Received 23/01/18

SOME INEQUALITIES OF OSTROWSKI AND TRAPEZOID TYPE FOR TRIGONOMETRICALLY ρ-CONVEX FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Ostrowski and Trapezoid type integral inequalities for trigonometrically ρ-convex functions.

1. Introduction

In 1938, A. Ostrowski [15], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1 (Ostrowski). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty(b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The following result of Ostrowski type for convex functions holds.

Theorem 2 (Dragomir, 2002 [7]). *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in[a,b]$ one has the inequality*

$$\begin{aligned}
&\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\\
&\le\int_a^b f(t)\,dt-(b-a)f(x)\\
&\le\frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right].
\end{aligned}\tag{1.2}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x=a$ or $x=b$.

1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

Key words and phrases. Convex functions, Trigonometrically ρ-Convex Functions, Ostrowski inequality, Trapezoid inequality, Integral inequalities.

RGMIA Res. Rep. Coll. 21 (2018), Art. 11, 13 pp.

In particular, for $x=\frac{a+b}{2}$*, we get the sharp inequalities*

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \\
&\leq \frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a).
\end{aligned} \tag{1.3}
$$

For various Ostrowski type inequalities see the recent survey paper [10] and the references therein.

The following trapezoid type inequality for convex functions also holds.

Theorem 3 (Dragomir, 2002 [7])**.** *Let* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *be a convex function on* $[a,b]$*. Then for any* $x\in[a,b]$ *one has the inequality*

$$
\begin{aligned}
&\frac{1}{2}\left[(b-x)^2 f'_{+}(x)-(x-a)^2 f'_{-}(x)\right] \\
&\leq (x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt \\
&\leq \frac{1}{2}\left[(b-x)^2 f'_{-}(b)-(x-a)^2 f'_{+}(a)\right].
\end{aligned} \tag{1.4}
$$

The constant $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $x=a$ *or* $x=b$*.*

In particular, for $x=\frac{a+b}{2}$*, we get the sharp inequalities*

$$
\begin{aligned}
0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a) \\
&\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&\leq \frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a).
\end{aligned} \tag{1.5}
$$

In the following we present the basic definitions and results concerning the class of trigonometrically ρ-convex function, see for example [12], [13] and [3], [5], [6], [11], [14], [16] and [17].

Following [1], we say that a function $f:I\to\mathbb{R}$ is *trigonometrically* ρ*-convex* on I if for any closed subinterval $[a,b]$ of I with $0<b-a<\frac{\pi}{\rho}$ we have

$$
f(x)\leq\frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]}f(a)+\frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]}f(b) \tag{1.6}
$$

for all $x\in[a,b]$.

If the inequality (1.6) holds with " $\geq$ ", then the function will be called *trigonometrically* ρ*-concave* on I.

Geometrically speaking, this means that the graph of f on $[a,b]$ lies nowhere above the ρ-trigonometric function determined by the equation

$$
H(x)=H(x;a,b,f):=A\cos(\rho x)+B\sin(\rho x)
$$

where A and B are chosen such that $H(a)=f(a)$ and $H(b)=f(b)$.

If we take $x = (1-t)a + tb \in [a,b]$, $t \in [0,1]$, then the condition (1.6) becomes

$$f((1-t)a+tb) \le \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]} f(b) \tag{1.7}$$

for any $t \in [0,1]$.

We have the following properties of trigonometrically ρ-convex on I, [1].

(i) A trigonometrically ρ-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \le f'_+(x)$. The function f is differentiable on I with the exception of an at most countable set.

(ii) A necessary and sufficient condition for the function $f : I \to \mathbb{R}$ to be trigonometrically ρ-convex function on I is that it satisfies the *gradient inequality*

$$f(y) \ge f(x)\cos[\rho(y-x)] + K_{x,f}\sin[\rho(y-x)] \tag{1.8}$$

for any x, $y \in I$ where $K_{x,f} \in \left[f'_-(x), f'_+(x)\right]$. If f is differentiable at the point x then $K_{x,f} = f'(x)$.

(iii) A necessary and sufficient condition for the function f to be a trigonometrically ρ-convex in I, is that the function

$$\varphi(x) = f'(x) + \rho^2 \int_a^x f(t)\,dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then f is trigonometrically ρ-convex on I if and only if for all $x \in I$ we have

$$f''(x) + \rho^2 f(x) \ge 0. \tag{1.9}$$

For other properties of trigonometrically ρ-convex functions, see [1].

As general examples of trigonometrically ρ-convex functions we can give the indicator function

$$h_F(\theta) := \limsup_{r\to\infty} \frac{\log\left|F\left(re^{i\theta}\right)\right|}{r^\rho},\ \theta \in (\alpha,\beta),$$

where F is an entire function of order $\rho \in (0,\infty)$.

If $0 < \beta - \alpha < \frac{\pi}{\rho}$, then, it was shown in 1908 by Phragmén and Lindelöf, see [12], that h_F is trigonometrically ρ-convex on (α,β).

Using the condition (1.9) one can also observe that any nonnegative twice differentiable and convex function on I is also trigonometrically ρ-convex on I for any $\rho > 0$.

There exists also concave functions on an interval that are trigonometrically ρ-convex on that interval for some $\rho > 0$.

Consider for example $f(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$f''(x) + \rho^2 f(x) = -\cos x + \rho^2 \cos x = \left(\rho^2 - 1\right)\cos x,$$

which shows that it is trigonometrically ρ-convex on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all $\rho > 1$ and trigonometrically ρ-concave for $\rho \in (0,1)$.

Consider the function $f:(0,\infty)\to(0,\infty)$, $f(x)=x^p$ with $p\in\mathbb{R}\setminus\{0\}$. If $p\in(-\infty,0)\cup[1,\infty)$ the function is convex and therefore trigonometrically ρ-convex for any $\rho>0$. If $p\in(0,1)$ then the function is concave and

$$f''(x)+\rho^2 f(x)=\rho^2 x^p-p(1-p)x^{p-2}=\rho^2 x^{p-2}\left(x^2-\frac{p(1-p)}{\rho^2}\right),\ x>0.$$

This shows that for $p\in(0,1)$ and $\rho>0$ the function $f(x)=x^p$ is trigonometrically ρ-convex on $\left(\frac{1}{\rho}\sqrt{p(1-p)},\infty\right)$ and trigonometrically ρ-concave on $\left(0,\frac{1}{\rho}\sqrt{p(1-p)}\right)$.

Consider the concave function $f:(0,\infty)\to\mathbb{R}$, $f(x)=\ln x$. We observe that

$$g(x):=f''(x)+\rho^2 f(x)=\rho^2\ln x-\frac{1}{x^2},\ x>0.$$

We have $g'(x)=\frac{2+\rho^2x^2}{x^2}>0$ for $x>0$ and $\lim_{x\to0+}g(x)=-\infty$, $\lim_{x\to\infty}g(x)=\infty$, showing that the function g is strictly increasing on $(0,\infty)$ and the equation $g(x)=0$ has a unique solution. Therefore $g(x)<0$ for $x\in(0,x_\rho)$ and $g(x)>0$ for $x\in(x_\rho,\infty)$, where x_ρ is the unique solution of the equation $\ln x=\frac{1}{\rho^2x^2}$.

In conclusion, if $\rho>0$, then the function $f(x)=\ln x$ is trigonometrically ρ-concave on $(0,x_\rho)$ and trigonometrically ρ-convex on (x_ρ,∞).

In this paper we establish some Ostrowski and Trapezoid type integral inequalities for trigonometrically ρ-convex functions.

2. Ostrowski Type Inequalities

We have:

Theorem 4. *Assume that the function $f:I\to\mathbb{R}$ is trigonometrically ρ-convex on I. Then for any $a,b\in I$ with $0<b-a<\frac{\pi}{\rho}$ and $x\in(a,b)$ we have*

$$\begin{aligned}(2.1)\quad &\frac{1}{2}\left[f'_+(x)(b-x)^2-f'_-(x)(x-a)^2\right]\\ &\leq\int_a^b f(t)\,dt+\frac{1}{2}\rho^2\left[\int_a^x(t-a)^2f(t)\,dt+\int_x^b(b-t)^2f(t)\,dt\right]\\ &\qquad-f(x)(b-a)\\ &\leq\frac{1}{2}\left[f'_-(b)(b-x)^2-f'_+(a)(x-a)^2\right]\\ &\qquad+\frac{1}{2}\rho^2\left[(x-a)^2\int_a^x f(t)\,dt+(b-x)^2\int_x^b f(t)\,dt\right].\end{aligned}$$

In particular, if f is differentiable in x, then we have

$$\begin{aligned}(2.2)\quad &f'(x)(b-a)\left(\frac{a+b}{2}-x\right)\\ &\leq\int_a^b f(t)\,dt+\frac{1}{2}\rho^2\left[\int_a^x(t-a)^2f(t)\,dt+\int_x^b(b-t)^2f(t)\,dt\right]\\ &\qquad-f(x)(b-a)\end{aligned}$$

$$\leq \frac{1}{2}\left[f'_{-}(b)(b-x)^{2}-f'_{+}(a)(x-a)^{2}\right] + \frac{1}{2}\rho^{2}\left[(x-a)^{2}\int_{a}^{x}f(t)\,dt+(b-x)^{2}\int_{x}^{b}f(t)\,dt\right].$$

Proof. We use the *Montgomery identity* for an absolutely continuous function $g : [a,b]\to\mathbb{C}$ that says that

$$g(x)(b-a)-\int_{a}^{b}g(s)\,ds=\int_{a}^{x}(s-a)\,g'(s)\,ds-\int_{x}^{b}(b-s)\,g'(s)\,ds \tag{2.3}$$

for $x\in(a,b)$. This can be proved in one line by integrating by parts on the second term.

Using the property (iii) from Introduction we have that

$$f'_{+}(a)\leq f'(s)+\rho^{2}\int_{a}^{s}f(t)\,dt\leq f'_{-}(x)+\rho^{2}\int_{a}^{x}f(t)\,dt \tag{2.4}$$

for a.e. $s\in[a,x]$.

This implies that

$$f'_{+}(a)(s-a)\leq\left[f'(s)+\rho^{2}\int_{a}^{s}f(t)\,dt\right](s-a) \leq\left[f'_{-}(x)+\rho^{2}\int_{a}^{x}f(t)\,dt\right](s-a),$$

that is equivalent to

$$f'_{+}(a)(s-a)-\rho^{2}(s-a)\int_{a}^{s}f(t)\,dt\leq f'(s)(s-a) \leq f'_{-}(x)(s-a)+\rho^{2}(s-a)\int_{a}^{x}f(t)\,dt-\rho^{2}(s-a)\int_{a}^{s}f(t)\,dt$$

for a.e. $s\in[a,x]$.

If we integrate this over $s\in[a,x]$ we get

$$f'_{+}(a)\int_{a}^{x}(s-a)\,ds-\rho^{2}\int_{a}^{x}(s-a)\left(\int_{a}^{s}f(t)\,dt\right)ds\leq\int_{a}^{x}f'(s)(s-a)\,ds$$
$$\leq f'_{-}(x)\int_{a}^{x}(s-a)\,ds+\rho^{2}\int_{a}^{x}(s-a)\,ds\int_{a}^{x}f(t)\,dt-\rho^{2}\int_{a}^{x}(s-a)\left(\int_{a}^{s}f(t)\,dt\right)ds,$$

that is equivalent to

$$\frac{1}{2}f'_{+}(a)(x-a)^{2}-\rho^{2}\int_{a}^{x}(s-a)\left(\int_{a}^{s}f(t)\,dt\right)ds\leq\int_{a}^{x}f'(s)(s-a)\,ds \leq\frac{1}{2}f'_{-}(x)(x-a)^{2}+\frac{1}{2}\rho^{2}(x-a)^{2}\int_{a}^{x}f(t)\,dt-\rho^{2}\int_{a}^{x}(s-a)\left(\int_{a}^{s}f(t)\,dt\right)ds, \tag{2.5}$$

for $x\in(a,b)$.

Using the property (iii) from Introduction we also have that

$$f'_{+}(x)+\rho^{2}\int_{a}^{x}f(t)\,dt\leq f'(s)+\rho^{2}\int_{a}^{s}f(t)\,dt\leq f'_{-}(b)+\rho^{2}\int_{a}^{b}f(t)\,dt \tag{2.6}$$

for a.e. $s\in[x,b]$.

This implies that

$$f'_+(x)(b-s)+\rho^2(b-s)\int_a^x f(t)\,dt \le \left(f'(s)+\rho^2\int_a^s f(t)\,dt\right)(b-s) \le f'_-(b)(b-s)+\rho^2(b-s)\int_a^b f(t)\,dt$$

for a.e $s \in [x,b]$.

If we integrate this over $s \in [x,b]$, we get

$$\frac{1}{2}f'_+(x)(b-x)^2+\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt \le \int_x^b \left(f'(s)+\rho^2\int_a^s f(t)\,dt\right)(b-s)\,ds \le \frac{1}{2}f'_-(b)(b-x)^2+\frac{1}{2}\rho^2(b-x)^2\int_a^b f(t)\,dt,$$

which is equivalent to

$$\frac{1}{2}f'_+(x)(b-x)^2+\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt-\rho^2\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds \le \int_x^b f'(s)(b-s)\,ds \le \frac{1}{2}f'_-(b)(b-x)^2+\rho^2\frac{1}{2}(b-x)^2\int_a^b f(t)\,dt-\rho^2\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds$$

or to

$$(2.7)\quad -\frac{1}{2}f'_-(b)(b-x)^2-\rho^2\frac{1}{2}(b-x)^2\int_a^b f(t)\,dt+\rho^2\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds \le -\int_x^b f'(s)(b-s)\,ds$$
$$-\frac{1}{2}f'_+(x)(b-x)^2-\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt+\rho^2\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds.$$

Now, if we add (2.5) with (2.7) and use Montgomery identity (2.3) we get

$$\begin{aligned}
(2.8)\quad & \frac{1}{2}f'_{+}(a)(x-a)^2-\rho^2\int_a^x (s-a)\left(\int_a^s f(t)\,dt\right)ds \\
& -\frac{1}{2}f'_{-}(b)(b-x)^2-\rho^2\frac{1}{2}(b-x)^2\int_a^b f(t)\,dt+\rho^2\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds \\
& \leq f(x)(b-a)-\int_a^b f(s)\,ds \\
& \leq \frac{1}{2}f'_{-}(x)(x-a)^2+\frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt-\rho^2\int_a^x (s-a)\left(\int_a^s f(t)\,dt\right)ds \\
& -\frac{1}{2}f'_{+}(x)(b-x)^2-\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt+\rho^2\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds.
\end{aligned}$$

Using the integration by parts, we have

$$\begin{aligned}
\int_a^x (s-a)\left(\int_a^s f(t)\,dt\right)ds &= \frac{1}{2}\int_a^x \left(\int_a^s f(t)\,dt\right)ds\left((s-a)^2\right) \\
&= \frac{1}{2}\left(\int_a^s f(t)\,dt\right)(s-a)^2\Big|_a^x-\frac{1}{2}\int_a^x (s-a)^2 f(s)\,ds \\
&= \frac{1}{2}(x-a)^2\int_a^x f(t)\,dt-\frac{1}{2}\int_a^x (s-a)^2 f(s)\,ds
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (b-s)\left(\int_a^s f(t)\,dt\right)ds &= -\frac{1}{2}\int_x^b \left(\int_a^s f(t)\,dt\right)d\left((b-s)^2\right) \\
&= -\frac{1}{2}\left(\int_a^s f(t)\,dt\right)(b-s)^2\Big|_x^b+\frac{1}{2}\int_x^b (b-s)^2 f(s)\,ds \\
&= \frac{1}{2}\int_x^b (b-s)^2 f(s)\,ds+\frac{1}{2}(b-x)^2\int_a^x f(t)\,dt.
\end{aligned}$$

Then by (2.8) we get

$$\begin{aligned}(2.9)\quad & \frac{1}{2}f'_{+}(a)(x-a)^{2}-\rho^{2}\left[\frac{1}{2}(x-a)^{2}\int_{a}^{x}f(t)\,dt-\frac{1}{2}\int_{a}^{x}(s-a)^{2}f(s)\,ds\right]\\ &-\frac{1}{2}f'_{-}(b)(b-x)^{2}-\rho^{2}\frac{1}{2}(b-x)^{2}\int_{a}^{b}f(t)\,dt\\ &+\rho^{2}\left[\frac{1}{2}\int_{x}^{b}(b-s)^{2}f(s)\,ds+\frac{1}{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt\right]\\ &\leq f(x)(b-a)-\int_{a}^{b}f(s)\,ds\\ &\leq\frac{1}{2}f'_{-}(x)(x-a)^{2}+\frac{1}{2}\rho^{2}(x-a)^{2}\int_{a}^{x}f(t)\,dt\\ &-\rho^{2}\left[\frac{1}{2}(x-a)^{2}\int_{a}^{x}f(t)\,dt-\frac{1}{2}\int_{a}^{x}(s-a)^{2}f(s)\,ds\right]\\ &-\frac{1}{2}f'_{+}(x)(b-x)^{2}-\frac{1}{2}\rho^{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt\\ &+\rho^{2}\left[\frac{1}{2}\int_{x}^{b}(b-s)^{2}f(s)\,ds+\frac{1}{2}(b-x)^{2}\int_{a}^{x}f(t)\,dt\right]\end{aligned}$$

or, equivalently

$$\begin{aligned}(2.10)\quad & \frac{1}{2}f'_{+}(a)(x-a)^{2}-\frac{1}{2}f'_{-}(b)(b-x)^{2}\\ &+\frac{1}{2}(b-x)^{2}\rho^{2}\int_{a}^{x}f(t)\,dt-\frac{1}{2}(x-a)^{2}\rho^{2}\int_{a}^{x}f(t)\,dt-\frac{1}{2}(b-x)^{2}\rho^{2}\int_{a}^{b}f(t)\,dt\\ &+\frac{1}{2}\rho^{2}\left[\int_{x}^{b}(b-s)^{2}f(s)\,ds+\int_{a}^{x}(s-a)^{2}f(s)\,ds\right]\\ &\leq f(x)(b-a)-\int_{a}^{b}f(s)\,ds\\ &\leq\frac{1}{2}f'_{-}(x)(x-a)^{2}-\frac{1}{2}f'_{+}(x)(b-x)^{2}\\ &+\frac{1}{2}\rho^{2}\left[\int_{a}^{x}(s-a)^{2}f(s)\,ds+\int_{x}^{b}(b-s)^{2}f(s)\,ds\right]\end{aligned}$$

for $x\in(a,b)$.

The inequality (2.10) can also be written as

$$\begin{aligned}&\frac{1}{2}f'_{+}(a)(x-a)^{2}-\frac{1}{2}f'_{-}(b)(b-x)^{2}\\ &-\frac{1}{2}\rho^{2}\left[(x-a)^{2}\int_{a}^{x}f(t)\,dt+(b-x)^{2}\int_{x}^{b}f(t)\,dt\right]\\ &\leq f(x)(b-a)-\int_{a}^{b}f(s)\,ds\end{aligned}$$

$$-\frac{1}{2}\rho^2\left[\int_a^x (s-a)^2 f(s)\,ds+\int_x^b (b-s)^2 f(s)\,ds\right] \\ \le \frac{1}{2}f'_-(x)(x-a)^2-\frac{1}{2}f'_+(x)(b-x)^2$$

for $x\in(a,b)$, which proves the desired inequality (2.1). □

Corollary 1. *With the assumptions of Theorem 4 we have*

$$0\le\frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\ \le\int_a^b f(t)\,dt+\frac{1}{2}\rho^2\left[\int_a^{\frac{a+b}{2}}(t-a)^2 f(t)\,dt+\int_{\frac{a+b}{2}}^b (b-t)^2 f(t)\,dt\right] \\ -f\left(\frac{a+b}{2}\right)(b-a) \\ \le\frac{1}{8}(b-a)^2\left[f'_-(b)-f'_+(a)\right]+\frac{1}{8}\rho^2(b-a)^2\int_a^b f(t)\,dt. \tag{2.11}$$

3. Trapezoid Type Inequalities

We have:

Theorem 5. *Assume that the function $f:I\to\mathbb{R}$ is trigonometrically ρ-convex on I. Then for any a, $b\in I$ with $0<b-a<\frac{\pi}{\rho}$ and $x\in(a,b)$ we have*

$$\frac{1}{2}f'_+(x)(b-x)^2-\frac{1}{2}f'_-(x)(x-a)^2 \\ -\frac{1}{2}\rho^2\left[(x-a)^2\int_a^x f(t)\,dt+(b-x)^2\left(\int_x^b f(t)\,dt\right)\right] \\ \le(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,ds-\frac{1}{2}\rho^2\int_a^b (x-s)^2 f(s)\,ds \\ \le\frac{1}{2}(b-x)^2 f'_-(b)-\frac{1}{2}f'_+(a)(x-a)^2. \tag{3.1}$$

In particular, if f is differentiable in x, then we have

$$f'(x)(b-a)\left(\frac{a+b}{2}-x\right) \\ -\frac{1}{2}\rho^2\left[(x-a)^2\int_a^x f(t)\,dt+(b-x)^2\left(\int_x^b f(t)\,dt\right)\right] \\ \le(x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,ds-\frac{1}{2}\rho^2\int_a^b (x-s)^2 f(s)\,ds \\ \le\frac{1}{2}(b-x)^2 f'_-(b)-\frac{1}{2}f'_+(a)(x-a)^2. \tag{3.2}$$

Proof. We use the following identity that holds for the absolutely continuous function $g:[a,b]\to\mathbb{C}$

$$(x-a)\,g(a)+(b-x)\,g(b)-\int_a^b g(s)\,dt = \int_a^b (s-x)\,g'(s)\,dt = \int_x^b (s-x)\,g'(s)\,ds - \int_a^x (x-s)\,g'(s)\,ds \tag{3.3}$$

for any $x\in[a,b]$. This can can be proved by integrating by parts in the second term.

Using the inequality (2.4) we get

$$f'_+(a)(x-s) \le f'(s)(x-s)+\rho^2(x-s)\int_a^s f(t)\,dt \le f'_-(x)(x-s)+\rho^2(x-s)\int_a^x f(t)\,dt$$

for a.e. $s\in[a,x]$.

Integrating on $[a,x]$, we have

$$\frac{1}{2}f'_+(a)(x-a)^2 \le \int_a^x f'(s)(x-s)\,ds+\rho^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds \le \frac{1}{2}f'_-(x)(x-a)^2+\frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt,$$

which is equivalent to

$$\rho^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds-\frac{1}{2}f'_-(x)(x-a)^2 - \frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt \le -\int_a^x f'(s)(x-s)\,ds \le \rho^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds-\frac{1}{2}f'_+(a)(x-a)^2 \tag{3.4}$$

for any $x\in(a,b)$.

From (2.6) we have

$$f'_+(x)(s-x)+\rho^2(s-x)\int_a^x f(t)\,dt \le (s-x)\,f'(s)+\rho^2(s-x)\int_a^s f(t)\,dt \le (s-x)\,f'_-(b)+\rho^2(s-x)\int_a^b f(t)\,dt$$

for a.e. $s\in[x,b]$.

Integrating on $[x,b]$ we get

$$\frac{1}{2}f'_{+}(x)(b-x)^2+\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt \le \int_x^b (s-x)f'(s)\,ds+\rho^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds \le \frac{1}{2}(b-x)^2 f'_{-}(b)+\frac{1}{2}\rho^2(b-x)^2\int_a^b f(t)\,dt,$$

which is equivalent to

$$\begin{aligned}(3.5)\quad &\frac{1}{2}f'_{+}(x)(b-x)^2 + \frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt-\rho^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds \\ &\le \int_x^b (s-x)f'(s)\,ds \\ &\le \frac{1}{2}(b-x)^2 f'_{-}(b)+\frac{1}{2}\rho^2(b-x)^2\int_a^b f(t)\,dt-\rho^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds,\end{aligned}$$

for any $x\in(a,b)$.

Adding (3.4) and (3.5) and using the identity (3.3) we get

$$\begin{aligned}&\rho^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds-\frac{1}{2}f'_{-}(x)(x-a)^2-\frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt \\ &+\frac{1}{2}f'_{+}(x)(b-x)^2+\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt-\rho^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds \\ &\le (x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,dt \\ &\le \rho^2\int_a^x (x-s)\left(\int_a^s f(t)\,dt\right)ds-\frac{1}{2}f'_{+}(a)(x-a)^2 \\ &+\frac{1}{2}(b-x)^2 f'_{-}(b)+\frac{1}{2}\rho^2(b-x)^2\int_a^b f(t)\,dt-\rho^2\int_x^b (s-x)\left(\int_a^s f(t)\,dt\right)ds\end{aligned}$$

that is equivalent to

$$\begin{aligned}(3.6)\quad &\frac{1}{2}f'_{+}(x)(b-x)^2-\frac{1}{2}f'_{-}(x)(x-a)^2-\frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt \\ &+\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt+\rho^2\int_a^b (x-s)\left(\int_a^s f(t)\,dt\right)ds \\ &\le (x-a)f(a)+(b-x)f(b)-\int_a^b f(s)\,dt \\ &\le \frac{1}{2}(b-x)^2 f'_{-}(b)-\frac{1}{2}f'_{+}(a)(x-a)^2+\frac{1}{2}\rho^2(b-x)^2\int_a^b f(t)\,dt \\ &+\rho^2\int_a^b (x-s)\left(\int_a^s f(t)\,dt\right)ds.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int_a^b (x-s)\left(\int_a^s f(t)\,dt\right)ds &= -\frac{1}{2}\int_a^b \left(\int_a^s f(t)\,dt\right) d\left((x-s)^2\right)\\ &= -\frac{1}{2}\left[(x-s)^2\left(\int_a^s f(t)\,dt\right)\Big|_a^b - \int_a^b (x-s)^2 f(s)\,ds\right]\\ &= \frac{1}{2}\int_a^b (x-s)^2 f(s)\,ds - \frac{1}{2}(b-x)^2\left(\int_a^b f(t)\,dt\right)\end{aligned}$$

and by (3.6) we get

$$\begin{aligned}&\frac{1}{2}f'_+(x)(b-x)^2 - \frac{1}{2}f'_-(x)(x-a)^2 - \frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt\\ &+\frac{1}{2}\rho^2(b-x)^2\int_a^x f(t)\,dt + \frac{1}{2}\rho^2\int_a^b (x-s)^2 f(s)\,ds - \frac{1}{2}(b-x)^2\rho^2\left(\int_a^b f(t)\,dt\right)\\ &\qquad\le (x-a)f(a) + (b-x)f(b) - \int_a^b f(s)\,dt\\ &\qquad\le \frac{1}{2}(b-x)^2 f'_-(b) - \frac{1}{2}f'_+(a)(x-a)^2 + \frac{1}{2}\rho^2(b-x)^2\int_a^b f(t)\,dt\\ &\qquad\qquad+\frac{1}{2}\rho^2\int_a^b (x-s)^2 f(s)\,ds - \frac{1}{2}(b-x)^2\rho^2\left(\int_a^b f(t)\,dt\right),\end{aligned}$$

namely

$$\begin{aligned}&\frac{1}{2}f'_+(x)(b-x)^2 - \frac{1}{2}f'_-(x)(x-a)^2\\ &\qquad-\frac{1}{2}\rho^2(x-a)^2\int_a^x f(t)\,dt - \frac{1}{2}(b-x)^2\rho^2\left(\int_x^b f(t)\,dt\right)\\ &\le (x-a)f(a) + (b-x)f(b) - \int_a^b f(s)\,dt - \frac{1}{2}\rho^2\int_a^b (x-s)^2 f(s)\,ds\\ &\qquad\qquad\le \frac{1}{2}(b-x)^2 f'_-(b) - \frac{1}{2}f'_+(a)(x-a)^2,\end{aligned}$$

which proves the desired result (3.1). □

Corollary 2. *With the assumptions of Theorem 5, we have*

$$\begin{aligned}(3.7)\quad 0 &\le \frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right]\\ &\le (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(s)\,ds + \frac{1}{2}\rho^2\int_a^b (b-s)(s-a)f(s)\,ds\\ &\le \frac{1}{8}(b-a)^2\left[f'_-(b) - f'_+(a)\right] + \frac{1}{8}\rho^2(b-a)^2\int_a^b f(s)\,ds.\end{aligned}$$

References

[1] M. S. S. Ali, On certain properties of trigonometrically ρ-convex functions, *Advances in Pure Mathematics*, 2012, 2, 337-340 http://dx.doi.org/10.4236/apm.2012.25047.

[2] M. S. S. Ali, On Hadamard's inequality for trigonometrically ρ-convex functions," accepted to appear in Theoretical Mathematics & Applications., March, 2013.

[3] E. F. Beckenbach, Convex functions, *Bulletin of the American Mathematical Society*, Vol. **54**, No. 5, 1948, pp. 439-460. doi:10.1090/S0002-9904-1948-08994-7

[4] M. Bessenyei, Hermite–Hadamard-type inequalities for generalized convex functions, *PhD Thesis*, Univ. Debrecen, Hungary, 2004. RGMIA Monograps, [http://rgmia.org/papers/monographs/dissertatio.pdf]

[5] F. F. Bonsall, The characterization of generalized convex functions,*The Quarterly Journal of Mathematics Oxford Series*, Vol. **1**, 1950, pp. 100-111. doi:10.1093/qmath/1.1.100

[6] A. M. Bruckner and E. Ostrow, Some functions classes related to the class of convex functions," *Pacific Journal of Mathematics*, Vol. **12**, 1962, pp. 1203-1215.

[7] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp. [Online http://www.emis.de/journals/JIPAM/article183.html?sid=183].

[8] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), no. 3,Article 35. [https://www.emis.de/journals/JIPAM/images/080_01_JIPAM/080_01.pdf].

[9] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[10] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[11] J. W. Green, Support, Convergence, and differentiability properties of generalized convex functions, *Proceedings of the American Mathematical Society*, Vol. **4**, No. 3, 1953, pp. 391-396. doi:10.1090/S0002-9939-1953-0056039-2

[12] B. Ya. Levin, *Lectures on Entire Functions*, American Mathematical Society, 1996.

[13] L. S. Maergoiz, *Asymptotic Characteristics of Entire Functions and Their Applications in Mathematics and Biophysics*, Kluwer Academic Publishers, New York, 2003.

[14] M. J. Miles, An extremum property of convex functions, *American Mathematical Monthly*, Vol. **76**, 1969, pp. 921-922. doi:10.2307/2317948

[15] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.

[16] M. M. Peixoto, On the existence of derivatives of generalized convex functions, *Summa Brasilian Mathematics,* Vol. 2, No. 3, 1948, pp. 35-42.

[17] M. M. Peixoto, Generalized convex functions and second order differential inequlities, *Bulletin of the American Mathematical Society,* Vol. **55**, No. 6, 1949, pp. 563- 572. doi:10.1090/S0002-9904-1949-09246-7

[18] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, 1973.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa