ON SOME GRÜSS' TYPE INEQUALITIES FOR THE COMPLEX INTEGRAL

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Abstract. Assume that f and g are continuous on γ, $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in [a,b]$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and the *complex Čebyšev functional* is defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz \frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under various assumptions for the functions f and g and provide a complex version for the well known Grüss inequality.

1. Introduction

For two Lebesgue integrable functions $f,\ g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [17] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.6) in the sense that it cannot be replaced by a smaller one.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|C(f,g)| \le \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_q \frac{1}{b-a}\left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt\right)^{\frac{1}{p}}, \\ \qquad\qquad \text{where } p > 1,\ 1/p + 1/q = 1. \end{cases} \tag{1.3}$$

For $\gamma = 0$, we get from the first inequality in (1.3)

$$|C(f,g)| \le \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.4}$$

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for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.3) we can deduce the following result obtained by Cheng and Sun [9]

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.5}$$

The constant $\frac{1}{2}$ is best in (1.5) as shown by Cerone and Dragomir in [7].

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

In order to extend Grüss' inequality to complex integral we need the following preparations.

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, and open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.6}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\,\ell(\gamma) \tag{1.7}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \geq 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz| \right)^{1/p}.$$

For $p=1$ we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by Hölder's inequality we have

$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t),\ t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$. If f and g are continuous on γ, we consider the *complex Čebyšev functional* defined by

$$\mathcal{D}_\gamma(f,g) := \frac{1}{w-u}\int_\gamma f(z)\,g(z)\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\,\frac{1}{w-u}\int_\gamma g(z)\,dz.$$

In this paper we establish some bounds for the magnitude of the functional $\mathcal{D}_\gamma(f,g)$ under various assumptions for the functions f and g and provide a complex version for the Grüss inequality (1.1).

2. General Results

We start with the following identity of interest:

Lemma 1. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t),\ t\in\gamma$ from $z(a)=u$ to $z(b)=w$ with $w\neq u$. If f and g are continuous on γ, then*

$$\begin{aligned}(2.1)\quad \mathcal{D}_\gamma(f,g) &= \frac{1}{2(w-u)^2}\int_\gamma\left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right)dz\\ &= \frac{1}{2(w-u)^2}\int_\gamma\left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dz\right)dw\\ &= \frac{1}{2(w-u)^2}\int_\gamma\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dzdw.\end{aligned}$$

Proof. For any $z\in\gamma$ the integral $\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw$ exists and

$$\begin{aligned}I(z) &:= \int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\\ &= \int_\gamma (f(z)g(z)+f(w)g(w)-g(z)f(w)-f(z)g(w))\,dw\\ &= f(z)g(z)\int_\gamma dw+\int_\gamma f(w)g(w)\,dw-g(z)\int_\gamma f(w)\,dw-f(z)\int_\gamma g(w)\,dw\\ &= (w-u)f(z)g(z)+\int_\gamma f(w)g(w)\,dw-g(z)\int_\gamma f(w)\,dw-f(z)\int_\gamma g(w)\,dw.\end{aligned}$$

The function $I(z)$ is also continuous on γ, then the integral $\int_\gamma I(z)\,dz$ exists and

$$\begin{aligned}\int_\gamma I(z)\,dz &= \int_\gamma \Big[(w-u) f(z) g(z) + \int_\gamma f(w) g(w)\,dw \\ &\qquad - g(z)\int_\gamma f(w)\,dw - f(z)\int_\gamma g(w)\,dw\Big]dz \\ &= (w-u)\int_\gamma f(z) g(z)\,dz + (w-u)\int_\gamma f(w) g(w)\,dw \\ &\qquad - \int_\gamma f(w)\,dw \int_\gamma g(z)\,dz - \int_\gamma g(w)\,dw\int_\gamma f(z)\,dz \\ &= 2(w-u)\int_\gamma f(z) g(z)\,dz - 2\int_\gamma f(z)\,dz\int_\gamma g(z)\,dz = 2(w-u)^2\,\mathcal{P}_\gamma(f,g),\end{aligned}$$

which proves the first equality in (2.1).

The rest follows in a similar manner and we omit the details. □

Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $f : \gamma \to \mathbb{C}$ a continuous function on γ. Define the quantity:

$$\begin{aligned}\mathcal{P}_\gamma\left(f,\overline{f}\right) &= \frac{1}{\ell(\gamma)}\int_\gamma |f(z)|^2\,|dz| - \left|\frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2 \\ &= \frac{1}{\ell(\gamma)}\int_\gamma \left|f(v) - \frac{1}{\ell(\gamma)}\int_\gamma f(z)\,|dz|\right|^2 |dv| \ge 0.\end{aligned} \tag{2.2}$$

We have:

Theorem 1. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ, then*

$$|\mathcal{D}_\gamma(f,g)| \le \frac{\ell^2(\gamma)}{|w-u|^2}\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}. \tag{2.3}$$

Proof. Taking the modulus in the first equality in (2.1), we get

$$\begin{aligned}|\mathcal{D}_\gamma(f,g)| &= \frac{1}{2|w-u|^2}\left|\int_\gamma\left(\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right)dz\right| \\ &\le \frac{1}{2|w-u|^2}\int_\gamma\left|\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right||dz| =: A\end{aligned}$$

Using the Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}&\left|\int_\gamma (f(z)-f(w))(g(z)-g(w))\,dw\right| \\ &\qquad \le \left(\int_\gamma |f(z)-f(w)|^2\,|dw|\right)^{1/2}\left(\int_\gamma |g(z)-g(w)|^2\,|dw|\right)^{1/2},\end{aligned}$$

which implies that

$$A \leq \frac{1}{2\left|w-u\right|^{2}}\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left|dz\right| =: B.$$

By the Cauchy-Bunyakovsky-Schwarz integral inequality, we also have

$$\begin{aligned}
&\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\left|dz\right| \\
&\qquad\leq\left(\int_{\gamma}\left[\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\right]^{2}\left|dz\right|\right)^{1/2} \\
&\qquad\times\left(\int_{\gamma}\left[\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)^{1/2}\right]^{2}\left|dz\right|\right)^{1/2} \\
&=\left(\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2}\left(\int_{\gamma}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2},
\end{aligned}$$

which implies that

$$\begin{aligned}
B \leq \frac{1}{2\left|w-u\right|^{2}}&\left(\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2} \\
&\times\left(\int_{\gamma}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|\right)^{1/2}.
\end{aligned} \tag{2.4}$$

Now, observe that

$$\begin{aligned}
&\int_{\gamma}\left(\int_{\gamma}\left|f\left(z\right)-f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right| \\
&=\int_{\gamma}\left(\int_{\gamma}\left(\left|f\left(z\right)\right|^{2}-2\operatorname{Re}\left(f\left(z\right)\overline{f\left(w\right)}\right)+\left|f\left(w\right)\right|^{2}\right)\left|dw\right|\right)\left|dz\right| \\
&=\int_{\gamma}\left(\ell\left(\gamma\right)\left|f\left(z\right)\right|^{2}-2\operatorname{Re}\left(f\left(z\right)\int_{\gamma}\overline{f\left(w\right)}\left|dw\right|\right)+\int_{\gamma}\left|f\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right| \\
&=\ell\left(\gamma\right)\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-2\operatorname{Re}\left(\int_{\gamma}f\left(z\right)\left|dz\right|\int_{\gamma}\overline{f\left(w\right)}\left|dw\right|\right)+\ell\left(\gamma\right)\int_{\gamma}\left|f\left(w\right)\right|^{2}\left|dw\right| \\
&=2\ell\left(\gamma\right)\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-2\operatorname{Re}\left(\int_{\gamma}f\left(z\right)\left|dz\right|\overline{\left(\int_{\gamma}f\left(w\right)\left|dw\right|\right)}\right) \\
&=2\left[\ell\left(\gamma\right)\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-\left|\int_{\gamma}f\left(z\right)\left|dz\right|\right|^{2}\right]=2\ell^{2}\left(\gamma\right)\mathcal{P}_{\gamma}\left(f,\overline{f}\right)
\end{aligned} \tag{2.5}$$

and, similarly

$$\int_{\gamma}\left(\int_{\gamma}\left|g\left(z\right)-g\left(w\right)\right|^{2}\left|dw\right|\right)\left|dz\right|=2\ell^{2}\left(\gamma\right)\mathcal{P}_{\gamma}\left(g,\overline{g}\right). \tag{2.6}$$

Making use of (2.5) and (2.6), we get

$$B \leq \frac{1}{2|w-u|^2}\left[2\ell^2(\gamma)\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}\left[2\ell^2(\gamma)\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2} = \frac{\ell^2(\gamma)}{|w-u|^2}\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2}\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2},$$

which proves the desired result (2.3). □

Remark 1. *For $g=f$ we have*

$$\mathcal{D}_\gamma(f,f) = \frac{1}{w-u}\int_\gamma f^2(z)\,dz - \left(\frac{1}{w-u}\int_\gamma f(z)\,dz\right)^2 \tag{2.7}$$

and by (2.3) we get

$$|\mathcal{D}_\gamma(f,f)| \leq \frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_\gamma\left(f,\overline{f}\right). \tag{2.8}$$

For $g=\bar{f}$ we have

$$\mathcal{D}_\gamma\left(f,\bar{f}\right) = \frac{1}{w-u}\int_\gamma |f(z)|^2\,dz - \frac{1}{w-u}\int_\gamma f(z)\,dz\frac{1}{w-u}\int_\gamma \overline{f(z)}dz \tag{2.9}$$

and by (2.3) we get

$$\left|\mathcal{D}_\gamma\left(f,\bar{f}\right)\right| \leq \frac{\ell^2(\gamma)}{|w-u|^2}\mathcal{P}_\gamma\left(f,\overline{f}\right). \tag{2.10}$$

3. Grüss' Type Inequalities

Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t\in\gamma$ from $z(a)=u$ to $z(b)=w$. Now, for ϕ, $\Phi\in\mathbb{C}$, define the sets of complex-valued functions

$$\bar{U}_\gamma(\phi,\Phi) := \left\{f:\gamma\to\mathbb{C}|\operatorname{Re}\left[(\Phi-f(z))\left(\overline{f(z)}-\overline{\phi}\right)\right]\geq 0 \text{ for each } z\in\gamma\right\}$$

and

$$\bar{\Delta}_\gamma(\phi,\Phi) := \left\{f:\gamma\to\mathbb{C}|\ \left|f(z)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for each } z\in\gamma\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_\gamma(\phi,\Phi)$ and $\bar{\Delta}_\gamma(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_\gamma(\phi,\Phi) = \bar{\Delta}_\gamma(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $w\in\mathbb{C}$ we have the equivalence

$$\left|w-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|w-\frac{\phi+\Phi}{2}\right|^2 = \operatorname{Re}\left[(\Phi-w)\left(\overline{w}-\overline{\phi}\right)\right]$$

that holds for any $w \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\bar{U}_\gamma(\phi,\Phi) = \{f : \gamma \to \mathbb{C} \mid (\operatorname{Re}\Phi - \operatorname{Re} f(z))(\operatorname{Re} f(z) - \operatorname{Re}\phi) + (\operatorname{Im}\Phi - \operatorname{Im} f(z))(\operatorname{Im} f(z) - \operatorname{Im}\phi) \geq 0 \text{ for each } z \in \gamma\}. \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_\gamma(\phi,\Phi) := \{f : \gamma \to \mathbb{C} \mid \operatorname{Re}(\Phi) \geq \operatorname{Re} f(z) \geq \operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi) \geq \operatorname{Im} f(z) \geq \operatorname{Im}(\phi) \text{ for each } z \in \gamma\}. \tag{3.3}$$

One can easily observe that $\bar{S}_\gamma(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_\gamma(\phi,\Phi) \subseteq \bar{U}_\gamma(\phi,\Phi). \tag{3.4}$$

We have the following simple facts:

Lemma 2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ, then for all $\lambda \in \mathbb{C}$ we have*

$$\mathcal{P}_\gamma(f,\overline{f}) = \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \right) \left(\overline{f(v)} - \lambda \right) |dv|. \tag{3.5}$$

Proof. We observe that

$$\begin{aligned}
&\frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \right) \left(\overline{f(v)} - \lambda \right) |dv| \\
&= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \frac{1}{\ell(\gamma)} \int_\gamma \overline{f(v)}\,|dv| \\
&- \lambda \frac{1}{\ell(\gamma)} \int_\gamma \left(f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \right) |dv| \\
&= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2 |dv| - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \frac{1}{\ell(\gamma)} \overline{\left(\int_\gamma f(v)\,|dv| \right)} \\
&= \frac{1}{\ell(\gamma)} \int_\gamma |f(v)|^2 |dv| - \left| \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \right|^2
\end{aligned}$$

for any $\lambda \in \mathbb{C}$, which proves (3.5). □

We have:

Lemma 3. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$. If f is continuous on γ and there exists $c \in \mathbb{C}$ and $\rho > 0$ such that*

$$f \in \overline{D}(c,\rho) := \{z \in \mathbb{C} \mid |z - c| \leq \rho\}, \tag{3.6}$$

then

$$0 \leq \mathcal{P}_\gamma(f,\overline{f}) \leq \rho \frac{1}{\ell(\gamma)} \left| \int_\gamma f(v) - \frac{1}{\ell(\gamma)} \int_\gamma f(z)\,|dz| \right| |dv| \tag{3.7}$$

and

$$0\leq\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\leq\rho^{2}. \tag{3.8}$$

Proof. For the equality (3.5) for $\lambda=\overline{c}$ we have

$$\begin{aligned}\mathcal{P}_{\gamma}\left(f,\overline{f}\right)&=\left|\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}\left(f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right)\left(\overline{f\left(v\right)}-\overline{c}\right)\left|dv\right|\right|\\&\leq\frac{1}{\ell\left(\gamma\right)}\left|\int_{\gamma}f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|\left|\overline{f\left(v\right)}-\overline{c}\right|\left|dv\right|\\&=\frac{1}{\ell\left(\gamma\right)}\left|\int_{\gamma}f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|\left|f\left(v\right)-c\right|\left|dv\right|\\&\leq\rho\frac{1}{\ell\left(\gamma\right)}\left|\int_{\gamma}f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|\left|dv\right|,\end{aligned}$$

which proves (3.7).

Using Cauchy-Bunyakovsky-Schwarz integral inequality, we have

$$\begin{aligned}&\frac{1}{\ell\left(\gamma\right)}\left|\int_{\gamma}f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|\left|dv\right| \qquad (3.9)\\&\leq\left(\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}\left|f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|^{2}\left|dv\right|\right)^{1/2}\\&=\left(\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}\left|f\left(z\right)\right|^{2}\left|dz\right|-\left|\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|^{2}\right)^{1/2},\end{aligned}$$

where for the last equality we used (2.2).

From (3.7) and (3.9) we have

$$0\leq\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\leq\rho\frac{1}{\ell\left(\gamma\right)}\left|\int_{\gamma}f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|\left|dv\right|\leq\rho\left[\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2},$$

which implies that

$$0\leq\left[\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\right]^{1/2}\leq\rho$$

proving the desired result (3.8). □

Corollary 2. *Suppose $\gamma\subset\mathbb{C}$ is a piecewise smooth path parametrized by $z\left(t\right)$, $t\in\gamma$ from $z\left(a\right)=u$ to $z\left(b\right)=w$. If f is continuous on γ and there exists $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$ such that $f\in\bar{\Delta}_{\gamma}\left(\phi,\Phi\right)$, then*

$$0\leq\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\leq\frac{1}{2}\left|\Phi-\phi\right|\frac{1}{\ell\left(\gamma\right)}\left|\int_{\gamma}f\left(v\right)-\frac{1}{\ell\left(\gamma\right)}\int_{\gamma}f\left(z\right)\left|dz\right|\right|\left|dv\right| \tag{3.10}$$

and

$$0\leq\mathcal{P}_{\gamma}\left(f,\overline{f}\right)\leq\frac{1}{4}\left|\Phi-\phi\right|^{2}. \tag{3.11}$$

The proof follows by Lemma 3 by choosing $c=\frac{\phi+\Phi}{2}$ and $\rho=\frac{1}{2}\left|\Phi-\phi\right|$.

We have the following Grüss' type result:

Theorem 2. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ and there exists $c \in \mathbb{C}$ and $\rho > 0$ such that (3.6) is valid, then*

$$|\mathcal{D}_\gamma(f,g)| \leq \frac{\ell^2(\gamma)}{|w-u|^2}\rho\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}. \tag{3.12}$$

The proof follows by Theorem 1 and Lemma 3.

If we take $g = f$ in (3.12), we get

$$|\mathcal{D}_\gamma(f,f)| \leq \frac{\ell^2(\gamma)}{|w-u|^2}\rho\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2} \leq \frac{\ell^2(\gamma)}{|w-u|^2}\rho^2 \tag{3.13}$$

and for $g = \overline{f}$ we get

$$\left|\mathcal{D}_\gamma\left(f,\overline{f}\right)\right| \leq \frac{\ell^2(\gamma)}{|w-u|^2}\rho\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2} \leq \frac{\ell^2(\gamma)}{|w-u|^2}\rho^2, \tag{3.14}$$

provided that f satisfies the condition (3.6).

Corollary 3. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ and there exists ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$ such that $f \in \bar{\Delta}_\gamma(\phi,\Phi)$, then*

$$|\mathcal{D}_\gamma(f,g)| \leq \frac{1}{2}|\Phi-\phi|\frac{\ell^2(\gamma)}{|w-u|^2}\left[\mathcal{P}_\gamma(g,\overline{g})\right]^{1/2}. \tag{3.15}$$

In particular, we have

$$|\mathcal{D}_\gamma(f,f)| \leq \frac{1}{2}|\Phi-\phi|\frac{\ell^2(\gamma)}{|w-u|^2}\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2} \leq \frac{1}{4}|\Phi-\phi|^2\frac{\ell^2(\gamma)}{|w-u|^2} \tag{3.16}$$

and

$$\left|\mathcal{D}_\gamma\left(f,\overline{f}\right)\right| \leq \frac{1}{2}|\Phi-\phi|\frac{\ell^2(\gamma)}{|w-u|^2}\left[\mathcal{P}_\gamma\left(f,\overline{f}\right)\right]^{1/2} \leq \frac{1}{4}|\Phi-\phi|^2\frac{\ell^2(\gamma)}{|w-u|^2}. \tag{3.17}$$

We have the following Grüss type inequality:

Corollary 4. *Suppose $\gamma \subset \mathbb{C}$ is a piecewise smooth path parametrized by $z(t)$, $t \in \gamma$ from $z(a) = u$ to $z(b) = w$ with $w \neq u$. If f and g are continuous on γ and there exists ϕ, Φ, ψ, $\Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_\gamma(\phi,\Phi)$ and $g \in \bar{\Delta}_\gamma(\psi,\Psi)$ then*

$$|\mathcal{D}_\gamma(f,g)| \leq \frac{1}{4}|\Phi-\phi||\Psi-\psi|\frac{\ell^2(\gamma)}{|w-u|^2}. \tag{3.18}$$

If the path γ is a segment $[u,w]$ connecting two distinct points u and w in $\mathbb{C}$ then we write $\int_\gamma f(z)\,dz$ as $\int_u^w f(z)\,dz$.

If f, g are continuous on $[u,w]$ and there exists ϕ, Φ, ψ, $\Psi \in \mathbb{C}$, $\phi \neq \Phi$, $\psi \neq \Psi$ such that $f \in \bar{\Delta}_{[u,w]}(\phi,\Phi)$ and $g \in \bar{\Delta}_{[u,w]}(\psi,\Psi)$ then

$$\left|\frac{1}{w-u}\int_u^w f(z)g(z)\,dz - \frac{1}{w-u}\int_u^w f(z)\,dz\frac{1}{w-u}\int_u^w g(z)\,dz\right| \leq \frac{1}{4}|\Phi-\phi||\Psi-\psi|. \tag{3.19}$$

4. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t), \ t \in [a,b].$$

If $[a,b] = [0,\pi]$ then we get a half circle while for $[a,b] = [0,2\pi]$ we get the full circle.

Since

$$\begin{aligned}\left|e^{is} - e^{it}\right|^2 &= \left|e^{is}\right|^2 - 2\operatorname{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2 \\ &= 2 - 2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,\ s \in \mathbb{R}$, then

$$\left|e^{is} - e^{it}\right|^r = 2^r \left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{4.1}$$

for any $t,\ s \in \mathbb{R}$ and $r>0$. In particular,

$$\left|e^{is} - e^{it}\right| = 2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,\ s \in \mathbb{R}$.

If $u = R\exp(ia)$ and $w = R\exp(ib)$ then

$$\begin{aligned}w - u &= R[\exp(ib) - \exp(ia)] = R[\cos b + i\sin b - \cos a - i\sin a] \\ &= R[\cos b - \cos a + i(\sin b - \sin a)].\end{aligned}$$

Since

$$\cos b - \cos a = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

and

$$\sin b - \sin a = 2\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right),$$

hence

$$\begin{aligned}w - u &= R\left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) + 2i\sin\left(\frac{b-a}{2}\right)\cos\left(\frac{a+b}{2}\right)\right] \\ &= 2R\sin\left(\frac{b-a}{2}\right)\left[-\sin\left(\frac{a+b}{2}\right) + i\cos\left(\frac{a+b}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{b-a}{2}\right)\left[\cos\left(\frac{a+b}{2}\right) + i\sin\left(\frac{a+b}{2}\right)\right] \\ &= 2Ri\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right].\end{aligned}$$

If $\gamma = \gamma_{[a,b],R}$, then the *circular complex Čebyšev functional* is defined by

$$\begin{aligned}(4.2)\quad \mathcal{C}_{[a,b],R}(f,g) &:= \mathcal{D}_{\gamma_{[a,b],R}}(f,g)\\ &= \frac{1}{2\sin\left(\frac{b-a}{2}\right)\exp\left[\left(\frac{a+b}{2}\right)i\right]}\int_a^b f(R\exp(it))\,g(R\exp(it))\exp(it)\,dt\\ &\quad-\frac{1}{4\sin^2\left(\frac{b-a}{2}\right)\exp\left[2\left(\frac{a+b}{2}\right)i\right]}\\ &\quad\times\int_a^b f(R\exp(it))\exp(it)\,dt\int_a^b g(R\exp(it))\exp(it)\,dt.\end{aligned}$$

We have the following result:

Proposition 2. *Let $\gamma_{[a,b],R}$ be a circular path centered in 0 and with radius $R>0$ and $[a,b]\subset[0,2\pi]$. If f and g are continuous on $\gamma_{[a,b],R}$ and there exists $c\in\mathbb{C}$ and $\rho>0$ such that (3.6) is valid, then*

$$\begin{aligned}(4.3)\quad \left|\mathcal{C}_{[a,b],R}(f,g)\right| &\le \frac{(b-a)^2}{4\sin^2\left(\frac{b-a}{2}\right)}\rho\\ &\quad\times\left[\frac{1}{b-a}\int_a^b |g(R\exp(it))|^2\,dt-\left|\frac{1}{b-a}\int_a^b g(R\exp(it))\,dt\right|^2\right]^{1/2}.\end{aligned}$$

We have the Grüss' type inequality for circular paths:

Corollary 5. *Let $\gamma_{[a,b],R}$ be a circular path centered in 0 and with radius $R>0$ and $[a,b]\subset[0,2\pi]$. If f and g are continuous on $\gamma_{[a,b],R}$ and there exists ϕ, Φ, ψ, $\Psi\in\mathbb{C}$, $\phi\ne\Phi$, $\psi\ne\Psi$ such that $f\in\bar{\Delta}_\gamma(\phi,\Phi)$ and $g\in\bar{\Delta}_\gamma(\psi,\Psi)$ then*

$$(4.4)\quad \left|\mathcal{C}_{[a,b],R}(f,g)\right| \le \frac{1}{16}|\Phi-\phi|\,|\Psi-\psi|\,\frac{(b-a)^2}{\sin^2\left(\frac{b-a}{2}\right)}.$$

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