)^{\alpha}\right]\bigvee_x^b(f)$$

$$\leq \frac{1}{2\Gamma(\alpha+1)}\begin{cases} \left[(b-a)^{\alpha}+\left|(x-a)^{\alpha}-(b-x)^{\alpha}\right|\right]\bigvee_a^b(f), \\ \\ (b-a)^{\alpha}\left[\bigvee_a^b(f)+\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]. \end{cases}$$

Proof. (i) By (2.1) we have

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{3.4}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left|f(t)-f(x)\right|dt$$

$$+\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\left|f(t)-f(x)\right|dt := B(x)$$

for any $x\in(a,b)$.

Since f is of bounded variation, then we have for $x\in(a,b)$ that

$$|f(t)-f(x)|\leq \bigvee_t^x(f)\leq \bigvee_a^x(f) \text{ for } a\leq t<x$$

and

$$|f(t)-f(x)|\leq \bigvee_x^t(f)\leq \bigvee_x^b(f) \text{ for } x<t\leq b.$$

Then

$$
\begin{aligned}
B(x) &\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x (x-t)^{\alpha-1}\bigvee_t^x(f)\,dt+\int_x^b (t-x)^{\alpha-1}\bigvee_x^t(f)\,dt\right]\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\bigvee_a^x(f)\int_a^x (x-t)^{\alpha-1}dt+\bigvee_x^b(f)\int_x^b (t-x)^{\alpha-1}dt\right]\\
&= \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\right],
\end{aligned}
$$

for any $x\in(a,b)$, which proves the first two inequalities in (3.1).

Now, by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq 0$

$$
mc+nd\leq\begin{cases}\max\{m,n\}(c+d);\\ \\ (m^p+n^p)^{1/p}(c^q+d^q)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}
$$

we have

$$
\begin{aligned}
&(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\\
&\leq\begin{cases}\max\{(x-a)^{\alpha},(b-x)^{\alpha}\}\bigvee_a^b(f);\\ ((x-a)^{\alpha p}+(b-x)^{\alpha p})^{1/p}\left((\bigvee_a^x(f))^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \max\left\{\bigvee_a^x(f),\bigvee_x^b(f)\right\}((x-a)^{\alpha}+(b-x)^{\alpha});\end{cases}\\
&=\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_a^b(f);\\ ((x-a)^{\alpha p}+(b-x)^{\alpha p})^{1/p}\left((\bigvee_a^x(f))^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]((x-a)^{\alpha}+(b-x)^{\alpha}),\end{cases}
\end{aligned}
$$

which proves the last part of (3.1).

(ii) By (2.2) we have

$$
\begin{aligned}
&\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x (t-a)^{\alpha-1}|f(t)-f(x)|\,dt+\int_x^b (b-t)^{\alpha-1}|f(t)-f(x)|\,dt\right]\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_x^b (b-t)^{\alpha-1}\bigvee_x^t(f)\,dt+\int_a^x (t-a)^{\alpha-1}\bigvee_t^x(f)\,dt\right]\\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\right]
\end{aligned}
$$

which proves the first two inequalities in (3.2).

The last part has been proved before.

(iii) From (2.3) we have

$$\begin{aligned}
&\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}\right| \\
&=\frac{1}{\Gamma(\alpha)}\left|\int_a^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}[f(t)-f(x)]\,dt\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left|\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}[f(t)-f(x)]\,dt\right| \\
&+\frac{1}{\Gamma(\alpha)}\left|\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}[f(t)-f(x)]\,dt\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}|f(t)-f(x)|\,dt \\
&+\frac{1}{\Gamma(\alpha)}\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}|f(t)-f(x)|\,dt \\
&\leq\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_t^x(f)\,dt \\
&+\frac{1}{\Gamma(\alpha)}\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_x^t(f)\,dt,
\end{aligned}$$

which proves the first inequality in (3.3).

We have

$$\begin{aligned}
&\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_t^x(f)\,dt \\
&\leq\bigvee_a^x(f)\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt \\
&=\frac{1}{2\alpha}\left[(b-a)^{\alpha}+(x-a)^{\alpha}-(b-x)^{\alpha}\right]\bigvee_a^x(f)
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_x^t(f)\,dt \\
&\leq\bigvee_x^b(f)\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt \\
&=\frac{1}{2\alpha}\left[(b-a)^{\alpha}+(b-x)^{\alpha}-(x-a)^{\alpha}\right]\bigvee_x^b(f),
\end{aligned}$$

which prove the second part of (3.3).

Since

$$\begin{aligned}
&\max\left\{(b-a)^{\alpha}+(x-a)^{\alpha}-(b-x)^{\alpha},(b-a)^{\alpha}+(b-x)^{\alpha}-(x-a)^{\alpha}\right\} \\
&=(b-a)^{\alpha}+\left|(x-a)^{\alpha}-(b-x)^{\alpha}\right|,
\end{aligned}$$

then we get the first branch of the last inequality.

The second branch of the last inequality in (3.3) is obvious. □

Remark 1. *For $\alpha = 1$ we have*

$$J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) = J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) = \int_a^b f(t)\,dt$$

for all $x \in (a,b)$. Then for either (3.1) or (3.2) we recapture (1.3).

Corollary 3. *Let $f : [a,b] \to \mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a,b]$. Then we have the following mid-point inequalities*

$$\left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \tag{3.5}$$
$$\leq \frac{1}{\Gamma(\alpha)} \times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha-1} \bigvee_t^{\frac{a+b}{2}} (f)\,dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right)^{\alpha-1} \bigvee_{\frac{a+b}{2}}^t (f)\,dt \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_a^b (f),$$

$$\left| J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \tag{3.6}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_{\frac{a+b}{2}}^b (b-t)^{\alpha-1} \bigvee_{\frac{a+b}{2}}^t (f)\,dt + \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} \bigvee_t^{\frac{a+b}{2}} (f)\,dt \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_a^b (f)$$

and

$$\left| \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) (b-a)^{\alpha} \right| \tag{3.7}$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \bigvee_t^{\frac{a+b}{2}} (f)\,dt$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \bigvee_{\frac{a+b}{2}}^t (f)\,dt$$
$$\leq \frac{1}{2\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_a^b (f)$$

4. Inequalities for Hölder's Continuous Functions

We say that the function $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$ if

$$|f(t) - f(s)| \leq H |t-s|^r \tag{4.1}$$

for any $t,\ s \in [a,b]$. If $r=1$ and $H=L$ we call the function *L-Lipschitzian* on $[a,b]$.

Theorem 4. *Assume that $f:[a,b]\to\mathbb{C}$ is r-H-Hölder continuous on $[a,b]$ with $r\in(0,1]$ and $H>0$. Then we have*

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \leq \frac{H}{(r+\alpha)\Gamma(\alpha)}\left[(x-a)^{r+\alpha}+(b-x)^{r+\alpha}\right] \tag{4.2}$$

and

$$\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \leq \frac{H}{\Gamma(\alpha)}B(\alpha,r+1)\left[(x-a)^{\alpha+r}+(b-x)^{\alpha+r}\right] \tag{4.3}$$

for any $x\in(a,b)$, where

$$B(\alpha,\beta)=\int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds,\ \ \alpha,\ \ \beta>0$$

is the Beta function.

If $f:[a,b]\to\mathbb{C}$ is L-Lipschitzian on $[a,b]$, then we have

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \leq \frac{L}{(1+\alpha)\Gamma(\alpha)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right] \tag{4.4}$$

and

$$\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \leq \frac{L}{\Gamma(\alpha+2)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right] \tag{4.5}$$

for any $x\in(a,b)$.

Proof. From the representation (2.1) we have

$$\begin{aligned}
&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left|f(t)-f(x)\right|dt \\
&+\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\left|f(t)-f(x)\right|dt \\
&\leq \frac{1}{\Gamma(\alpha)}H\left[\int_a^x (x-t)^{\alpha-1}(x-t)^r dt+\int_x^b (t-x)^{\alpha-1}(t-x)^r dt\right] \\
&=\frac{1}{\Gamma(\alpha)}H\left[\frac{(x-a)^{r+\alpha}+(b-x)^{r+\alpha}}{r+\alpha}\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves (4.2).

From (2.2) we have for $x \in (a, b)$ that

$$\begin{aligned} (4.6) \qquad & \left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} |f(t) - f(x)| \, dt \\ & + \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} |f(t) - f(x)| \, dt \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_x^b (b-t)^{\alpha-1} (t-x)^r \, dt + \int_a^x (t-a)^{\alpha-1} (x-t)^r \, dt \right]. \end{aligned}$$

We observe that, by using the change of variable $t = (1-s) a + sb$ we have for α, $\beta > 0$ that

$$\begin{aligned} \int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \, dt &= (b-a)^{\alpha+\beta-1} \int_0^1 s^{\alpha-1} (1-s)^{\beta-1} \, ds \\ &= (b-a)^{\alpha+\beta-1} B(\alpha, \beta), \end{aligned}$$

where $B(\cdot, \cdot)$ is Beta function.

Therefore

$$\int_x^b (b-t)^{\alpha-1} (t-x)^r \, dt = (b-x)^{\alpha+r} B(\alpha, r+1)$$

and

$$\int_a^x (t-a)^{\alpha-1} (x-t)^r \, dt = (x-a)^{\alpha+r} B(\alpha, r+1),$$

for $\alpha > 0$ and $r \in (0, 1]$, which, by (4.6) produces the desired result (4.3). □

Corollary 4. *With the assumptions of Theorem 4, we have*

$$\begin{aligned} (4.7) \qquad & \left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{H}{(r+\alpha) 2^{r+\alpha-1} \Gamma(\alpha)} (b-a)^{r+\alpha} \end{aligned}$$

and

$$\begin{aligned} (4.8) \qquad & \left| J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{H}{2^{r+\alpha-1}\Gamma(\alpha)} B(\alpha, r+1) (b-a)^{r+\alpha}. \end{aligned}$$

In particular,

$$\begin{aligned} (4.9) \qquad & \left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{L}{(1+\alpha) 2^{\alpha} \Gamma(\alpha)} (b-a)^{\alpha+1} \end{aligned}$$

and

$$\left|J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right| \le \frac{L}{2^{\alpha}\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{4.10}$$

Remark 2. *For $\alpha=1$ we recapture from (4.2) and (4.4) the corresponding Ostrowski type inequalities for Hölder continuous functions and Lipschitzian functions, respectively, see for instance* [7] *and the survey paper* [12].

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