FURTHER OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR THE GENERALIZED Riemann-Liouville Fractional Integrals of Functions with Bounded Variation

SILVESTRU SEVER DRAGOMIR

Abstract. In this paper we establish some Ostrowski and generalized trapezoid type inequalities for the Generalized Riemann-Liouville fractional integrals of functions of bounded variation. Applications for midpoint and trapezoid inequalities are provided as well. Some examples concerning the Hadamard and Harmonic fractional integrals are also given.

1. Introduction

Let \( f : [a, b] \rightarrow \mathbb{C} \) be a complex valued Lebesgue integrable function on the real interval \([a, b]\). The Riemann-Liouville fractional integrals are defined for \( \alpha > 0 \) by

\[
 J_{a+}^\alpha f (x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) \, dt
\]

for \( a < x \leq b \) and

\[
 J_{b-}^\alpha f (x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) \, dt
\]

for \( a \leq x < b \), where \( \Gamma \) is the Gamma function. For \( \alpha = 0 \), they are defined as

\[
 J_{a+}^0 f (x) = J_{b-}^0 f (x) = f(x) \quad \text{for } x \in (a, b).
\]

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[6], [17]-[28] and the references therein.

The following Ostrowski type inequalities for functions of bounded variation generalize the corresponding results for the Riemann integral obtained in [9], [11], [10] and have been established recently by the author in [15]:

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{C} \) be a complex valued function of bounded variation on the real interval \([a, b]\). For any \( x \in (a, b) \) we have

\[
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\( J_a^\alpha f(x) + J_b^\alpha f(x) - \frac{f(x)}{\Gamma(\alpha + 1)} [(x - a)^\alpha + (b - x)^\alpha] \)

\[
\leq \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (x - t)^{\alpha-1} x f(t) \ dt + \int_x^b (t - x)^{\alpha-1} t x f(t) \ dt \right] \\
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ (x - a)^\alpha a x f(x) + (b - x)^\alpha b x f(x) \right] \\
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ \frac{1}{2} a x f(x) + \frac{1}{2} b x f(x) \right] ((x - a)^\alpha + (b - x)^\alpha),
\]

and

\( J_{x}^\alpha f(b) + J_{x}^\alpha f(a) - \frac{f(x)}{\Gamma(\alpha + 1)} [(x - a)^\alpha + (b - x)^\alpha] \)

\[
\leq \frac{1}{\Gamma(\alpha)} \left[ \int_x^b (b - t)^{\alpha-1} x f(t) \ dt + \int_x^a (t - x)^{\alpha-1} t x f(t) \ dt \right] \\
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ (x - a)^\alpha a x f(x) + (b - x)^\alpha b x f(x) \right] \\
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ \frac{1}{2} a x f(x) + \frac{1}{2} b x f(x) \right] ((x - a)^\alpha + (b - x)^\alpha),
\]

The following mid-point inequalities that can be derived from Theorem 1 are of interest as well:
In order to extend this result for other fractional integrals, we need the following definitions.

Let \((a, b)\) with \(-\infty < a < b < \infty\) be a finite or infinite interval of the real line \(\mathbb{R}\) and \(\alpha\) a complex number with \(\text{Re}(\alpha) > 0\). Also let \(g\) be a strictly increasing function on \((a, b)\), having a continuous derivative \(g'\) on \((a, b)\): Following [19, p. 100], we introduce the generalized left- and right-sided Riemann-Liouville fractional integrals of a function \(f\) with respect to another function \(g\) on \([a, b]\) by

\[
I^\alpha_{a+g} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{[g(x) - g(t)]^{1-\alpha}}, \quad a < x \leq b
\]

and

\[
I^\alpha_{b-g} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{[g(t) - g(x)]^{1-\alpha}}, \quad a < x < b.
\]

For \(g(t) = t\) we have the classical Riemann-Liouville fractional integrals defined above while for the logarithmic function \(g(t) = \ln t\) we have the Hadamard fractional integrals [19, p. 111]

\[
H^\alpha_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[ \ln \left( \frac{x}{t} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x \leq b
\]

and

\[
H^\alpha_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[ \ln \left( \frac{t}{x} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x < b.
\]
One can consider the function \( g(t) = -t^{-1} \) and define the "Harmonic fractional integrals" by

\[
R_{a+}^\alpha f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t) \, dt}{(x-t)^{\alpha+1}}, \quad 0 \leq a < x \leq b
\]

and

\[
R_{b-}^\alpha f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t) \, dt}{(t-x)^{\alpha+1}}, \quad 0 \leq a < x < b.
\]

Also, for \( g(t) = t^p, \ p > 0 \), we have the \( p \)-Riemann-Liouville fractional integrals

\[
J_{a+}^\alpha f(x) := \frac{p}{\Gamma(\alpha)} \int_a^x \frac{t^{p-1} f(t) \, dt}{(x-t)^{1-\alpha}}, \quad 0 \leq a < x \leq b
\]

and

\[
J_{b-}^\alpha f(x) := \frac{p}{\Gamma(\alpha)} \int_x^b \frac{t^{p-1} f(t) \, dt}{(t-x)^{1-\alpha}}, \quad 0 \leq a \leq x < b.
\]

Motivated by the above results, in this paper we establish some new Ostrowski and generalized trapezoid type inequalities for the Generalized Riemann-Liouville fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples concerning the Hadamard and Harmonic fractional integrals are also given.

2. Some Identities of Interest

We have the following results:

**Lemma 1.** Let \( f : [a, b] \rightarrow \mathbb{C} \) be a function of bounded variation on \([a, b]\). Also let \( g \) be a strictly increasing function on \((a, b)\), having a continuous derivative \( g' \) on \((a, b)\).

(i) For any \( x \in (a, b) \) we have

\[
(2.1) \quad I_{a+}^\alpha f(x) + I_{b-}^\alpha f(x)
= \frac{1}{\Gamma(\alpha+1)} \left[ (g(x) - g(a))^\alpha f(a) + (g(b) - g(x))^\alpha f(b) \right]
+ \frac{1}{\Gamma(\alpha+1)} \left[ \int_a^x (g(x) - g(t))^\alpha df(t) - \int_a^b (g(t) - g(x))^\alpha df(t) \right].
\]

(ii) For any \( x \in (a, b) \) we have

\[
(2.2) \quad I_{a-}^\alpha f(a) + I_{x+}^\alpha f(b)
= \frac{1}{\Gamma(\alpha+1)} \left[ (g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha \right] f(x)
+ \frac{1}{\Gamma(\alpha+1)} \left[ \int_x^b (g(b) - g(t))^\alpha df(t) - \int_a^x (g(t) - g(a))^\alpha df(t) \right].
\]
(iii) We have the trapezoid equality

\[
\frac{I^\alpha_{b-g} f(a) + I^\alpha_{a+g} f(b)}{2} = \frac{1}{\Gamma(\alpha+1)} \left( g(b) - g(a) \right)^\alpha \frac{f(b) + f(a)}{2} + \frac{1}{\Gamma(\alpha+1)} \int_a^b \frac{(g(b) - g(t))^{\alpha} - (g(t) - g(a))^{\alpha}}{2} df(t).
\]

Proof. (i) Since \( f: [a, b] \to \mathbb{C} \) is of bounded variation on \([a, b]\) and \( g \) is continuous on \([a, b]\), then the Riemann-Stieltjes integrals

\[
\int_a^x (g(x) - g(t))^\alpha df(t) \quad \text{and} \quad \int_x^b (g(t) - g(x))^\alpha df(t)
\]

exist and integrating by parts, we have

\[
\frac{1}{\Gamma(\alpha+1)} \int_a^x (g(x) - g(t))^\alpha df(t)
= \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) dt - \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^\alpha f(a)
= I^\alpha_{a+g} f(x) - \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^\alpha f(a)
\]

for \( a < x \leq b \) and

\[
\frac{1}{\Gamma(\alpha+1)} \int_x^b (g(t) - g(x))^\alpha df(t)
= \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^\alpha f(b) - \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t) dt
= \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^\alpha f(b) - I^\alpha_{b-g} f(x)
\]

for \( a \leq x < b \).

From (2.4), we then have

\[
I^\alpha_{a+g} f(x) = \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^\alpha f(a)
+ \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(x) - g(t))^\alpha df(t)
\]

for \( a < x \leq b \) and from (2.5) we have

\[
I^\alpha_{b-g} f(x) = \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^\alpha f(b)
- \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(t) - g(x))^\alpha df(t),
\]

for \( a \leq x < b \), which by addition give (2.1).

(ii) We have

\[
I^\alpha_{x+g} f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(b) - g(t))^{\alpha-1} g'(t) f(t) dt
\]
for \( a \leq x < b \) and
\[
I_{x^{-},g}^{\alpha} f (a) = \frac{1}{\Gamma (\alpha)} \int_{a}^{x} (g (t) - g (a))^{\alpha - 1} g' (t) f (t) \, dt
\]
for \( a < x \leq b \).

Since \( f : [a, b] \to \mathbb{C} \) is of bounded variation on \([a, b] \) and \( g \) is continuous on \([a, b] \), then the Riemann-Stieltjes integrals
\[
\int_{a}^{x} (g (t) - g (a))^{\alpha} df (t) \quad \text{and} \quad \int_{x}^{b} (g (b) - g (t))^{\alpha} df (t)
\]
exist and integrating by parts, we have
\[
(2.6) \quad \frac{1}{\Gamma (\alpha + 1)} \int_{a}^{x} (g (t) - g (a))^{\alpha} df (t) = \frac{1}{\Gamma (\alpha + 1)} \int_{a}^{x} (g (t) - g (a))^{\alpha - 1} g' (t) f (t) \, dt
\]
for \( a < x \leq b \) and
\[
(2.7) \quad \frac{1}{\Gamma (\alpha + 1)} \int_{x}^{b} (g (b) - g (t))^{\alpha} df (t) = \frac{1}{\Gamma (\alpha + 1)} \int_{x}^{b} (g (b) - g (t))^{\alpha - 1} g' (t) f (t) \, dt - \frac{1}{\Gamma (\alpha + 1)} (g (b) - g (x))^{\alpha} f (x)
\]
for \( a \leq x < b \).

From (2.6) we have
\[
(2.8) \quad I_{x^{-},g}^{\alpha} f (a) = \frac{1}{\Gamma (\alpha + 1)} \int_{a}^{x} (g (t) - g (a))^{\alpha - 1} g' (t) f (t) \, dt
\]
for \( a < x \leq b \) and from (2.7)
\[
(2.9) \quad I_{x^{+},g}^{\alpha} f (b) = \frac{1}{\Gamma (\alpha + 1)} (g (b) - g (x))^{\alpha} f (x) + \frac{1}{\Gamma (\alpha + 1)} \int_{x}^{b} (g (b) - g (t))^{\alpha} df (t),
\]
for \( a \leq x < b \), which by addition produce (2.2).

(iii) For \( x = b \) in (2.8) we have
\[
(2.10) \quad I_{b^{-},g}^{\alpha} f (a) = \frac{1}{\Gamma (\alpha + 1)} (g (b) - g (a))^{\alpha} f (b)
\]
\[- \frac{1}{\Gamma (\alpha + 1)} \int_{a}^{b} (g (t) - g (a))^{\alpha} df (t)
\]
while from (2.9) we have for $x = a$ that

$$I_{a+g} f(b) = \frac{1}{\Gamma(\alpha + 1)} (g(b) - g(a))^\alpha f(a)$$

$$+ \frac{1}{\Gamma(\alpha + 1)} \int_a^b (g(b) - g(t))^\alpha df(t).$$

If we add these two equalities and divide by 2, we get (2.3).

**Corollary 1.** With the assumptions of Lemma 1, we have

$$I_{a+g} f\left(\frac{a+b}{2}\right) + I_{b-g} f\left(\frac{a+b}{2}\right)$$

$$= \frac{1}{\Gamma(\alpha + 1)} \left[\left(g\left(\frac{a+b}{2}\right) - g(a)\right)^\alpha f(a) + \left(g(b) - g\left(\frac{a+b}{2}\right)\right)^\alpha f(b)\right]$$

$$+ \frac{1}{\Gamma(\alpha + 1)} \int_{\frac{a+b}{2}}^b (g\left(\frac{a+b}{2}\right) - g(t))^\alpha df(t)$$

$$- \frac{1}{\Gamma(\alpha + 1)} \int_{\frac{a+b}{2}}^a (g(t) - g\left(\frac{a+b}{2}\right))^\alpha df(t).$$

If $g$ is a function which maps an interval $I$ of the real line to the real numbers, and is both continuous and injective then we can define the $g$-mean of two numbers $a, b \in I$ by

$$M_g(a, b) := g^{-1}\left(\frac{g(a) + g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the identity function, then $M_g(a, b) = A(a, b) := \frac{a+b}{2}$, the arithmetic mean. If $I = (0, \infty)$ and $g(t) = \ln t$, then $M_g(a, b) = G(a, b) := \sqrt{ab}$, the geometric mean. If $I = (0, \infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a, b) = H(a, b) := \frac{2ab}{a + b}$, the harmonic mean. If $I = (0, \infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a, b) = M_p(a, b) := \left(\frac{a^p + b^p}{2}\right)^{1/p}$, the power mean with exponent $p$. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a, b) = LME(a, b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the LogMeanExp function.
Corollary 2. With the assumptions of Lemma 1, we have

\[ I_\alpha^a f (M_g (a, b)) + I_\alpha^b f (M_g (a, b)) \]
\[ = \frac{1}{2^{\alpha-1} \Gamma (\alpha + 1)} (g (b) - g (a))^\alpha f (a) + f (b) \]
\[ + \frac{1}{\Gamma (\alpha + 1)} \int_a^b \left( \frac{g (a) + g (b)}{2} - g (t) \right)^\alpha df (t) \]
\[ - \frac{1}{\Gamma (\alpha + 1)} \int_a^b \left( g (t) - \frac{g (a) + g (b)}{2} \right)^\alpha df (t) \]

From a complementary view point we also have:

\[ I_\alpha^a f (a) + I_\alpha^b f (b) \]
\[ = \frac{1}{2^{\alpha-1} \Gamma (\alpha + 1)} (g (b) - g (a))^\alpha f (M_g (a, b)) \]
\[ + \frac{1}{\Gamma (\alpha + 1)} \int_a^b (g (b) - g (t))^\alpha df (t) \]
\[ - \frac{1}{\Gamma (\alpha + 1)} \int_a^b (g (t) - g (a))^\alpha df (t) \]

Lemma 2. With the assumptions of Lemma 1, we have

\[ \frac{1}{2} \Gamma (\alpha + 1) \left[ \frac{I_\alpha^a f (x)}{(g (x) - g (a))^\alpha} + \frac{I_\alpha^b f (x)}{(g (b) - g (x))^\alpha} \right] = \frac{f (a) + f (b)}{2} \]
\[ + \frac{1}{2 (g (x) - g (a))^\alpha} \int_a^x (g (x) - g (t))^\alpha df (t) \]
\[ - \frac{1}{2 (g (b) - g (x))^\alpha} \int_x^b (g (t) - g (x))^\alpha df (t) \]

and

\[ \frac{1}{2} \Gamma (\alpha + 1) \left[ \frac{I_\alpha^b f (a)}{(g (x) - g (a))^\alpha} + \frac{I_\alpha^a f (b)}{(g (b) - g (x))^\alpha} \right] = f (x) \]
\[ + \frac{1}{2 (g (b) - g (x))^\alpha} \int_a^b (g (b) - g (t))^\alpha df (t) \]
\[ - \frac{1}{2 (g (x) - g (a))^\alpha} \int_a^x (g (t) - g (a))^\alpha df (t) \]

for any \( x \in (a, b) \).

Proof. By the above equalities (2.4) and (2.5) we have

\[ \frac{I_\alpha^a f (x)}{(g (x) - g (a))^\alpha} = \frac{1}{\Gamma (\alpha + 1)} \left[ f (a) \right. \]
\[ + \frac{1}{\Gamma (\alpha + 1)} \frac{1}{(g (x) - g (a))^\alpha} \int_a^x (g (x) - g (t))^\alpha df (t) \]
for $a < x \leq b$ and

$$\frac{I_{b-a}^\alpha f(x)}{(g(b) - g(x))^{\alpha}} = \frac{1}{\Gamma(\alpha + 1)} f(b) \quad \frac{1}{\Gamma(\alpha + 1)} \int_a^b (g(t) - g(x))^{\alpha} df(t)$$

and $a \leq x < b$.

If we add these two equalities and multiply by $\frac{1}{2} \Gamma(\alpha + 1)$ we get (2.14).

By the equalities (2.6) and (2.7)

$$\frac{I_{b-a}^\alpha f(a)}{(g(x) - g(a))^{\alpha}} = \frac{1}{\Gamma(\alpha + 1)} f(x) - \frac{1}{\Gamma(\alpha + 1)} \int_a^x (g(t) - g(a))^{\alpha} df(t)$$

for $a < x \leq b$ and

$$\frac{I_{a+b}^\alpha f(b)}{(g(b) - g(x))^{\alpha}} = \frac{1}{\Gamma(\alpha + 1)} f(x) + \frac{1}{\Gamma(\alpha + 1)} \int_a^b (g(b) - g(t))^{\alpha} df(t)$$

for $a \leq x < b$.

If we add these two equalities and multiply by $\frac{1}{2} \Gamma(\alpha + 1)$ we get (2.15). □

**Corollary 3.** With the assumptions of Lemma 1, we have

$$\int_{a+b}^\alpha \frac{I_{a+b}^\alpha f \left( \frac{a+b}{2} \right)}{(g \left( \frac{a+b}{2} \right) - g(a))^{\alpha}} + \frac{I_{b-a}^\alpha f \left( \frac{a+b}{2} \right)}{(g(b) - g \left( \frac{a+b}{2} \right))^{\alpha}} = \frac{f(a) + f(b)}{2}$$

and

$$\int_{a+b}^\alpha \frac{I_{b-a}^\alpha f \left( \frac{a+b}{2} \right)}{(g \left( \frac{a+b}{2} \right) - g(a))^{\alpha}} + \frac{I_{a+b}^\alpha f \left( \frac{a+b}{2} \right)}{(g(b) - g \left( \frac{a+b}{2} \right))^{\alpha}} = f \left( \frac{a+b}{2} \right)$$

for any $x \in (a, b)$.

**Remark 1.** If we take $x = M_g(a, b) = g^{-1} \left( \frac{g(a) + g(b)}{2} \right)$ in Lemma 2, then we get the same equalities that have been stated in Corollary 2.
3. Some General Inequalities

The following lemma is of interest in itself as well [2, p. 177], see also [12] for a
generalization.

Lemma 3. Let \( f, u : [a, b] \to \mathbb{C} \). If \( f \) is continuous on \([a, b]\) and \( u \) is of bounded
variation on \([a, b]\), then the Riemann-Stieltjes integral \( \int_a^b f(t) \, du(t) \) exists and

\[
\left| \int_a^b f(t) \, du(t) \right| \leq \int_a^b |f(t)| \, d \left( \int_a^t (u) \right) \leq \max_{t \in [a, b]} |f(t)| \int_a^b (u),
\]

where \( \int_a^b (u) \) denotes the total variation of \( u \) on \([a, b]\), \( t \in [a, b] \).

We have:

Theorem 2. Let \( f : [a, b] \to \mathbb{C} \) be a function of bounded variation on \([a, b]\). Also
let \( g \) be a strictly increasing function on \([a, b]\), having a continuous derivative \( g' \)
on \([a, b]\). Then we have

\[
\left| \int_a^b f(x) \, du(x) + \int_a^b f(x) \right| \leq \frac{1}{\Gamma(\alpha + 1)} \left[ (g(x) - g(a))^\alpha f(a) + (g(b) - g(x))^\alpha f(b) \right]
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ (g(x) - g(a))^\alpha \int_a^x (f(t)) dt + \int_a^b (g(t) - g(x))^\alpha \int_a^b (f(t)) dt \right]
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ (g(x) - g(a))^\alpha \int_a^x (f(t)) dt + (g(b) - g(x))^\alpha \int_a^b (f(t)) dt \right]
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ \frac{1}{2} (g(b) - g(a)) + \left( g(x) - \frac{g(a) + g(b)}{2} \right)^\alpha \right] \int_a^b (f(t)) dt;
\]

with \( p, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \);

where

\[
\begin{align*}
\left[ \frac{1}{2} \int_a^b (f(t)) dt + \frac{1}{2} \int_a^b (f(t)) dt \right]
&= (g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha
\end{align*}
\]

and

\[
\left| \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{\Gamma(\alpha + 1)} \left[ \frac{\int_a^b f(x)}{(g(x) - g(a))^\alpha} + \frac{\int_a^b f(x)}{(g(b) - g(x))^\alpha} \right]
\leq \frac{\alpha}{2(g(x) - g(a))^\alpha} \int_a^b (g(x) - g(t))^{\alpha - 1} g'(t) \int_a^t (f(t)) dt
\leq \frac{\alpha}{2(g(b) - g(x))^\alpha} \int_a^b (g(t) - g(x))^{\alpha - 1} g'(t) \int_a^t (f(t)) dt
\leq \frac{1}{2} \int_a^b (f(t)) dt
\]
Proof. By using Lemma 3 we have

$$
\left| \int_a^x (g(x) - g(t))^\alpha \, df(t) \right| \leq \int_a^x (g(x) - g(t))^\alpha \, d\left( \sqrt[\alpha]{f} \right)
$$

for any $x \in (a, b)$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$
\int_a^x (g(x) - g(t))^\alpha \, df(t) = (g(x) - g(t))^\alpha \sqrt[\alpha]{f} \bigg|_a^x + \alpha \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \sqrt[\alpha]{f} \, dt
$$

and

$$
\int_a^b (g(x) - g(t))^\alpha \, df(t) = (g(b) - g(x))^\alpha \sqrt[\alpha]{f} \bigg|_x^b - \alpha \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \sqrt[\alpha]{f} \, dt
$$

for any $x \in (a, b)$. 

By taking the modulus in the equality (2.1) we have

$$\left| I_{a+g}^\alpha f(x) + I_{b-g}^\alpha f(x) \right|$$

$$\leq \frac{1}{\Gamma(\alpha + 1)} \left[ \left| \int_a^x (g(x) - g(t))^\alpha \, df(t) \right| + \left| \int_x^b (g(t) - g(x))^\alpha \, df(t) \right| \right]$$

$$\leq \frac{1}{\Gamma(\alpha + 1)} \int_a^x (g(x) - g(t))^\alpha \, d\left( \frac{t}{x} \right)$$

$$+ \frac{1}{\Gamma(\alpha + 1)} \int_x^b (g(t) - g(x))^\alpha \, d\left( \frac{t}{x} \right)$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \, \frac{t}{x} \, df(t)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \, \frac{b}{x} \, df(t)$$

for any \( x \in (a, b) \), which proves the first part of (3.2).

Moreover, since \( \nabla_t^a f(t) \leq \nabla_a^x f(t) \) for \( a \leq t \leq x \) and \( \nabla_t^b f(t) \leq \nabla_x^b f(t) \) for \( x \leq t \leq b \), then

$$\frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \, \frac{t}{x} \, df(t) + \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \, \frac{b}{x} \, df(t) \right]$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[ \nabla_a^x (f) \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \, dt + \nabla_x^b (f) \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \, dt \right]$$

$$= \frac{1}{\Gamma(\alpha + 1)} \left[ (g(x) - g(a))^{\alpha} \nabla_a^x (f) + (g(b) - g(x))^{\alpha} \nabla_x^b (f) \right]$$

for any \( x \in (a, b) \), which proves the second part of (3.2).

The last part of (3.2) is obvious by making use of the elementary Hölder type inequalities for positive real numbers \( c, d, m, n \geq 0 \)

$$mc + nd \leq \begin{cases} \max \{m, n\} (c + d); \\
(m^p + n^p)^{1/p} (c^q + d^q)^{1/q} \quad \text{with} \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$
By the equality (2.14) we also have

\[
\frac{f(a) + f(b)}{2} - \frac{1}{2} \Gamma(\alpha + 1) \left[ \frac{I_{a+g}^\alpha f(x)}{(g(x) - g(a))^{\alpha}} + \frac{I_{b-g}^\alpha f(x)}{(g(b) - g(x))^{\alpha}} \right] \\
\leq \frac{1}{2} \left( \frac{g(x) - g(a)}{g(b) - g(x)} \right)^{\alpha} \int_a^x (g(x) - g(t))^{\alpha} df(t) \\
+ \frac{1}{2} \left( \frac{g(x) - g(a)}{g(b) - g(x)} \right)^{\alpha} \int_x^b (g(t) - g(x))^{\alpha} df(t) \\
\leq \frac{1}{2} \left( \frac{g(x) - g(a)}{g(b) - g(x)} \right)^{\alpha} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \sqrt[\alpha]{f(t)} \, dt \\
+ \frac{1}{2} \left( \frac{g(x) - g(a)}{g(b) - g(x)} \right)^{\alpha} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \sqrt[\alpha]{f(t)} \, dt \\
\leq \frac{1}{2} \int_a^b \sqrt[\alpha]{f(t)} \, dt + \frac{1}{2} \int_a^b \sqrt[\alpha]{f(t)} \, dt = \frac{1}{2} \int_a^b \sqrt[\alpha]{f(t)} \, dt,
\]

which proves the inequality (3.3). \(\square\)

**Remark 2.** The inequality (3.2) was obtained by a different technique in the earlier paper [16].

**Corollary 4.** With the assumptions of Theorem 2, we have

\[
(3.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(g(b) - g(a))^{\alpha}} \left[ I_{a+g}^\alpha f(M_g(a,b)) + I_{b-g}^\alpha f(M_g(a,b)) \right] \right| \\
\leq \frac{2^{\alpha-1} \alpha}{(g(b) - g(a))^{\alpha}} \left[ \int_{M_g(a,b)}^a \left( \frac{g(a) + g(b)}{2} \right)^{\alpha-1} g'(t) \sqrt[\alpha]{f(t)} \, dt \\
+ \int_{M_g(a,b)}^b \left( \frac{g(a) + g(b)}{2} \right)^{\alpha-1} g'(t) \sqrt[\alpha]{f(t)} \, dt \right] \\
\leq \frac{1}{2} \int_a^b \sqrt[\alpha]{f(t)} \, dt.
\]

The proof follows by either the inequality (3.2) or (3.3) by taking \(x = x = M_g(a,b) = g^{-1} \left( \frac{g(a) + g(b)}{2} \right) \).
Theorem 3. With the assumptions of Theorem 2, we have

\begin{equation}
\left| I_{x-}^{a}f(a) + I_{x+}^{a}f(b) \right| - \frac{1}{\Gamma(\alpha+1)} \left( (g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha \right) f(x)
\end{equation}

\begin{align*}
& \leq \frac{1}{\Gamma(\alpha+1)} \left[ \int_{a}^{x} (g(t) - g(a))^{\alpha-1} g'(t) \left( \frac{t}{x} \right)^{\alpha-1} f(t) dt + \int_{x}^{b} (g(b) - g(t))^{\alpha-1} g'(t) \left( \frac{t}{x} \right)^{\alpha-1} f(t) dt \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \left[ (g(x) - g(a))^\alpha \left( \int_{a}^{x} f(t) dt \right) + (g(b) - g(x))^\alpha \left( \int_{x}^{b} f(t) dt \right) \right]
\end{align*}

\begin{align*}
& \quad + \frac{1}{\Gamma(\alpha+1)} \left[ \frac{1}{2} g(b) - g(a) + \left( \frac{g(x) - g(a) + g(b)}{2} \right)^{\alpha} \right] \frac{1}{\Gamma(\alpha+1)} \left[ \frac{1}{2} \left( \left( (g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha} \right)^{1/p} \times \left( \left( \left( \frac{\alpha}{p} \right) + \left( \frac{\alpha}{q} \right) \right)^{1/q} \right) \right) \right] \\
& \quad \left( \left( (g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha \right) \right)
\end{align*}

and

\begin{align*}
& \left| \frac{1}{2} \Gamma(\alpha+1) \left[ \frac{I_{x-}^{a}f(a) + I_{x+}^{a}f(b)}{(g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha} - f(x) \right] \right| \\
& \quad \leq \frac{\alpha}{2(\alpha+1)} \int_{x}^{b} (g(b) - g(t))^{\alpha-1} g'(t) \left( \frac{t}{x} \right)^{\alpha-1} f(t) dt \\
& \quad + \frac{\alpha}{2(\alpha+1)} \int_{x}^{b} (g(t) - g(a))^{\alpha-1} g'(t) \left( \frac{t}{x} \right)^{\alpha-1} f(t) dt \\
& \quad \leq \frac{1}{2} \int_{a}^{b} \left( f(t) \right)
\end{align*}

for any \( x \in (a, b) \).

Proof. By using Lemma 3 we have

\[ \left| \int_{a}^{x} (g(t) - g(a))^\alpha df(t) \right| \leq \int_{a}^{x} (g(t) - g(a))^\alpha d \left( \int_{a}^{t} f(u) du \right) \]

and

\[ \left| \int_{x}^{b} (g(b) - g(t))^\alpha df(t) \right| \leq \int_{x}^{b} (g(b) - g(t))^\alpha d \left( \int_{x}^{t} f(u) du \right). \]
Integrating by parts in the Riemann-Stieltjes integral, we have

\[
\int_a^x (g(t) - g(a))^\alpha \, d\left(\frac{t}{a} \int_a^t (f)\right) = (g(t) - g(a))^\alpha \left. \frac{t}{a} \int_a^t (f) \right|_a^x - \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt
\]

\[
= (g(x) - g(a))^\alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt - \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt
\]

\[
= \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt - \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt
\]

\[
= \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt
\]

\[
= \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt
\]

\[
= \alpha \int_a^x (g(t) - g(a))^{\alpha - 1} g'(t) \frac{t}{a} \int_a^t (f) \, dt
\]

\[
\text{and}
\]

\[
\int_b^x (g(b) - g(t))^\alpha \, d\left(\frac{t}{x} \int_x^t (f)\right) = (g(b) - g(t))^\alpha \left. \frac{t}{x} \int_x^t (f) \right|_x^b + \alpha \int_x^b (g(b) - g(t))^{\alpha - 1} g'(t) \frac{t}{x} \int_x^t (f) \, dt
\]

\[
= \alpha \int_x^b (g(b) - g(t))^{\alpha - 1} g'(t) \frac{t}{x} \int_x^t (f) \, dt + \alpha \int_x^b (g(b) - g(t))^{\alpha - 1} g'(t) \frac{t}{x} \int_x^t (f) \, dt
\]

for any \( x \in (a, b) \).

Using the equality (2.2) we have

\[
I_{x-}^\alpha f(a) + I_{x+}^\alpha f(b) = \frac{1}{\Gamma(\alpha + 1)} \left( (g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha \right) f(x)
\]

\[
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ \int_x^b (g(b) - g(t))^{\alpha} \, df(t) + \int_x^x (g(t) - g(a))^{\alpha} \, df(t) \right]
\]

\[
\leq \frac{1}{\Gamma(\alpha + 1)} \int_x^x (g(t) - g(a))^{\alpha} \, d\left(\frac{t}{a} \int_a^t (f)\right) + \frac{1}{\Gamma(\alpha + 1)} \int_x^b (g(b) - g(t))^{\alpha} \, d\left(\frac{t}{x} \int_x^t (f)\right)
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (g(t) - g(a))^{\alpha-1} g'(t) \left( \int_{t}^{x} f(t) \, dt \right) \, dt + \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (g(b) - g(t))^{\alpha-1} g'(t) \left( \int_{x}^{t} f(t) \, dt \right) \, dt
\]

for \( x \in (a, b) \), which proves (3.5).

By the equality (2.15) we also have

\[
\left| \frac{1}{2} \Gamma(\alpha + 1) \left[ \frac{I_{a}^{\alpha} f(a)}{(x - g(a))^{\alpha}} + \frac{I_{b}^{\alpha} f(b)}{(g(b) - g(x))^{\alpha}} \right] - f(x) \right|
\leq \frac{1}{2} \frac{1}{(g(b) - g(x))^{\alpha}} \left| \int_{x}^{b} (g(b) - g(t))^{\alpha} \, df(t) \right|
\]

\[
+ \frac{1}{2} \frac{1}{(g(x) - g(a))^{\alpha}} \left| \int_{a}^{x} (g(t) - g(a))^{\alpha} \, df(t) \right|
\]

\[
\leq \frac{1}{2} \frac{1}{(g(b) - g(x))^{\alpha}} \left| \int_{x}^{b} (g(b) - g(t))^{\alpha} \, df(t) \right|
\]

\[
+ \frac{1}{2} \frac{1}{(g(x) - g(a))^{\alpha}} \left| \int_{a}^{x} (g(t) - g(a))^{\alpha} \, df(t) \right|
\]

\[
\leq \frac{1}{2} \frac{1}{(g(b) - g(x))^{\alpha}} \int_{x}^{b} (g(b) - g(t))^{\alpha-1} g'(t) \left( \int_{x}^{t} f(t) \, dt \right) \, dt
\]

\[
+ \frac{1}{2} \frac{1}{(g(x) - g(a))^{\alpha}} \int_{a}^{x} (g(t) - g(a))^{\alpha-1} g'(t) \left( \int_{a}^{t} f(t) \, dt \right) \, dt
\]

\[
\leq \frac{1}{2} \left( \int_{a}^{x} f(t) \, dt + \int_{x}^{b} f(t) \, dt \right) = \frac{1}{2} \left( \int_{a}^{b} f(t) \, dt \right),
\]

which proves (3.6). \( \square \)

**Remark 3.** The inequality (3.5) was obtained by a different technique in the earlier paper [16].
Corollary 5. With the assumptions of Theorem 2, we have

\begin{equation}
\left| \frac{2^\alpha - 1}{(b - a)^\alpha} \left[ I_{M_{g}(a,b)}^\alpha f(a) + I_{M_{g}(a,b)+g}^\alpha f(b) \right] - f(M_{g}(a,b)) \right| \leq \frac{2^\alpha - 1}{(b - a)^\alpha} \left[ \int_{M_{g}(a,b)}^b (g(t) - g(a))^{\alpha-1} g'(t) \int_{M_{g}(a,b)}^t (f) \, dt \right. \\
\left. + \int_a^{M_{g}(a,b)} (g(t) - g(a))^{\alpha-1} g'(t) \int_{M_{g}(a,b)}^t (f) \, dt \right] \leq \frac{1}{2} \int_a^b (f) \, dt.
\end{equation}

The proof follows by either the inequality (3.5) or (3.6) by taking \( x = M_{g}(a,b) = g^{-1} \left( \frac{g(a) + g(b)}{2} \right) \).

4. Some Examples

If we take \( g(t) = t \), \( t \in [a, b] \) in (3.2) and (3.5), then we recapture the inequalities from Theorem 1. From (3.3) we get for the classical Riemann-Liouville fractional integrals the following inequalities

\begin{equation}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{2} \Gamma(\alpha + 1) \left[ J_{a+}^\alpha f(x) \right] + \frac{1}{(b-a)^\alpha} \left[ J_{b-}^\alpha f(x) \right] \right| \leq \frac{\alpha}{2} \left[ \int_a^x (x-t)^{\alpha-1} \int_t^b (f) \, dt + \int_x^b (t-x)^{\alpha-1} \int_t^b (f) \, dt \right] \leq \frac{1}{2} \int_a^b (f) \, dt
\end{equation}

while from (3.6) we get

\begin{equation}
\left| \frac{1}{2} \Gamma(\alpha + 1) \left[ J_{a+}^\alpha f(a) \right] + \frac{1}{(b-a)^\alpha} \left[ J_{b-}^\alpha f(b) \right] - f(x) \right| \leq \frac{\alpha}{2} \left[ \int_a^x (t-a)^{\alpha-1} \int_t^b (f) \, dt + \int_x^b (b-t)^{\alpha-1} \int_t^b (f) \, dt \right] \leq \frac{1}{2} \int_a^b (f) \, dt
\end{equation}

for any \( x \in (a, b) \).
Consider the function \( g(t) = \ln t, \ t \in [a, b] \subset (0, \infty) \), then by (3.3) we have for Hadamard fractional integrals

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{2} \Gamma(\alpha + 1) \left[ \frac{H^\alpha_{a+} f(x)}{\ln \left( \frac{x}{a} \right)} + \frac{H^\alpha_{b-} f(x)}{\ln \left( \frac{x}{b} \right)} \right] \right|
\leq \frac{\alpha}{2} \left[ \frac{1}{\ln \left( \frac{x}{a} \right)} \int_a^x \ln \left( \frac{t}{x} \right) \alpha^{-1} \frac{t}{x} \sqrt{f(t)} dt \ight.

\left. + \frac{1}{\ln \left( \frac{x}{b} \right)} \int_x^b \ln \left( \frac{t}{x} \right) \alpha^{-1} \frac{b}{x} \sqrt{f(t)} dt \right]
\leq \frac{1}{2} \sqrt{f(x)}
\]

while from (3.6) we get

\[
\left| \frac{1}{2} \Gamma(\alpha + 1) \left[ \frac{H^\alpha_{a-} f(a)}{\ln \left( \frac{x}{a} \right)} + \frac{H^\alpha_{b+} f(b)}{\ln \left( \frac{x}{b} \right)} \right] - f(x) \right|
\leq \frac{\alpha}{2} \left[ \frac{1}{\ln \left( \frac{x}{b} \right)} \int_a^b \ln \left( \frac{b}{x} \right) \alpha^{-1} \frac{b}{x} \sqrt{f(t)} dt \ight.

\left. + \frac{1}{\ln \left( \frac{x}{a} \right)} \int_x^b \ln \left( \frac{t}{x} \right) \alpha^{-1} \frac{x}{t} \sqrt{f(t)} dt \right]
\leq \frac{1}{2} \sqrt{f(x)}
\]

for any \( x \in (a, b) \).

If we take the function \( g(t) = -t^{-1}, \ t \in [a, b] \subset (0, \infty) \), then by (3.3) we have for Harmonic fractional integrals

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{2} \Gamma(\alpha + 1) x^\alpha \left[ \frac{a^\alpha R^\alpha_{a+} f(x)}{(x-a)^{\alpha+1}} + \frac{b^\alpha R^\alpha_{b-} f(x)}{(b-x)^{\alpha+1}} \right] \right|
\leq \frac{x^{\alpha+1}}{\Gamma(\alpha+1)} \int_a^x \left( x - t \right)^{\alpha-1} \sqrt{f(t)} dt + \frac{b^{\alpha}}{(b-x)^{\alpha+1}} \int_x^b \left( t - x \right)^{\alpha-1} \sqrt{f(t)} dt
\leq \frac{1}{2} \sqrt{f(x)}
\]
while from (3.6) we get

\[
\frac{1}{2} \left( \frac{a^\alpha R_a^\alpha f(a)}{(x - a)^\alpha} + \frac{b^\alpha R_b^\alpha f(b)}{(b - x)^\alpha} \right) - f(x) \leq \frac{\alpha x^\alpha}{2} \left( b \int_x^b \frac{(b - t)^{\alpha - 1}}{t^{\alpha + 1}} \int_x^t f(t) \, dt \, dt + \frac{a}{(x - a)^\alpha} \int_a^x \frac{(t - a)^{\alpha - 1}}{t^{\alpha + 1}} \int_x^t f(t) \, dt \, dt \right)
\]

\[
\leq \frac{1}{2} \int_a^b f(t) \, dt,
\]

for any \( x \in (a, b) \).

References


1Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

2DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa