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OSTROWSKI TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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Abstract. In this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

1. Introduction

Let (a,b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [16, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)\, dt}{[g(x) - g(t)]^{1-\alpha}},\ a < x \leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)\, dt}{[g(t) - g(x)]^{1-\alpha}},\ a \leq x < b. \tag{1.2}$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\alpha}},\ a < x \leq b \tag{1.3}$$

and

$$J_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\alpha}},\ a \leq x < b, \tag{1.4}$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [16, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \leq a < x \leq b \tag{1.5}$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \leq a < x < b. \tag{1.6}$$

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One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.7}$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.8}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \frac{\exp(\beta t) f(t)\,dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}},\ a < x \le b \tag{1.9}$$

and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b \frac{\exp(\beta t) f(t)\,dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}},\ a \le x < b. \tag{1.10}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [14]-[25] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski type inequalities for the generalized Riemann-Liouville fractional integrals of functions of bounded variation, of Hölder continuous functions and of Lipschitzian functions. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

2. Some Identities of Interest

We have:

Lemma 1. *Let $f : [a, b] \to \mathbb{C}$ be Lebesgue integrable on $[a, b]$ and g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b).*

(i) For any $x \in (a, b)$ we have the representation

$$\begin{aligned} I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t) - f(x)]\,dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t) - f(x)]\,dt}{[g(t) - g(x)]^{1-\alpha}} \right] \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) &= \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \\ &+ \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t) - f(x)]\,dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t) - f(x)]\,dt}{[g(b) - g(t)]^{1-\alpha}} \right]. \end{aligned} \tag{2.2}$$

(ii) We have

$$\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}+\frac{1}{2\Gamma(\alpha)}\left[\int_a^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right]. \tag{2.3}$$

Proof. (i) We observe that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}}\\&=I^{\alpha}_{a+,g}f(x)-f(x)\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}\\&=I^{\alpha}_{a+,g}f(x)-\frac{\left[g(x)-g(a)\right]^{\alpha}}{\alpha\Gamma(\alpha)}f(x)=I^{\alpha}_{a+,g}f(x)-\frac{\left[g(x)-g(a)\right]^{\alpha}}{\Gamma(\alpha+1)}f(x)\end{aligned} \tag{2.4}$$

for $a<x\le b$ and, similarly,

$$\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}=I^{\alpha}_{b-,g}f(x)-\frac{\left[g(b)-g(x)\right]^{\alpha}}{\Gamma(\alpha+1)}f(x) \tag{2.5}$$

for $a\le x<b$.

If $x\in(a,b)$, then by adding the equalities (2.4) and (2.5) we get the representation (2.1).

By the definition of fractional integrals we have

$$I^{\alpha}_{x+,g}f(b):=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)\,f(t)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}},\ a\le x<b$$

and

$$I^{\alpha}_{x-,g}f(a):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)\,f(t)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}},\ a<x\le b.$$

Then

$$\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}=I^{\alpha}_{x+,g}f(b)-\frac{\left[g(b)-g(x)\right]^{\alpha}}{\Gamma(\alpha+1)}f(x) \tag{2.6}$$

for $a\le x<b$ and

$$\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}=I^{\alpha}_{x-,g}f(a)-\frac{\left[g(x)-g(a)\right]^{\alpha}}{\Gamma(\alpha+1)}f(x) \tag{2.7}$$

for $a<x\le b$.

If $x\in(a,b)$, then by adding the equalities (2.6) and (2.7) we get the representation (2.1).

If we take $x=b$ in (2.4) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}=I^{\alpha}_{a+,g}f(b)-\frac{\left[g(b)-g(a)\right]^{\alpha}}{\Gamma(\alpha+1)}f(b) \tag{2.8}$$

while from $x=a$ in (2.5) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}=I^{\alpha}_{b-,g}f(a)-\frac{\left[g(b)-g(a)\right]^{\alpha}}{\Gamma(\alpha+1)}f(a). \tag{2.9}$$

If we add (2.8) with (2.9) and divide by 2 we get (2.3). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\, b\in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Corollary 1. *Let $f:[a,b]\to\mathbb{C}$ be Lebesgue integrable on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the equalities*

$$\begin{aligned}(2.10)\quad & I^{\alpha}_{a+,g}f(M_g(a,b))+I^{\alpha}_{b-,g}f(M_g(a,b))\\ &=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}f(M_g(a,b))\\ &+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)[f(t)-f(M_g(a,b))]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^{b}\frac{g'(t)[f(t)-f(M_g(a,b))]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right],\end{aligned}$$

and

$$\begin{aligned}(2.11)\quad & I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)\\ &=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}f(M_g(a,b))\\ &+\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)[f(t)-f(M_g(a,b))]\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^{b}\frac{g'(t)[f(t)-f(M_g(a,b))]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right].\end{aligned}$$

Remark 1. *If we take $x=\frac{a+b}{2}$ in Lemma 1 we also have the mid-point equalities of interest*

$$\begin{aligned}(2.12)\quad & I^{\alpha}_{a+,g}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b-,g}f\left(\frac{a+b}{2}\right)\\ &=\frac{1}{\Gamma(\alpha+1)}\left(\left[g\left(\frac{a+b}{2}\right)-g(a)\right]^{\alpha}+\left[g(b)-g\left(\frac{a+b}{2}\right)\right]^{\alpha}\right)f\left(\frac{a+b}{2}\right)\\ &+\frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\frac{g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt}{\left[g\left(\frac{a+b}{2}\right)-g(t)\right]^{1-\alpha}}+\int_{\frac{a+b}{2}}^{b}\frac{g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt}{\left[g(t)-g\left(\frac{a+b}{2}\right)\right]^{1-\alpha}}\right],\end{aligned}$$

and

$$I^{\alpha}_{\frac{a+b}{2}-,g}f(a)+I^{\alpha}_{\frac{a+b}{2}+,g}f(b) \tag{2.13}$$
$$=\frac{1}{\Gamma(\alpha+1)}\left(\left[g\left(\frac{a+b}{2}\right)-g(a)\right]^{\alpha}+\left[g(b)-g\left(\frac{a+b}{2}\right)\right]^{\alpha}\right)f\left(\frac{a+b}{2}\right)$$
$$+\frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\frac{g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{\frac{a+b}{2}}^{b}\frac{g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt}{[g(b)-g(t)]^{1-\alpha}}\right].$$

3. Inequalities for Functions of Bounded Variation

We have the following result:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

(i) For any $x\in(a,b)$ we have the inequalities

$$\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right| \tag{3.1}$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\bigvee_t^x(f)\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)\bigvee_x^t(f)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$
$$\le\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_a^x(f)+[g(b)-g(x)]^{\alpha}\bigvee_x^b(f)\right]$$
$$\le\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right],\end{cases}$$

and

$$\left|I^{\alpha}_{x-,g}f(a)+I^{\alpha}_{x+,g}f(b)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right| \tag{3.2}$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\bigvee_t^x(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b\frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$
$$\le\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_a^x(f)+[g(b)-g(x)]^{\alpha}\bigvee_x^b(f)\right]$$
$$\le\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right].\end{cases}$$

(ii) We have

$$\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \tag{3.3}$$
$$\leq\frac{1}{2\Gamma(\alpha)}\left[\int_a^b\frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b\frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha}\bigvee_a^b(f)\,dt.$$

Proof. (i) By the representation (2.1) and the properties of modulus, we have

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left(\left[g(x)-g(a)\right]^{\alpha}+\left[g(b)-g(x)\right]^{\alpha}\right)f(x)\right| \tag{3.4}$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\left|\int_a^x\frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}}\right|+\int_x^b\left|\frac{g'(t)\left[f(t)-f(x)\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right|\right]$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\left|f(t)-f(x)\right|dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b\frac{g'(t)\left|f(t)-f(x)\right|dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right]=:B(x).$$

Since f is of bounded variation, then we have for $x\in(a,b)$ that

$$|f(t)-f(x)|\leq\bigvee_t^x(f)\leq\bigvee_a^x(f)\ \text{ for } a\leq t<x$$

and

$$|f(t)-f(x)|\leq\bigvee_x^t(f)\leq\bigvee_x^b(f)\ \text{ for } x<t\leq b.$$

Then

$$\begin{aligned}B(x)&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\bigvee_t^x(f)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b\frac{g'(t)\bigvee_x^t(f)\,dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right]\\&\leq\frac{1}{\Gamma(\alpha)}\left[\bigvee_a^x(f)\int_a^x\frac{g'(t)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\bigvee_x^b(f)\int_x^b\frac{g'(t)\,dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right]\\&=\frac{1}{\Gamma(\alpha+1)}\left[\left[g(x)-g(a)\right]^{\alpha}\bigvee_a^x(f)+\left[g(b)-g(x)\right]^{\alpha}\bigvee_x^b(f)\right],\end{aligned}$$

for any $x\in(a,b)$, which proves the first two inequalities in (3.1).

Now, by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq 0$

$$mc+nd\leq\begin{cases}\max\{m,n\}(c+d);\\ \\ (m^p+n^p)^{1/p}(c^q+d^q)^{1/q}\ \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}$$

we have

$$\left[g(x)-g(a)\right]^{\alpha}\bigvee_a^x(f)+\left[g(b)-g(x)\right]^{\alpha}\bigvee_x^b(f)$$

$$\leq \begin{cases} \max\left\{(g(x)-g(a))^{\alpha},(g(b)-g(x))^{\alpha}\right\}\bigvee_{a}^{b}(f); \\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \max\left\{\bigvee_{a}^{x}(f),\bigvee_{x}^{b}(f)\right\}\left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right); \end{cases}$$

$$= \begin{cases} \left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f); \\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right], \end{cases}$$

which proves the last part of (3.1).

By (2.2) we have

$$\begin{aligned} &\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}\frac{g'(t)\left|f(t)-f(x)\right|dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{x}^{b}\frac{g'(t)\left|f(t)-f(x)\right|dt}{[g(b)-g(t)]^{1-\alpha}}\right] \\ &\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}\frac{g'(t)\bigvee_{t}^{x}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{x}^{b}\frac{g'(t)\bigvee_{x}^{t}(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right] \\ &\leq \frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_{a}^{x}(f)+[g(b)-g(x)]^{\alpha}\bigvee_{x}^{b}(f)\right] \end{aligned}$$

which proves the first two inequalities in (3.2).

The last part has been proved before.

(ii) From (2.3) we also have for $x\in[a,b]$ that

$$\begin{aligned} &\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\ &\leq \frac{1}{2\Gamma(\alpha)}\left[\int_{a}^{b}\frac{g'(t)\left|f(t)-f(b)\right|dt}{[g(b)-g(t)]^{1-\alpha}}+\int_{a}^{b}\frac{g'(t)\left|f(t)-f(a)\right|dt}{[g(t)-g(a)]^{1-\alpha}}\right] \\ &=: C. \end{aligned} \tag{3.5}$$

Moreover, we observe that

$$\begin{aligned} \int_{a}^{b}\frac{g'(t)\left|f(t)-f(b)\right|dt}{[g(b)-g(t)]^{1-\alpha}} &\leq \int_{a}^{b}\frac{g'(t)\bigvee_{t}^{b}(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\leq \bigvee_{a}^{b}(f)\,dt\int_{a}^{b}\frac{g'(t)\,dt}{[g(b)-g(t)]^{1-\alpha}} \\ &= \frac{(g(b)-g(a))^{\alpha}}{\alpha}\bigvee_{a}^{b}(f)\,dt \end{aligned}$$

and

$$\int_{a}^{b}\frac{g'(t)\left|f(t)-f(a)\right|dt}{[g(t)-g(a)]^{1-\alpha}}\leq \int_{a}^{b}\frac{g'(t)\bigvee_{a}^{t}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}\leq \frac{(g(b)-g(a))^{\alpha}}{\alpha}\bigvee_{a}^{b}(f)\,dt$$

By adding these inequalities we get

$$C\leq\int_a^b\frac{g'(t)\bigvee_t^b(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b\frac{g'(t)\bigvee_a^t(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}\leq\frac{2(g(b)-g(a))^{\alpha}}{\alpha}\bigvee_a^b(f)\,dt$$

which proves the first and the second inequality in (3.3). □

The following particular case is of interest:

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right|\tag{3.6}$$

$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)\bigvee_{M_g(a,b)}^t(f)\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]$$

$$\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f);$$

and

$$\left|I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right|\tag{3.7}$$

$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$

$$\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f).$$

Remark 2. *If we take in Theorem 1 $x=\frac{a+b}{2}$, then we obtain similar mid-point inequalities, however the details are not presented here.*

4. Inequalities for Hölder's Continuous Functions

We say that the function $f:[a,b]\to\mathbb{C}$ is *r-H-Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $H>0$ if

$$|f(t)-f(s)|\leq H|t-s|^{r}\tag{4.1}$$

for any $t,\ s\in[a,b]$. If $r=1$ and $H=L$ we call the function *L-Lipschitzian* on $[a,b]$.

Theorem 2. *Assume that $f:[a,b]\to\mathbb{C}$ is r-H-Hölder continuous on $[a,b]$ with $r\in(0,1]$ and $H>0$, and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

(i) For any $x\in(a,b)$ we have the inequalities

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right|\tag{4.2}$$

$$\leq\frac{H}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)(x-t)^{r}\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)(t-x)^{r}\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$

$$\leq\frac{H}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}(x-a)^{r}+[g(b)-g(x)]^{\alpha}(b-x)^{r}\right]$$

and

$$\tag{4.3} \begin{aligned} &\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \right| \\ &\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(x-t)^r \, dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)(t-x)^r \, dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha+1)} \left[[g(x) - g(a)]^{\alpha} (x-a)^r + [g(b) - g(x)]^{\alpha} (b-x)^r \right]. \end{aligned}$$

(ii) We have the inequalities

$$\tag{4.4} \begin{aligned} &\left| \frac{I^{\alpha}_{b-,g} f(a) + I^{\alpha}_{a+,g} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^{\alpha} \frac{f(b) + f(a)}{2} \right| \\ &\leq \frac{H}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t)(b-t)^r \, dt}{[g(b) - g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)(t-a)^r \, dt}{[g(t) - g(a)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha+1)} (b-a)^r (g(b) - g(a))^{\alpha}. \end{aligned}$$

Proof. (i) By the representation (2.1) and the properties of modulus, we have

$$\begin{aligned} &\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\left| \int_a^x \frac{g'(t)[f(t) - f(x)] \, dt}{[g(x) - g(t)]^{1-\alpha}} \right| + \int_x^b \left| \frac{g'(t)[f(t) - f(x)] \, dt}{[g(t) - g(x)]^{1-\alpha}} \right| \right] \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)|f(t) - f(x)| \, dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)|f(t) - f(x)| \, dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(x-t)^r \, dt}{[g(x) - g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)(t-x)^r \, dt}{[g(t) - g(x)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha+1)} \left[[g(x) - g(a)]^{\alpha} (x-a)^r + [g(b) - g(x)]^{\alpha} (b-x)^r \right], \end{aligned}$$

which proves (4.2).

By (2.2) we have that

$$\begin{aligned} &\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{1}{\Gamma(\alpha+1)} \left([g(x) - g(a)]^{\alpha} + [g(b) - g(x)]^{\alpha} \right) f(x) \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)|f(t) - f(x)| \, dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)|f(t) - f(x)| \, dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(x-t)^r \, dt}{[g(t) - g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)(t-x)^r \, dt}{[g(b) - g(t)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha+1)} \left[(x-a)^r [g(x) - g(a)]^{\alpha} + (b-x)^r [g(b) - g(x)]^{\alpha} \right], \end{aligned}$$

which proves (4.3).

(ii) From (2.3) we also have for $x \in [a,b]$ that

$$\begin{aligned}
&\left|\frac{I^{\alpha}_{b-,g}f(a)+I^{\alpha}_{a+,g}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
&\le \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)|f(t)-f(b)|\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b \frac{g'(t)|f(t)-f(a)|\,dt}{[g(t)-g(a)]^{1-\alpha}}\right] \\
&\le \frac{H}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)(b-t)^r\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b \frac{g'(t)(t-a)^r\,dt}{[g(t)-g(a)]^{1-\alpha}}\right] \\
&\le \frac{H}{2\Gamma(\alpha)}\left[(b-a)^r\int_a^b \frac{g'(t)\,dt}{[g(b)-g(t)]^{1-\alpha}}+(b-a)^r\int_a^b \frac{g'(t)(t-a)^r\,dt}{[g(t)-g(a)]^{1-\alpha}}\right] \\
&= \frac{H}{\Gamma(\alpha+1)}(b-a)^r(g(b)-g(a))^{\alpha}.
\end{aligned}$$

□

We have:

Corollary 3. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
(4.5)\quad &\left|I^{\alpha}_{a+,g}f(M_g(a,b))+I^{\alpha}_{b-,g}f(M_g(a,b))-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right| \\
&\le \frac{H}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)(M_g(a,b)-t)^r\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b \frac{g'(t)(t-M_g(a,b))^r\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right] \\
&\le \frac{H}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left[(M_g(a,b)-a)^r+(b-M_g(a,b))^r\right],
\end{aligned}$$

and

$$\begin{aligned}
(4.6)\quad &\left|I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right| \\
&\le \frac{H}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)(M_g(a,b)-t)^r\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b \frac{g'(t)(t-M_g(a,b))^r\,dt}{[g(b)-g(t)]^{1-\alpha}}\right] \\
&\le \frac{H}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left[(M_g(a,b)-a)^r+(b-M_g(a,b))^r\right].
\end{aligned}$$

Remark 3. *If we take in Theorem 2 $x=\frac{a+b}{2}$, then we obtain similar mid-point inequalities, however the details are not presented here.*

The above results provide various inequalities for particular fractional integrals by taking for g various examples of strictly increasing functions on (a,b), having continuous derivatives g' on (a,b). The case $g(t)=t$ was considered in details in the recent paper [13].

5. Applications for Hadamard Fractional Integrals

If we take $g(t)=\ln t$ and $0\le a<x\le b$, then by Theorem 1 for Hadamard fractional integrals H^{α}_{a+} and H^{α}_{b-} we have for $f:[a,b]\to\mathbb{C}$ a function of bounded

variation on $[a,b]$ that

$$\begin{aligned}
(5.1)\quad & \left|H_{a+}^{\alpha}f(x)+H_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left(\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}\right)f(x)\right| \\
& \leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{\left(\ln\left(\frac{x}{t}\right)\right)^{\alpha-1}\bigvee_t^x(f)\,dt}{t}+\int_x^b \frac{\left(\ln\left(\frac{t}{x}\right)\right)^{\alpha-1}\bigvee_x^t(f)\,dt}{t}\right] \\
& \leq \frac{1}{\Gamma(\alpha+1)}\left[\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}\bigvee_a^x(f)+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\bigvee_x^b(f)\right] \\
& \leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\bigvee_a^b(f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right],\end{cases}
\end{aligned}$$

$$\begin{aligned}
(5.2)\quad & \left|H_{x-}^{\alpha}f(a)+H_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)f(x)\right| \\
& \leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{\left(\ln\left(\frac{t}{a}\right)\right)^{\alpha-1}\bigvee_t^x(f)\,dt}{t}+\int_x^b \frac{\left(\ln\left(\frac{b}{t}\right)\right)^{\alpha-1}\bigvee_x^t(f)\,dt}{t}\right] \\
& \leq \frac{1}{\Gamma(\alpha+1)}\left[\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}\bigvee_a^x(f)+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\bigvee_x^b(f)\right] \\
& \leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\bigvee_a^b(f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha}+\left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right],\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(5.3)\quad & \left|\frac{H_{b-}^{\alpha}f(a)+H_{a+}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
& \leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}\bigvee_t^b(f)\,dt}{t}+\int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\bigvee_a^t(f)\,dt}{t}\right] \\
& \leq \frac{1}{\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha}\bigvee_a^b(f)\,dt.
\end{aligned}$$

The following particular case of interest for $x = G(a,b) = \sqrt{ab}$ also holds:

$$
\begin{aligned}
(5.4)\quad & \left| H^{\alpha}_{a+}f(G(a,b)) + H^{\alpha}_{b-}f(G(a,b)) - \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(G(a,b)) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{G(a,b)} \left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} \frac{1}{t} \bigvee_{t}^{G(a,b)} (f)\, dt \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{G(a,b)}^{b} \left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} \frac{1}{t} \bigvee_{G(a,b)}^{t} (f)\, dt \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left(\ln\left(\frac{b}{a}\right)\right)^{\alpha} \bigvee_{a}^{b}(f);
\end{aligned}
$$

and

$$
\begin{aligned}
(5.5)\quad & \left| H^{\alpha}_{G(a,b)-}f(a) + H^{\alpha}_{G(a,b)+}f(b) - \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(G(a,b)) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{G(a,b)} \left(\ln\left(\frac{t}{a}\right)\right)^{\alpha-1} \frac{1}{t} \bigvee_{t}^{G(a,b)} (f)\, dt \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{G(a,b)}^{b} \left(\ln\left(\frac{b}{t}\right)\right)^{\alpha-1} \frac{1}{t} \bigvee_{G(a,b)}^{t} (f)\, dt \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left(\ln\left(\frac{b}{a}\right)\right)^{\alpha} \bigvee_{a}^{b}(f).
\end{aligned}
$$

Similar inequalities may be stated in the case of r-K-Hölder continuous functions $f : [a,b] \to \mathbb{C}$ with $r \in (0,1]$ and $K > 0$. For instance, if we write the inequalities (4.5)-(4.6) for the function $g(t) = \ln t$ and $0 \leq a < b$, then we get

$$
\begin{aligned}
(5.6)\quad & \left| H^{\alpha}_{a+}f(G(a,b)) + H^{\alpha}_{b-}f(G(a,b)) - \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(G(a,b)) \right| \\
&\leq \frac{H}{\Gamma(\alpha)} \int_{a}^{G(a,b)} (G(a,b)-t)^{r} \left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} \frac{dt}{t} \\
&\quad + \frac{H}{\Gamma(\alpha)} \int_{G(a,b)}^{b} (G(a,b)-t)^{r} \left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} \frac{dt}{t} \\
&\leq \frac{H}{2^{\alpha}\Gamma(\alpha+1)} \left[(G(a,b)-a)^{r} + (b-G(a,b))^{r}\right] \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha},
\end{aligned}
$$

and

$$\begin{aligned}
(5.7) \quad & \left| H^{\alpha}_{G(a,b)-} f(a) + H^{\alpha}_{G(a,b)+} f(b) - \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(G(a,b)) \right| \\
& \leq \frac{H}{\Gamma(\alpha)} \int_a^{G(a,b)} (G(a,b)-t)^r \left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} \frac{dt}{t} \\
& + \frac{H}{\Gamma(\alpha)} \int_{G(a,b)}^b (t-G(a,b))^r \left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} \frac{dt}{t} \\
& \leq \frac{H}{2^{\alpha}\Gamma(\alpha+1)} \left[(G(a,b)-a)^r + (b-G(a,b))^r\right] \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}.
\end{aligned}$$

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